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I. g.

For the Royal Society of Arts & Sciences, Edinburg

from the Translator.

MÉCANIQUE CÉLESTE.

MÉCANIQUE CÉLESTE.

BY THE

MARQUIS DE LA PLACE,

PEER OF FRANCE; GRAND CROSS OF THE LEGION OF HONOR; MEMBER OF THE FRENCH ACADEMY, OF THE ACADEMY
OF SCIENCES OF PARIS, OF THE BOARD OF LONGITUDE OF FRANCE, OF THE ROYAL SOCIETIES OF
LONDON AND GÖTTINGEN, OF THE ACADEMIES OF SCIENCES OF RUSSIA, DENMARK,
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AMERICAN ACADEMY OF ARTS AND SCIENCES; ETC.

TRANSLATED, WITH A COMMENTARY,

BY

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FELLOW OF THE ROYAL SOCIETIES OF LONDON, EDINBURGH, AND DUBLIN; OF THE PHILOSOPHICAL SOCIETY HELD
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VOLUME I.

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DISTRICT OF MASSACHUSETTS, TO WIT: *District Clerk's Office.*

Be it remembered, that on the sixteenth day of October, A. D. 1829, in the fifty-fourth year of the Independence of the United States of America, Nathaniel Bowditch, of the said district, has deposited in this office the title of a book, the right whereof he claims as Proprietor, in the words following, to wit: "Mécanique Céleste. By the Marquis De La Place, Peer of France; Grand Cross of the Legion of Honor; Member of the French Academy, of the Academy of Sciences of Paris, of the Board of Longitude of France, of the Royal Societies of London and Göttingen, of the Academies of Sciences of Russia, Denmark, Sweden, Prussia, Holland, and Italy; Member of the American Academy of Arts and Sciences; etc. Translated, with a Commentary, by Nathaniel Bowditch, LL. D., Fellow of the Royal Societies of London, Edinburgh, and Dublin; of the Philosophical Society held at Philadelphia; of the American Academy of Arts and Sciences; etc." In conformity to the Act of the Congress of the United States, entitled, "An Act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned:" and also to an Act, entitled, "An Act, supplementary to an Act, entitled, An Act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned; and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

JNO. W. DAVIS, *Clerk of the District of Massachusetts.*

INTRODUCTION BY THE TRANSLATOR.

THE object of the author, in composing this work, as stated by him in his preface, was to reduce all the known phenomena of the system of the world to the law of gravity, by strict mathematical principles; and to complete the investigations of the motions of the planets, satellites, and comets, begun by Newton in his Principia. This he has accomplished, in a manner deserving the highest praise, for its symmetry and completeness; but from the abridged manner, in which the analytical calculations have been made, it has been found difficult to be understood by many persons, who have a strong and decided taste for mathematical studies, on account of the time and labour required, to insert the intermediate steps of the demonstrations, necessary to enable them easily to follow the author in his reasoning. To remedy, in some measure, this defect, has been the chief object of the translator in the notes. It is hoped that the facility, arising from having the work in our own language, with the aid of these explanatory notes, will render it more accessible to persons who have been unable to prepare themselves for this study by a previous course of reading, in those modern publications, which contain the many important discoveries in analysis, made since the time of Newton. It is expected that the reader should be acquainted with the common principles of spherical trigonometry, conic sections, and those branches of the fluxionary, or differential calculus, usually found in elementary treatises on this subject, in this country; and as frequent use is made of the rules for the products of the sines and cosines of angles, &c., it was thought expedient to collect together, at the end of this Introduction, such formulas as are of frequent use. The demonstrations of these formulas may be found in most treatises of trigonometry, and some of them occur in the notes on this work; the methods in which these demonstrations may be obtained, as well as those of the common problems of spherical trigonometry, are also briefly pointed out, in the appendix, at the end of this volume, which may be referred to, in cases where it may be found necessary. Some of the most important theorems in conic sections are also demonstrated in the second book.

The notation of the author has been strictly adhered to, and the double parentheses, which he has used to denote the partial differentials, have been retained, though at present many mathematicians reject them.

For the sake of a more easy method of reference, to any particular part of the work, or to any single formula, the marginal numbers are inserted. These are frequently referred to, in the translation, and in the notes. The introduction of these numbers is the only alteration which has been made in the original work. In other respects it will be found, that the translation has been as nearly literal, as is consistent with a faithful interpretation of the sense of the author. These marginal references might supersede the use of those made by the author, in a few of the most important formulas, but it was thought best to retain them, because they might possibly be referred to, in quoting from the original work. It must be observed that in citing a single formula, the marginal reference will be found on the same line with the formula; but in referring to a particular sentence, or paragraph, it will generally be on the middle line of it.

As the author has supposed the quadrant of a circle to be divided into 100 degrees, each degree into 100 minutes, each minute into 100 seconds, &c., and has applied the usual marks $^{\circ}\ '\ ''$ &c., to these quantities; it has been found convenient, in the notes, when the sexagesimal division is used, to employ the letters *d*, *m*, *s*, &c., to denote degrees, minutes, seconds, &c., of the common sexagesimal notation; so that $1000''$ is equivalent to 324^{s}. This distinction will be adhered to throughout the work.

The notes were written at the time of reading the volumes, as they were successively published. The translation was made between the years 1815 and 1817, at which time the four first volumes, with the several appendices and notes, were ready for publication. Soon afterwards, the American Academy of Arts and Sciences liberally offered to print the work at their expense, but this proposal was not accepted. One of the reasons for not printing it at that time, was the expectation that the author would publish another edition, in which he might modify the first volume, by the introduction of the matter contained in the appendix to the third volume, depending on the improvements made by Mr. Poisson, in the demonstration of the permanency of the mean motions of the planets; and might also correct the second volume, on account of the defects in some parts of the theory of the calculation of the attraction of spheroids, and make other alterations, on account of the improvements in the calculation of the attraction of an ellipsoid, first pointed out by Mr. Ivory.

The notes are adapted in some respects to the state of the elementary publications on scientific subjects in this country, and a greater number have been given, than would have been necessary, if the elementary principles of some of the methods, used by the author, had been in common use in our schools and colleges. They might in some cases have been abridged, by small alterations in the original work, but it was thought best to adhere strictly to the method of the author.

It may be advisable for a young person, in reading this volume for the first time, to pass over the eighth chapter of the first book, which treats of the motion of fluids, being rather more difficult than the rest of this volume; he may also pass over the fourth and sixth chapters of the same book. After reading the second book, which contains all the most interesting principles of the motions of the heavenly bodies, he can return with additional force, to these chapters, before entering on the calculation of the figures of the heavenly bodies in the second volume.

Since this work was prepared for publication, there have been printed in England, two translations of the first book, with notes, by Mr. Toplis and Dr. Young, which were seen before this volume was printed, and occasional use has been made of them. It is understood that Mr. Harte is now printing a translation, but no copy of it has yet been received in this part of the country.

The second volume of this translation is now in the press, and will be published in the course of the next year. These two volumes will finish the first part of the work, which may be considered as forming a complete treatise in itself. If it should be found expedient, the whole work will be printed, in five or six volumes, as soon as it can be done with convenience, taking sufficient time to ensure typographical accuracy, in the execution, and the whole will probably be completed in four or five years. This time has been considered necessary on account of the laborious occupation of the translator, which affords him but little leisure to attend to the revision and publication of the work.

The following formulas are much used in the course of this work. They are to be found in most treatises on Trigonometry, and may be demonstrated by the method given in the appendix to this volume.

[1] Int. $\sin.^2 z = -\frac{1}{2}\cdot\left\{\cos. 2z - 1\right\},$ Radius $= 1$.

[2] " $\sin.^3 z = -\frac{1}{2^2}\cdot\left\{\sin. 3z - 3 \,.\, \sin. z\right\},$

[3] " $\sin.^4 z = \frac{1}{2^3}\cdot\left\{\cos. 4z - 4 \,.\, \cos. 2z + \tfrac{1}{2}\cdot\frac{4.3}{1.2}\right\},$

[4] " $\sin.^5 z = \frac{1}{2^4}\cdot\left\{\sin. 5z - 5 \,.\, \sin. 3z + \frac{5.4}{1.2}\cdot\sin. z\right\},$

[5] " $\sin.^6 z = -\frac{1}{2^5}\cdot\left\{\cos. 6z - 6 \,.\, \cos. 4z + \frac{6.5}{1.2}\cdot\cos. 2z - \tfrac{1}{2}\cdot\frac{6.5.4}{1.2.3}\right\},$

[6] " $\cos.^2 z = \frac{1}{2}\cdot\left\{\cos. 2z + 1\right\},$

[7] " $\cos.^3 z = \frac{1}{2^2}\cdot\left\{\cos. 3z + 3 \,.\, \cos. z\right\},$

[8] " $\cos.^4 z = \frac{1}{2^3}\cdot\left\{\cos. 4z + 4 \,.\, \cos. 2z + \tfrac{1}{2}\cdot\frac{4.3}{1.2}\right\},$

[9] " $\cos.^5 z = \frac{1}{2^4}\cdot\left\{\cos. 5z + 5 \,.\, \cos. 3z + \frac{5.4}{1.2}\cdot\cos. z\right\},$

[10] " $\cos.^6 z = \frac{1}{2^5}\cdot\left\{\cos. 6z + 6 \,.\, \cos. 4z + \frac{6.5}{1.2}\cdot\cos. 2z + \tfrac{1}{2}\cdot\frac{6.5.4}{1.2.3}\right\},$

[11] " $\sin. z = \frac{c^{z.\sqrt{-1}} - c^{-z.\sqrt{-1}}}{2.\sqrt{-1}},$ Hyp. log. $c = 1$.

[12] " $\cos. z = \frac{c^{z.\sqrt{-1}} + c^{-z.\sqrt{-1}}}{2},$

[13] " $c^{z.\sqrt{-1}} = \cos. z + \sqrt{-1}\,.\,\sin. z,$

[14] " $c^{-z.\sqrt{-1}} = \cos. z - \sqrt{-1}\,.\,\sin. z,$

$\{\cos. z + \sqrt{-1} . \sin. z\}^n = c^{nz.\sqrt{-1}} = \cos. nz + \sqrt{-1} . \sin. nz,$ [15] Int.

$\{\cos. z - \sqrt{-1} . \sin. z\}^n = c^{-nz.\sqrt{-1}} = \cos. nz - \sqrt{-1} . \sin. nz,$ [16] ”

$\sin. a . \sin. b = \tfrac{1}{2} . \cos. (a-b) - \tfrac{1}{2} . \cos. (a+b),$ [17] ”

$\sin. a . \cos. b = \tfrac{1}{2} . \sin. (a+b) + \tfrac{1}{2} . \sin. (a-b),$ [18] ”

$\sin. a . \cos. b = \tfrac{1}{2} . \sin. (a+b) - \tfrac{1}{2} . \sin. (b-a),$ [19] ”

$\cos. a . \cos. b = \tfrac{1}{2} . \cos. (a+b) + \tfrac{1}{2} . \cos. (a-b),$ [20] ”

$\sin. (a+b) = \sin. a . \cos. b + \cos. a . \sin. b,$ [21] ”

$\sin. (a-b) = \sin. a . \cos. b - \cos. a . \sin. b,$ [22] ”

$\cos. (a+b) = \cos. a . \cos. b - \sin. a . \sin. b,$ [23] ”

$\cos. (a-b) = \cos. a . \cos. b + \sin. a . \sin. b,$ [24] ”

$\sin. a + \sin. b = 2 . \sin. \tfrac{1}{2} . (a+b) . \cos. \tfrac{1}{2} . (a-b),$ [25] ”

$\sin. a - \sin. b = 2 . \sin. \tfrac{1}{2} . (a-b) . \cos. \tfrac{1}{2} . (a+b),$ [26] ”

$\cos. a + \cos. b = 2 . \cos. \tfrac{1}{2} . (a+b) . \cos. \tfrac{1}{2} . (a-b),$ [27] ”

$\cos. a - \cos. b = 2 . \sin. \tfrac{1}{2} . (a+b) . \sin. \tfrac{1}{2} . (b-a),$ [28] ”

$\text{tang.} (a+b) = \dfrac{\text{tang.} a + \text{tang.} b}{1 - \text{tang.} a . \text{tang.} b},$ [29] ”

$\text{tang.} (a-b) = \dfrac{\text{tang.} a - \text{tang.} b}{1 + \text{tang.} a . \text{tang.} b},$ [30] ”

$\text{tang.} 2a = \dfrac{2 . \text{tang.} a}{1 - \text{tang.}^2 a},$ [30'] ”

$\sin. 2a = \dfrac{2 . \text{tang.} a}{1 + \text{tang.}^2 a},$ [30''] ”

$\sin. 2a = 2 . \sin. a . \cos. a,$ [31] ”

$\cos. 2a = \cos.^2 a - \sin.^2 a,$ [32] ”

$\cos. 2a = 1 - 2 . \sin.^2 a,$ [33] ”

$\cos. 2a = 2 . \cos.^2 a - 1,$ [34] ”

$\text{tang.} a = \dfrac{\sin. a}{\cos. a} = \dfrac{1}{\cot. a},$ [34'] ”

$\dfrac{1}{\cos. a} = \sec. a,$ [34''] ”

$\dfrac{1}{\cos.^2 a} = \sec.^2 a = 1 + \text{tang.}^2 a,$ [34'''] ”

[35] Int. $\frac{\sin. a - \sin. b}{\sin. a + \sin. b} = \frac{\text{tang.} \frac{1}{2}(a-b)}{\text{tang.} \frac{1}{2}(a+b)}$,

[36] " $\frac{\sin. a - \sin. b}{\cos. a + \cos. b} = \text{tang.} \frac{1}{2} . (a - b)$,

[37] " $\frac{\sin. a + \sin. b}{\cos. a + \cos. b} = \text{tang.} \frac{1}{2} . (a + b)$,

[38] " $\frac{\sin. a + \sin. b}{\cos. a - \cos. b} = - \cot. \frac{1}{2} . (a - b)$,

[39] " $\frac{\cos. a - \cos. b}{\cos. a + \cos. b} = \text{tang.} \frac{1}{2} . (a + b) . \text{tang.} \frac{1}{2} . (b - a)$,

[40] " $\frac{1 - \cos. b}{1 + \cos. b} = \text{tang.}^2 \frac{1}{2} b$,

[41] " $\frac{\sin. a}{1 + \cos. a} = \text{tang.} \frac{1}{2} a$,

[41′] " $\frac{1 + \cos. a}{\sin. a} = \cot. \frac{1}{2} a$,

[42] " $\frac{\sin. a}{1 - \cos. a} = \cot. \frac{1}{2} a$,

[42′] " $\frac{1 - \cos. a}{\sin. a} = \text{tang.} \frac{1}{2} a$,

[43] " $\sin. z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7} + \&c.$, [607*d*],

[44] " $\cos. z = 1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6} + \&c.$, [607*e*],

[45] " $\text{tang.} z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \&c.$,

[46] " $z = \text{arc.} (\sin. y) = y + \frac{1}{6} y^3 + \frac{3}{40} y^5 + \&c.$,

[47] " $z = \text{arc.} (\cos. x) = (1 - x) + \frac{1}{6} . (1 - x^3) + \frac{3}{40} . (1 - x^5) + \&c.$,

[48] " $z = \text{arc.} (\text{tang.} t) = t - \frac{1}{3} t^3 + \frac{1}{5} t^5 + \&c.$,

[49] " $dz = d . (\text{arc.} \sin. y) = \frac{dy}{\sqrt{1 - yy}}$,

[50] " $dz = d . (\text{arc.} \cos. x) = \frac{-dx}{\sqrt{1 - xx}}$,

[51] " $dz = d . (\text{arc.} \text{tang.} t) = \frac{dt}{1 + tt}$,

$d . \sin. z = dz . \cos. z,$ [52] Int.

$d . \cos. z = - dz . \sin. z,$ [53] ”

$d . \text{tang}. z = \frac{dz}{\cos.^2 z} = dz . (1 + \text{tang}.^2 z),$ [54] ”

$c^z = 1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.,$ Hyp. log. $c = 1$, [607*c*], [55] ”

$c^{-z} = 1 - z + \frac{z^2}{1.2} - \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} - \&c.,$ [56] ”

$a^x = 1 + x . \log. a + \frac{x^2}{1.2} . (\log. a)^2 + \frac{x^3}{1.2.3} . (\log. a)^3,$ [607*b*], [56'] ”

$d . c^z = dz . c^z,$ [57] ”

$\text{hyp. log.} (1 + x) = x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 - \tfrac{1}{4} x^4 + \&c.,$ [58] ”

$d . \text{hyp. log.} y = \frac{dy}{y},$ [59] ”

$\sin. (z + \alpha) = \sin. z + \alpha . \cos. z,$ neglecting $\alpha^2, \alpha^3,$ &c., [60] ”

$\cos. (z + \alpha) = \cos. z - \alpha . \sin. z,$ ibid. [61] ”

In the notes may be found several formulas, definitions, &c., some of which will be often referred to, namely,

Change of rectangular co-ordinates from one system to another, [171*a*—172*i*].

Composition and resolution of rotatory motions, [230*r*—231*c*].

Conic Sections, [378*a*—379*e*, 603*a*, 726—750].

Curve of double curvature, [25*b*].

Differentials, partial, complete, exact, [13*a*—14*a*].

Elliptical Functions of Le Gendre, [82*a*],

$$F . (c, \varphi) = \int \frac{d\varphi}{\Delta . (c, \varphi)}; \qquad E . (c, \varphi) = \int d\varphi . \Delta . (c, \varphi),$$

$$\Pi . (n, c, \varphi) = \int \frac{d\varphi}{(1 + n . \sin.^2 \varphi) . \Delta . (c, \varphi)}.$$

The last of these functions is inaccurately printed in [82*a*], the factor $\Delta . (c, \varphi)$, ought to be in the denominator.

Equations of a right line, [19b', 19b''].
" cycloid, [102a].
" plane, [19c, d].
" spherical surface, [19e].
Integral formulas, $\int d\theta \,.\, \sin.^n \theta$, [84$a$—$e$].
Linear functions, [125a].
Mechanical powers, [114a, &c.]
Plane triangle CAM, in the second figure page 292,

[62] Int. $$AM^2 = CM^2 - 2\, CM \,.\, AC \,.\, \cos.\, ACM + AC^2, \qquad [471].$$

Principle of the least squares, [849k].
Radius of curvature, [53a, b, c].
Spherics. The fundamental theorem,

[63] " $$\cos.\, AG = \cos.\, AB \,.\, \cos.\, BG + \sin.\, AB \,.\, \sin.\, BG \,.\, \cos.\, ABG,$$

[172i], corresponding to a spherical triangle ABG, is used in the appendix, page 729, &c., to demonstrate the most useful propositions in spherical trigonometry.

Theorems of Maclaurin, [607a].
" Taylor, [617].
" La Grange, [629c].
Variations. Principles of this method, [36a—k].

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SECOND BOOK.

ON THE LAW OF UNIVERSAL GRAVITATION, AND THE MOTIONS OF THE CENTRES OF GRAVITY OF THE HEAVENLY BODIES.

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The mean longitude of the first satellite, minus three times that of the second, plus twice that of the third, is invariably equal to two right angles, [1239^{vi}].

These two theorems take place, notwithstanding the alterations which the mean motions of the satellites may suffer, either from a cause similar to that which alters the mean motion of the moon, or from the resistance of a very rare medium. These theorems give rise to an arbitrary inequality, which differs for each of the three satellites, only by its coefficient. This inequality is insensible by observation, [1240—1242^{v}]. . . . . § 66

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PREFACE BY THE AUTHOR.

Towards the end of the seventeenth century, Newton published his discovery of universal gravitation. Mathematicians have, since that epoch, succeeded in reducing to this great law of nature all the known phenomena of the system of the world, and have thus given to the theories of the heavenly bodies, and to astronomical tables, an unexpected degree of precision. My object is to present a connected view of these theories, which are now scattered in a great number of works. The whole of the results of gravitation, upon the equilibrium and motions of the fluid and solid bodies, which compose the solar system, and the similar systems, existing in the immensity of space, constitute the object of *Celestial Mechanics*, or the application of the principles of mechanics to the motions and figures of the heavenly bodies. Astronomy, considered in the most general manner, is a great problem of mechanics, in which the elements of the motions are the arbitrary constant quantities. The solution of this problem depends, at the same time, upon the accuracy of the observations, and upon the perfection of the analysis. It is very important to reject every empirical process, and to complete the analysis, so that it shall not be necessary to derive from observations any but indispensable data. The intention of this work is to obtain, as much as may be in my power, this interesting result. I hope, in consideration of the difficulty and importance of the subject, that mathematicians and astronomers will receive it with indulgence, and that they will find the results sufficiently simple to be used in their researches.

It will be divided into two parts. In the first part, I shall give the methods and formulas, to determine the motions of the centres of gravity of the heavenly bodies, the figures of those bodies, the oscillations of the fluids which cover them, and the motions about their centres of gravity. In the second part, I shall apply the formulas found in the first, to the planets, satellites, and comets; and I shall conclude the work, with an examination of several questions relative to the system of the world, and with an historical account of the labors of mathematicians upon this subject. I shall adopt the decimal division of the right angle, and of the day, and shall refer the linear measures to the length of the metre, determined by the arc of the terrestrial meridian comprised between Dunkirk and Barcelona.

FIRST PART.

GENERAL THEORY OF THE MOTIONS AND FIGURES OF THE HEAVENLY BODIES.

FIRST BOOK.

ON THE GENERAL LAWS OF EQUILIBRIUM AND MOTION.

It is my intention to give in this book the general principles of the equilibrium and motion of bodies, and to solve those problems of mechanics which are indispensable in the theory of the system of the world.

CHAPTER I.

ON THE EQUILIBRIUM AND COMPOSITION OF FORCES WHICH ACT ON A MATERIAL POINT.

1. A BODY appears to us to be in motion, when it changes its relative situation with respect to a system of bodies supposed to be at rest; but as all bodies, even those which appear in the most perfect repose, may be in motion; a space is conceived of, without bounds, immoveable, and penetrable by the particles of matter; and we refer in our minds the position of bodies to the parts of this real, or ideal space, supposing the bodies to be in motion, when they correspond, in successive moments, to different parts of this space.

The nature of that singular modification, by means of which a body is transported from one place to another, is now, and always will be, unknown; it is denoted by the name of *Force*. We can only ascertain its effects, and the laws of its action. The effect of a force acting upon a material point, or particle, is to put it in motion, if no obstacle is opposed; the direction of the force is the right line which it tends to make the point describe. It is evident, that if two forces act in the same direction, the resultant is the sum of the two forces; but if they act in contrary directions, the point is affected by the **Force.**

difference of the forces. If their directions form an angle with each other, the force which results will have an intermediate direction between the two proposed forces. We shall now investigate the quantity and direction of this resulting force.

For this purpose, let us consider two forces, x and y, acting at the same moment upon a material point M, in directions forming a right angle with each other. Let z be their resultant, and θ the angle which it makes with the direction of the force x. The two forces x and y being given, the angle θ and the quantity z must have determinate values, so that there will exist, between the three quantities x, z and θ, a relation which is to be investigated.

Suppose in the first place that the two forces x and y are infinitely small, and equal to the differentials dx, dy. Then suppose that x becomes successively dx, $2dx$, $3dx$, &c., and y becomes dy, $2dy$, $3dy$, &c., it is evident that the angle θ will remain constant, and the resultant z will become successively dz, $2dz$, $3dz$, &c., and in the successive increments of the three forces x, y and z, the ratio of x to z will be constant, and may be expressed by a function of θ, which we shall denote by $\varphi(\theta)$;* we shall therefore have $x = z.\varphi(\theta)$, in which equation we may change x into y, provided we also change the angle θ into $\frac{\pi}{2} - \theta$, π being the semi-circumference of a circle whose radius is unity.

Now we may consider the force x as the resultant of two forces x' and x'', of which the first x' is directed along the resultant z, and the second x'' is perpendicular to it.† The force x, which results from these two new forces,

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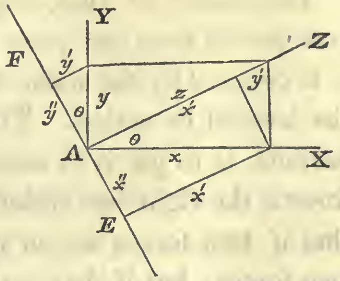
* A quantity z is said to be a function of another quantity x, when it depends on it in any manner. Thus, if z, y be variable, a, b, c, &c. constant, and we have either of the following expressions, $z = ax + b$, $z = ax^2 + bx + c$; $z = a^x$, $z = \sin. ax$, &c. z will be a function
[1a] of x; and if the precise form of the function is known, as in these examples, it is called an *explicit* function. If the form is not known, but must be found by some algebraical process, it is called an *implicit* function.

† (2) For illustration, suppose the forces x and y to act at the point A, in the directions AX, AY, respectively, and that the resultant z is in the direction AZ, forming with AX, AY, the angles $ZAX = \theta$, $ZAY = \frac{\pi}{2} - \theta$. Then as above, we have $x = z.\varphi(\theta)$, $y = z.\varphi\left(\frac{\pi}{2} - \theta\right)$. Draw EAF perpendicular to AZ, and suppose the force x in the direction AX to be resolved into two forces, x', x'', in the



forms the angle θ with the force x', and the angle $\frac{\pi}{2}-\theta$ with the force x''; we shall therefore have

$$x'=x.\varphi(\theta)=\frac{x^2}{z}; \qquad x''=x.\varphi\left(\frac{\pi}{2}-\theta\right)=\frac{xy}{z}; \qquad [1]$$

and we may substitute these two forces instead of the force x. We may likewise substitute for the force y two new forces, y' and y'', of which the first is equal to $\frac{y^2}{z}$ in the direction z, and the second equal to $\frac{xy}{z}$ perpendicular to z; we shall thus have, instead of the two forces x and y, the four following:

$$\frac{x^2}{z},\ \frac{y^2}{z},\ \frac{xy}{z},\ \frac{xy}{z}; \qquad [2]$$

the two last, acting in contrary directions, destroy each other;* the two first, acting in the same direction, are to be added together, and produce the resultant z; we shall therefore have†

$$x^2+y^2=z^2; \qquad [3]$$

whence it follows, that the resultant of the two forces x and y is represented in magnitude, by the diagonal of the rectangle whose sides represent those forces.

---

directions AZ, AE, respectively, so that the angle $ZAX=\theta$, and $XAE=\frac{\pi}{2}-\theta$. Then, in the same manner in which the above values of x, y, are obtained from z, we may get $x'=x.\varphi(\theta)$; $x''=x.\varphi\left(\frac{\pi}{2}-\theta\right)$. If in these we substitute the values $\varphi(\theta)=\frac{x}{z}$; $\varphi\left(\frac{\pi}{2}-\theta\right)=\frac{y}{z}$, deduced from the above equations, we obtain $x'=\frac{x^2}{z}$; $x''=\frac{xy}{x}$. In like manner, if the force y, in the direction AY, be resolved into the two forces y', y'', in the directions AZ, AF, making the angle $YAZ=\frac{\pi}{2}-\theta$, $YAF=\theta$, we shall have $y'=y.\varphi\left(\frac{\pi}{2}-\theta\right)$; $y''=y.\varphi(\theta)$; which, by substituting the above values of $\varphi\left(\frac{\pi}{2}-\theta\right)$, $\varphi(\theta)$, become $y'=\frac{y^2}{z}$, $y''=\frac{xy}{z}$, as above.

* (3) For, by the preceding note, the force $x''=\frac{xy}{z}$, is in the direction AE, and the force $y''=\frac{xy}{z}$, is in the *opposite* direction AF, and as they are equal they must destroy each other.

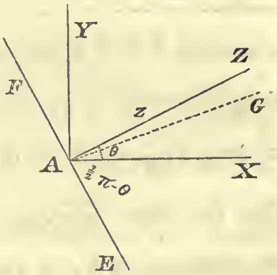
† (4) The sum of the two forces $x'=\frac{x^2}{z}$, $y'=\frac{y^2}{z}$, in the direction AZ, being put equal to the resultant z, gives $\frac{x^2}{z}+\frac{y^2}{z}=z$, which multiplied by z becomes $x^2+y^2=z^2$.

Let us now determine the angle θ. If we increase the force x by the differential dx, without varying the force y, that angle will be diminished by the infinitely small quantity $d\theta$;* now we may conceive the force dx to be resolved into two other forces, the one dx' in the direction z, and the other dx'' perpendicular to z; the point M will then be acted upon by the two forces $z+dx'$ and dx'', perpendicular to each other, and the resultant of these two forces, which we shall call z', will make with dx'' the angle $\frac{\pi}{2}-d\theta$;† we shall thus have, by what precedes,

[4] $$dx''=z'.\varphi\left(\frac{\pi}{2}-d\theta\right);$$

consequently the function $\varphi\left(\frac{\pi}{2}-d\theta\right)$ is infinitely small, and of the form $-k\,d\theta$, k being a constant quantity, independent of the angle θ;‡ we shall therefore have

[5] $$\frac{dx''}{z'}=-k\,d\theta,$$



[*] (5) The resultant of the forces x, y, is, by hypothesis, in the direction AZ, and, by increasing the force x by dx, the forces become equal to z in the direction AZ, and dx in the direction AX, and the resulting force z', must evidently fall between AZ, AX, on a line as AG, forming with AZ an infinitely small angle ZAG, represented by $d\theta$. Then the force dx, in the direction AX, may be resolved into two forces, the one dx' in the direction AZ, the other dx'' in the direction AE, and as this last force is inclined to AX by the angle $XAE=\frac{\pi}{2}-\theta$, we shall have as above $dx''=dx.\varphi\left(\frac{\pi}{2}-\theta\right)$; or by substituting the preceding value of $\varphi\left(\frac{\pi}{2}-\theta\right)=\frac{y}{z}$, $dx''=\frac{y\,dx}{z}$.

[†] (6) This angle is equal to $GAE.=\frac{\pi}{2}-d\theta$; and if the force z' in the direction AG is resolved into two forces in the directions AZ, AE, the last will (by the nature of the function φ) be represented by $z'.\varphi\left(\frac{\pi}{2}-d\theta\right)$.

[‡] (7) Because $\varphi\left(\frac{\pi}{2}-d\theta\right)$ contains only the quantities $\frac{\pi}{2}$, $d\theta$, but does not explicitly contain θ. Moreover, the function $\varphi\left(\frac{\pi}{2}-d\theta\right)$ being developed in the usual manner, according

z' differs from z but by an infinitely small quantity; again, $d\,x''$ forming with $d\,x$, the angle $\frac{\pi}{2}-\theta$, we shall have*

$$d\,x''=d\,x.\varphi\left(\frac{\pi}{2}-\theta\right)=\frac{y\,d\,x}{z}; \qquad [6]$$

therefore†

$$d\,\theta=\frac{-y\,d\,x}{k.z^2}. \qquad [7]$$

If we vary the force y by $d\,y$, supposing x constant, we shall have the corresponding variation of the angle θ, by changing in the preceding equation x into y, y into x, θ into $\frac{\pi}{2}-\theta$, which gives‡

$$d\,\theta=\frac{x\,d\,y}{k.z^2}; \qquad [8]$$

supposing, therefore, x and y to vary at the same time, the whole variation of the angle θ will be $\frac{x\,d\,y-y\,d\,x}{k.z^2}$; and we shall have

$$\frac{x\,d\,y-y\,d\,x}{z^2}=k\,d\,\theta. \qquad [9]$$

Substituting for z^2 its value x^2+y^2, and integrating, we shall have§

$$\frac{y}{x}=\text{tang.}\,(k\,\theta+\rho),$$

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to the powers of $d\,\theta$, by Taylor's Theorem [617], or by any other way, will be of the form $A-k.d\,\theta+k'.d\,\theta^2-\text{etc.}$ — k, k', etc. being constant quantities, dependant on the first, second, etc. differentials of $\varphi\,(\frac{1}{2}\pi)$. By this means, $d\,x''$ [4] will become $d\,x''=z'.\{A-k\,d\,\theta+k'\,d\,\theta^2-\text{etc.}\}$. Now it is evident, that when $d\,x=0$, the quantities $d\,x''$ and $d\,\theta$ must also vanish; and the preceding expression will, in this case, become $0=z.A$, or $A=0$. Substituting this value of A, we get generally $d\,x''=z'\{-k\,d\,\theta+k'\,d\,\theta^2-\text{etc.}\}$; and by neglecting the second and higher powers of $d\,\theta$, it becomes as above, $d\,x''=-k\,d\,\theta\,.\,z'$.

* (8) As in note 5.

† (9) By putting the two values of $d\,x''$ [5, 6,] equal to each other, and deducing therefrom the value of $d\,\theta$.

‡ (10) As θ is changed into $\frac{\pi}{2}-\theta$, the differential $d\,\theta$ changes into $-d\,\theta$.

§ (11) By the substitution of x^2+y^2 for z^2, the equation becomes $\frac{x\,d\,y-y\,d\,x}{x\,x+y\,y}=k\,d\,\theta$; or, as it may be written, $\frac{d.\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=k\,d\,\theta$, for the differential of the numerator of the first

ρ being an arbitrary constant quantity. This equation, combined with $x^2+y^2=z^2$, gives

[10] $$x=z.\cos.(k\theta+\rho).$$

It remains to determine the two constant quantities k and ρ; now if we suppose y to be nothing, we shall evidently have $z=x$, and $\theta=0$; therefore $\cos.\rho=1$, and $x=z.\cos.k\theta$. If we suppose $x=0$, we shall have* $z=y$, and $\theta=\frac{1}{2}\pi$; $\cos.k\theta$ being then equal to nothing, k ought to be equal to† $2n+1$, n being a whole number, and in this case, x will be nothing whenever θ is equal to $\frac{\frac{1}{2}\pi}{2n+1}$; but x being nothing, we evidently have $\theta=\frac{1}{2}\pi$; therefore $2n+1=1$, or $n=0$, consequently

[11] $$x=z.\cos.\theta.$$

Composition of two Forces. Whence it follows that the diagonal of the parallelogram, constructed upon the right lines which represent the two forces x and y, represents not only the quantity, but also the direction of their resultant. Therefore we may, for any force, substitute two other forces which form the sides of a parallelogram, of which the proposed force is the diagonal; whence it is easy to infer that a force may be resolved into three others, forming the sides of a rectangular parallelopiped, of which the proposed force is the diagonal.‡

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member indicated by the sign d, being taken, and the numerator and denominator multiplied by x^2, it becomes identical with the proposed. The integral of this equation is, (by 51 Int.)

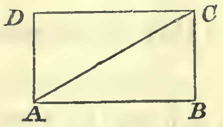
$$\text{arc.}\left(\text{tang.}\frac{y}{x}\right)=k\theta+\rho,\ \text{or}\ \frac{y}{x}=\text{tang.}(k\theta+\rho).\ \text{Hence}\ y^2=x^2.\{\text{tang.}^2(k\theta+\rho)\},$$

and $z^2=y^2+x^2=x^2.\{1+\text{tang.}^2(k\theta+\rho)\}=\frac{x^2}{\cos.^2(k\theta+\rho)}$, whence $z=\frac{x}{\cos.(k\theta+\rho)}$, or $x=z.\cos.(k\theta+\rho)$. This calculation might have been much simplified if the Author had supposed x constant, or $dx=0$.

* (12) Because the line AZ then falls upon AY.

† (13) Because the cosine of any uneven multiple of $\frac{1}{2}\pi$ is equal to nothing.

‡ (13*a*) In any parallelogram, $ABCD$, whose diagonal is AC, we have $AB=AC.\cos.BAC$; and if $AC=z$, $BAC=\theta$, this will become $AB=z.\cos.\theta$, or by [11] $AB=x$. In like manner we find $BC=y$; consequently the forces x, y, are equal to the sides of the parallelogram whose diagonal is z.



Let a, b, c, be the three rectangular co-ordinates of the extremity of the right line which represents any force, whose origin is the same as that of the co-ordinates; this force will be expressed by the function $\sqrt{a^2+b^2+c^2}$, [11′] and by resolving it into directions parallel to the axes of a, b, c, the partial forces will be expressed respectively by these co-ordinates.

Let a', b', c', be the co-ordinates of a second force; $a+a'$, $b+b'$, $c+c'$, will be the co-ordinates of the resultant of the two forces, and will represent the partial forces into which it may be resolved parallel to the three axes; whence it is easy to conclude that this resultant is the diagonal of the parallelogram constructed upon these two forces.

Composition of any number of Forces.

In general a, b, c; a', b', c'; a'', b'', c'' &c.; being the co-ordinates of any number of forces; $a+a'+a''+$&c.; $b+b'+b''+$&c.; $c+c'+c''+$&c.; will be the co-ordinates of the resultant, whose square will be the sum of the [11″] squares of these last co-ordinates; we shall thus have the magnitude and position of this resulting force.

2. From any point of the direction of a force S, taken as the origin of this force, suppose a right line s to be drawn to a material point or particle M; let x, y, z, be the three rectangular co-ordinates which determine the position

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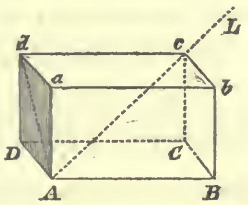


Again, suppose a rectangular parallelopiped to be formed upon the base $ABCD$, having the diagonal Ac, and the equal and parallel edges Aa, Bb, Cc, Dd, perpendicular to the plane $ABCD$; forming the rectangular triangles ADd, Adc. Then if the force z in the direction Ac be represented by Ac, it may, by what has been just said, be resolved into the two forces Ad, dc, perpendicular to each other, and the former force, may be resolved, as above, into the two forces AD, Dd; consequently the force Ac, represented by the diagonal, may be resolved into the three forces AD, Dd, dc, corresponding to the sides of the rectangular parallelopiped. The rectangular triangle ADd gives $Ad^2=AD^2+Dd^2$, and the rectangular triangle Adc gives $Ac^2=Ad^2+dc^2$; hence by substituting Ad^2, we have $Ac^2=AD^2+Dd^2+dc^2$, which, by putting $AD=a$, $Dd=b$, $dc=c$, gives as [11a] above $Ac=\sqrt{a^2+b^2+c^2}$. If the forces AD, Dd, dc, are supposed to be respectively equal to $a+a'+a''+$&c.; $b+b'+b''+$&c.; $c+c'+c''+$&c.; the corresponding force Ac must be found in the same manner, by taking the square root of the sum of their squares.

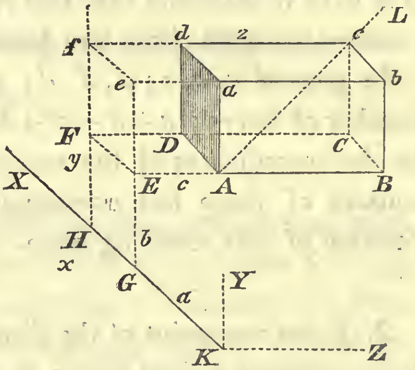
of the point M, and a, b, c, the co-ordinates of the origin of the force, we shall have*

[12] $$s = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Resolution of Forces.

If we resolve the force S into three others in directions parallel to the axes of x, y, z; these partial forces will be, by the preceding article,†

[13] $$S.\frac{(x-a)}{s};\quad S.\frac{(y-b)}{s};\quad S.\frac{z-c}{s};\quad \text{or } S.\left(\frac{\delta s}{\delta x}\right);\quad S.\left(\frac{\delta s}{\delta y}\right);\quad S.\left(\frac{\delta s}{\delta z}\right);$$



* (13b) Let K be the origin of the co-ordinates, A the origin of the force S, c the place of the point M. Draw, as in the last note, the lines AD, Dd, dc, parallel to the axes x, y, z, represented by KX, KY, KZ, respectively, and complete the parallelopiped $ABCD\ abcd$. Continue the lines ba, BA, cd, CD, till they meet the plane YKX, in the points e, E, f, F. Draw the lines fF, eE, to meet the axis KX perpendicularly in H, G. Then, by the above notation, the co-ordinates of the point A are $KG = a$, $GE = b$, $EA = c$. The co-ordinates of the point c are $KH = x$, $Hf = y$, $fc = z$, and $Ac = s$. From this construction it follows that $AD = EF = GH = KH - KG = x - a$; $Dd = Cc = Ff = Hf - HF = Hf - GE = y - b$; DC or $dc = fc - fd = fc - EA = z - c$, substituting these values in $Ac = \sqrt{AD^2 + Dd^2 + dc^2}$, found as in the last note, it becomes $s = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, as in [12].

[12a] † (13c) If the line Ac represents the force S, it might be resolved, as in note 13a, into three forces, AD, Dd, dc, parallel to the axes x, y, z, respectively; consequently these

[13a] three forces will be represented by $S.\frac{AD}{Ac}$; $S.\frac{Dd}{Ac}$; $S.\frac{dc}{Ac}$; which, by substituting the values of Ac, AD, Dd, dc, given in the last note, become $S.\frac{(x-a)}{s}$; $S.\frac{(y-b)}{s}$; $S.\frac{(z-c)}{s}$, respectively, as in [13]. They may be put under a different form, by means of the *partial differentials or variations* of s. The *partial* differential of a quantity denotes its differential supposing only part of the quantities of which it is composed to be variable. Thus the *partial* differential of $s = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, taken relative to x, is

Partial Differentials.

$\frac{(x-a)\,dx}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}$, x only being considered variable, this is usually denoted by $\left(\frac{ds}{dx}\right)dx$ including the quantities considered as variable between the parentheses. In the same

$\left(\frac{\delta s}{\delta x}\right)$, $\left(\frac{\delta s}{\delta y}\right)$, $\left(\frac{\delta s}{\delta z}\right)$ expressing according to the usual notation the coefficients of the variations δx, δy, δz, in the variation of the preceding expression of s.

If, in like manner, we put s' for the distance of M from any point of the direction of another force S', taken as the origin of that force; $S'.\left(\frac{\delta s'}{\delta x}\right)$, will be the part of this force resolved into the direction parallel to x, and in the same manner for others; the sum of the forces S, S', S'', &c., resolved into directions parallel to x, will be $\Sigma.S.\left(\frac{\delta s}{\delta x}\right)$; the symbol Σ of finite integrals, denoting the sum of the terms $S.\left(\frac{\delta s}{\delta x}\right)$, $S'.\left(\frac{\delta s'}{\delta x}\right)$, &c.

Symbol Σ.

Let V be the resultant of all the forces S, S', &c., and u the distance of the point M, from a point in the direction of this resultant taken as its origin; $V.\left(\frac{\delta u}{\delta x}\right)$ will be the expression of the part of this resultant resolved into a direction parallel to the axis of x; we shall therefore have, by the preceding article,*

$$V.\left(\frac{\delta u}{\delta x}\right)=\Sigma.S.\left(\frac{\delta s}{\delta x}\right). \qquad [14]$$

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manner the *partial* differential of s taken relative to y is denoted by $\left(\frac{ds}{dy}\right)dy$; &c. When the differential is taken supposing all the variable quantities x, y, z, to be noticed, it is [13b] called the *complete* differential. The term *variation* is used above instead of *differential*. The difference between these expressions is fully pointed out in note (17*g*). A *partial* or *complete* variation is found in precisely the same manner as a *partial* or *complete* differential, changing the characteristic d into δ, so that the partial variation of s relative to x, is

Complete Differential.

$$\left(\frac{\delta s}{\delta x}\right).\delta x=\frac{(x-a).\delta x}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}=\frac{(x-a)}{s}.\delta x;$$

hence $\left(\frac{\delta s}{\delta x}\right)=\frac{x-a}{s}$; and in like manner $\left(\frac{\delta s}{\delta y}\right)=\frac{y-b}{s}$; $\left(\frac{\delta s}{\delta z}\right)=\frac{z-c}{s}$. These [13c] being substituted in the three first expressions [13] give the three last values of [13].

* (13*d*) The formulas [14, 15] necessarily follow from the principles proved in [11″] and [13]. By multiplying the three equations [14, 15] respectively by δx, δy, δz, and adding them together, we have

$$V.\left\{\left(\frac{\delta u}{\delta x}\right).\delta x+\left(\frac{\delta u}{\delta y}\right).\delta y+\left(\frac{\delta u}{\delta z}\right)\delta z\right\}=\Sigma.S.\left\{\left(\frac{\delta s}{\delta x}\right).\delta x+\left(\frac{\delta s}{\delta y}\right).\delta y+\left(\frac{\delta s}{\delta z}\right)\delta z\right\}.$$

Now it is evident that the complete variation of u is equal to the sum of its partial variations

We shall have in like manner

[15] $$V.\left(\frac{\delta u}{\delta y}\right)=\Sigma.S.\left(\frac{\delta s}{\delta y}\right); \qquad V.\left(\frac{\delta u}{\delta z}\right)=\Sigma.S.\left(\frac{\delta s}{\delta z}\right).$$

By multiplying these equations respectively by δx, δy, δz, and adding them together, we shall have

[16] $$V.\delta u=\Sigma.S.\delta s; \quad (a)$$

This equation exists whatever be the variations δx, δy, δz, and it is equivalent to the three preceding. If its second member is an exact variation* of a function φ, we shall have

[16′] $$V.\delta u=\delta\varphi;$$

consequently

[17] $$V.\left(\frac{\delta u}{\delta x}\right)=\left(\frac{\delta \varphi}{\delta x}\right);$$

that is, the sum of all the forces S, S', &c., resolved into directions parallel to the axis of x, is equal to the partial differential $\left(\frac{\delta\varphi}{\delta x}\right)$. This is generally the case, when these forces are respectively functions of their distances from their origin to the point M.† In order therefore to compute the resultant of all these forces, resolved into a direction parallel to any right line, we must find the integral $\Sigma.\int.S.\delta s$, and calling it φ, we shall consider it as a function of x, and of two other right lines perpendicular to each other, and to the line

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relative to x, y, z, that is $\delta u=\left(\frac{\delta u}{\delta x}\right).\delta x+\left(\frac{\delta u}{\delta y}\right).\delta y+\left(\frac{\delta u}{\delta z}\right).\delta z$, also

[14a] $$\delta s=\left(\frac{\delta s}{\delta x}\right).\delta x+\left(\frac{\delta s}{\delta y}\right).\delta y+\left(\frac{\delta s}{\delta z}\right).\delta z; \text{ \&c.}$$

which, being substituted, give $V.\delta u=\Sigma.S.\delta s$, [16].

Exact Differential.

* (13e) An expression is said to be an *exact variation* or *differential*, when as it then exists its integral is possible. Thus $x\,dy+y\,dx$ is an *exact* differential, because its integral is xy. But $x\,dy+2y\,dx$ is not an exact differential, because no finite quantity can in general be found, whose differential will produce that expression. The same remarks will apply to expressions of any order of differentials. Now having as above $V.\delta u=\delta\varphi$, whatever be the variations δx, δy, δz, we shall obtain by noticing only the variation δx, the expression [17].

† (13f) If S is a function of s, the quantity $\int S.\delta s$ is evidently integrable; and if S' is a function of s', $\int S'.\delta s'$ will be integrable, &c.; and in this case the sum of all these quantities, $\int S.\delta s$, $\int S'.\delta s'$, &c., or $\Sigma.\int S.\delta s$ is integrable, or is an exact variation.

x; the partial differential $\left(\frac{\delta\varphi}{\delta x}\right)$ will be the resultant of the forces S, S', &c. resolved into a direction parallel to the right line x.

Equation of Equilibrium.

3. If the point M is in equilibrium, by means of all the forces which act upon it, the resultant will be nothing, and the equation (*a*) will become

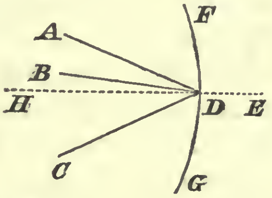
$$0 = \Sigma . S . \delta s ; \qquad (b)$$ [18]

which shows that in the case of the equilibrium of a point, acted upon by any number of forces, the sum of the products of each force by the element of its direction is nothing.

Pressure upon curved surfaces.

If the point M is forced to remain upon a curved surface, it will be affected by the reaction of the surface, by a quantity which we shall denote by R. This reaction is equal and directly opposed to the pressure which the point exerts against the surface; for if we suppose the point to be acted upon by the two forces R and $-R$, we may conceive that the force $-R$ is destroyed by the reaction of the surface, and that the point M presses the surface with the force $-R$; now the pressure of a point upon a surface is perpendicular to it, otherwise it might be resolved into two forces, the one of which would be perpendicular to the surface, and would be destroyed by it; the other parallel to the surface, and by means of which the point would not have any action upon that surface; which is contrary to the hypothesis; putting therefore r for the perpendicular to the surface drawn through the point M, and terminated at any point of its direction, the force R will act in the direction of this perpendicular; we must therefore add $R . \delta r$ to the second member of the equation* (*b*), which will thus become

$$0 = \Sigma . S . \delta s + R . \delta r ; \qquad (c)$$ [19]



*(14) To illustrate this by a simple example, we shall suppose that the point M is forced to move upon a curve line FDG, and that all the forces act in the plane of this curve; these forces being S, S', S'', in the directions $AD (=s)$, $BD (=s')$, $CD (=s'')$, producing a pressure $-R$ upon the curve in the direction HDE, which, by what is said above, must be perpendicular to the curve at the point D. This pressure must be destroyed by the reaction R of the curve in the direction ED. We may therefore suppose the curve FDG to be taken away, and the body to be acted upon by the forces S, S', S'', R, in the directions AD, BD, CD, ED, or s, s', s'', r, respectively. In this case the equation [16] will become

$$V.\delta u = S.\delta s + S'.\delta s' + S''.\delta s'' + R.\delta r.$$

—R being then the resultant of all the forces S, S', &c., it will be perpendicular to the surface.

Equation of the perpendicular to a surface.

If we suppose the arbitrary variations δx, δy, δz, to appertain to the curved surface upon which the point is forced to remain; we shall have by the nature of the perpendicular to that surface, $\delta r = 0$,* which makes $R.\delta r$ vanish from

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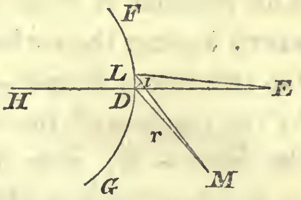
If we suppose the point to be kept in equilibrium by means of these forces S, S', S'' and the resistance of the curve, the resulting force V will be nothing, and we shall have

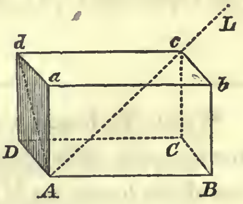
$$0 = S.\delta s + S'.\delta s' + S''.\delta s'' + R.\delta r;$$

or by including under the symbol Σ, all the forces S, S', S'', it will become $0 = \Sigma.S.\delta s + R.\delta r$, as in [19].



* (14a) Let $GDLF$ be the proposed surface, to which the line HDE is perpendicular, at D; E being the origin of the force R, put $ED = r$, and let DL be infinitely small, and join EL; then as DL is perpendicular to ED, we shall have $EL = \sqrt{ED^2 + DL^2}$, and by neglecting the infinitely small quantity of the *second* order DL^2, we shall have $EL = ED$, consequently in this case the variation of ED, which is equal to $EL - ED$, would be nothing, that is $\delta r = 0$. This would not be the case if ED was inclined to the curve, as is the case with the line MD; for by drawing the lines MD, ML, and letting fall on ML the perpendicular Dl, we have nearly $ML - MD = Ll = LD.\cos.MLD$, which is of the same order as LD, except $\cos.MLD = 0$, which excepted case corresponds to that where DM falls upon DE, or where DM is perpendicular to the surface. Therefore the equation $\delta r = 0$
[19a] is the equation of the perpendicular to the surface.

Equation of a line passing through the origin.



(14b) I shall in this and the three following notes, investigate the equations of a right line, a plane, and a spherical surface, which will frequently be wanted in the course of this work. Let AcL be a right line, the origin of whose co-ordinates is A, putting $AD = x$, $Dd = y$, $dc = z$ for the rectangular co-ordinates of any point c of this line. The projection of AcL upon the plane $ADda$, corresponds to the diagonal Ad, and, at whatever point of the line AL the point c is taken, the angle DAd will be the same, and by putting its tangent equal to A, and observing that by trigonometry $Dd = AD.\text{tang}.DAd$, we shall have $y = Ax$. In like manner by projecting Ac upon the plane $ABCD$, and putting $\text{tang}.DAC = B$, we shall have $DC = AD.\text{tang}.DAC$, or $z = Bx$. These values of y, z, give $z = Cy$, putting
[19b] $C = \frac{B}{A}$, which last equation might be found like the two former, by projecting the line Ac upon
the plane $CDdc$, and putting $\text{tang}.dDc = C$. Hence the equations of a right line passing

the preceding equation: therefore the equation (b) takes place also in this case, provided that one of the three variations δx, δy, δz, be exterminated by means of the equation of the surface; but then the equation (b) which, in

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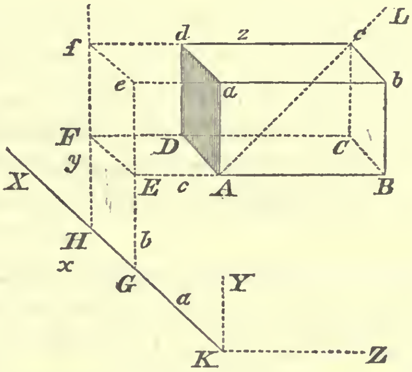
through the origin of the co-ordinates are

$$y = Ax; \qquad z = Bx; \qquad z = Cy. \qquad [19b']$$

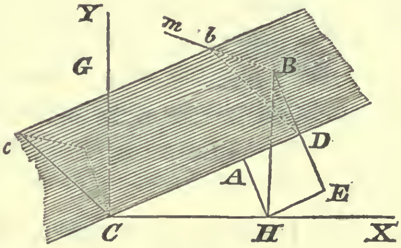


When the line does not pass through the origin of the co-ordinates, as is the case in the annexed figure, where that origin is K, these equations will be varied a little. For in this case the co-ordinates of the point c are $KH = x$, $Hf = y$, $fc = z$; those of the point A are $KG = a$, $GE = b$, $EA = c$, whence $AD = x - a$, $Dd = y - b$, $dc = z - c$, as in note (13b). These being substituted in the three preceding equations, $Dd = AD\,.\,\text{tang.}\,DAd$; $DC = AD\,.\,\text{tang.}\,DAC$; $dc = Dd\,.\,\text{tang.}\,dDc$, give $y - b = A\,.\,(x - a)$; $z - c = B\,.\,(x - a)$; $z - c = C\,.\,(y - b)$; which, by transposing the terms b, c, of the first members, and putting $b - A\,.\,a = A'$, $c - B\,.\,a = B'$, $c - C\,.\,b = C'$ give the general equations of a right line

General Equation of a right line.

$$y = Ax + A'; \qquad z = Bx + B'; \qquad z = Cy + C'; \qquad [19b'']$$

which, like [19b′] are equivalent to two distinct equations, and any two of the co-ordinates, as y, z, are given by means of the third.



(14c) Let C be the origin of the co-ordinates, CHX the axis of x, CGY that of y, the axis of z being perpendicular to the plane of the figure, and let the plane whose equation is required be $CDbc$, intersecting the plane $YCHB$ in the right line CAD. From any point b of this plane let fall the perpendicular bB upon the plane $YCHB$. Draw BDE perpendicular to CD; BH perpendicular to CX; HA parallel to DE, HE parallel to AD. Then the co-ordinates of the point b are $CH = x$, $HB = y$, $Bb = z$. And if we put the angle $ACH = HBE = \varepsilon$, and the angle bDB, which denotes the inclination of the planes $YCHB$, $CDbc$, equal to φ, we shall have in the rectangular triangles CAH, BEH, DBb, the following equations, [19b‴]

$$AH(= DE) = CH\,.\,\text{sin.}\,ACH = x\,.\,\text{sin.}\,\varepsilon; \qquad BE = HB\,.\,\text{cos.}\,HBE = y\,.\,\text{cos.}\,\varepsilon;$$

$$Bb = BD\,.\,\text{tang.}\,bDB.$$

The two former equations make $BD = BE - DE = y\,.\,\text{cos.}\,\varepsilon - x\,.\,\text{sin.}\,\varepsilon$; which, being substituted in the last, gives $Bb = z = (y\,.\,\text{cos.}\,\varepsilon - x\,.\,\text{sin.}\,\varepsilon)\,.\,\text{tang.}\,\varphi$; and by putting general, is equivalent to three equations, furnishes only two distinct equations, which may be obtained by puting each of the co-efficients of the two remaining differentials equal to nothing. Let $u=0$ be the equation of the surface, the [19′]

Equation of a plane surface passing through the origin.

$\cos.\varepsilon\,\text{tang}.\varphi=B$; $-\sin.\varepsilon\,\text{tang}.\varphi=A$, we obtain the following equation of a plane passing through the origin of the co-ordinates

[19c] $$z=Ax+By.$$

If the plane do not pass through the origin of the co-ordinates, and we put a, b, c, for the co-ordinates of the point C, measured in the directions x, y, z, respectively, we must, in the above equation, change x, y, z, into $x-a$, $y-b$, $z-c$, respectively, as is evident by proceeding as in note (13*b*). Substituting these values, we get $z-c=A(x-a)+B(y-b)$; or $0=Ax+By-z+(c-Aa-Bb)$; which, for the sake of symmetry, may be multiplied by $-C'$, putting $A'=-C'A$, $B'=-C'B$, $D'=-C'(c-Aa-Bb)$, and it becomes

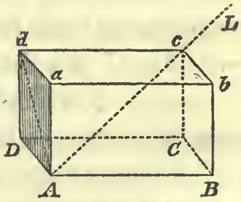
General equation of a plane.

[19d] $$0=A'x+B'y+C'z+D',$$

which is the general equation of a plane surface, and when it is compared with the general form $u=0$, assumed in [19′], we shall find that the function u corresponding to a plane surface is $A'x+B'y+C'z+D'$.

Equation of a sphere.

(14*d*) The equation of a spherical surface, the origin of whose rectangular co-ordinates is at the centre of the sphere, is easily computed by supposing A to be the centre of a sphere whose radius is r', and c to be any point of its surface, so that $Ac=r'$; the rectangular co-ordinates of the point c being $AD=x$; $Dd(=Cc)=y$; $dc(=DC)=z$. Then by [11*a*], $Ac^2=AD^2+Dd^2+dc^2$, which in symbols is $r'^2=x^2+y^2+z^2$; consequently the equation of this surface may be thus written,



[19e] $$0=x^2+y^2+z^2-r'^2. \qquad \text{or} \qquad 0=r'^2-x^2-y^2-z^2;$$

This is a particular case of the equation of the surfaces of the second order, given in Book III, § 2, [1363], and is what in [19′] is called $u=0$ for a spherical surface, u being the function $x^2+y^2+z^2-r'^2$.

(14*e*) From the two preceding examples we may perceive the method to be used in finding the equation of *any* surface; and it appears from the formulas [19*d*, *e*,] that the equation of a plane or a spherical surface gives *one* of the co-ordinates, as z, by means of the other *two*, x, y; and the same remark will apply to any other surface. If we represent, therefore, as in [19′], the equation of this surface by $u=0$, u will be a function of x, y, z. The differential, or rather the variation of this function will correspond to the infinitely small plane which touches the proposed surface in the point whose co-ordinates are x, y, z; and the equation of this plane will be of the form $\delta u=A'\delta x+B'\delta y+C'\delta z=0$; in which the rectangular co-ordinates of the plane are δx, δy, δz, parallel to x, y, z, respectively, the origin being at the point of the surface corresponding to x, y, z, where the variations δx, δy, δz, are nothing. [19f]

two equations $\delta r = 0$ and* $\delta u = 0$ will exist at the same time, which requires that

$$\delta r = N . \delta u, \qquad [19']$$

N being a function of x, y, z. To find this function, let a, b, c, be the co-ordinates of the origin of r, we shall have†

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}; \qquad [20]$$

(14f) From the variation of the equation of a surface $\delta u = 0$, we may deduce the value of *one* of the variations of the co-ordinates, as δz, by means of the other *two*, δx, δy. Thus for a sphere in which the radius r' is constant, [19e], $\delta u = 0$ becomes $0 = x \delta x + y \delta y + z \delta z$, which gives δz, by means of δx, δy, at any point of the surface whose co-ordinates, x, y, z, are known. On the contrary, the variations of the equations of a right line, [19b''], are $\delta y = \mathcal{A} . \delta x$; $\delta z = B . \delta x$; $\delta z = C . \delta y$; which give any *two* of the variations, as δy, δz, by means of the remaining *one*, δx.

[*] (15) It follows from [19f], that $\delta u = 0$, is in general of the form $0 = \mathcal{A}' . \delta x + B' . \delta y + C' . \delta z$; and the variation of r [20] put equal to nothing is of the form $0 = \mathcal{A}'' . \delta x + B'' . \delta y + C'' . \delta z$; $\mathcal{A}'$, B', C', $\mathcal{A}''$, B''; C'', being functions of the co-ordinates x, y, z, and constant quantities, independent of δx, δy, δz. Now if these equations were not multiples of each other, we might, in general, by the usual rules of extermination, find any *two* of the quantities, as δy, δz, in functions of the *other*, δx, so that we should have $\delta y = \mathcal{A} . \delta x$; $\delta z = B . \delta x$; $\mathcal{A}$, B, being functions of $\mathcal{A}'$, B', C', $\mathcal{A}''$, B'', C''. These equations correspond to those of any infinitely small right line whose origin is at the point of the surface, whose co-ordinates are x, y, z, as is evident by writing δx, δy, δz, for x, y, z, in the two first equations of a *right line* [19b'], by which means they become like the preceding; and it is evident that this right line must be a tangent to the surface, because by hypothesis, δx, δy, δz, correspond to that surface. It would therefore follow, if $\delta r = 0$ is not a multiple of $\delta u = 0$, that the point could not be moved, except in the direction of that *line*, thus putting a limit to the direction of the motion, even when there is none by the nature of the question, and when the point is left free to move in *any* direction upon the proposed surface; therefore we must necessarily have δr equal to a multiple of δu, which may be denoted by $\delta r = \mathcal{N} . \delta u$.

[†] (15a) This value of r is equal to that of s [12], found as in note (13b). Its partial variations found as in note (13c), give $\left(\frac{\delta r}{\delta x}\right) = \frac{x-a}{r}$; $\left(\frac{\delta r}{\delta y}\right) = \frac{y-b}{r}$; $\left(\frac{\delta r}{\delta z}\right) = \frac{z-c}{r}$; the sum of whose squares is $\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2 + \left(\frac{\delta r}{\delta z}\right)^2 = \frac{(x-a)^2 + (y-b)^2 + (z-c)^2}{r^2}$; and as the numerator and denominator of the second member are equal to each other, the first member will be equal 1, that is $\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2 + \left(\frac{\delta r}{\delta z}\right)^2 = 1$. Again, since by hypothesis we have $\delta r = \mathcal{N} . \delta u$, its partial variations give $\left(\frac{\delta r}{\delta x}\right) = \mathcal{N} . \left(\frac{\delta u}{\delta x}\right)$; $\left(\frac{\delta r}{\delta y}\right) = \mathcal{N} . \left(\frac{\delta u}{\delta y}\right)$;

whence we deduce $\left(\frac{\delta r}{\delta x}\right)^2+\left(\frac{\delta r}{\delta y}\right)^2+\left(\frac{\delta r}{\delta z}\right)^2=1$; consequently

[21] $$N^2.\left\{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\}=1;$$

$\left(\frac{\delta r}{\delta z}\right)=N.\left(\frac{\delta u}{\delta z}\right)$. The sum of whose squares is

$$\left(\frac{\delta r}{\delta x}\right)^2+\left(\frac{\delta r}{\delta y}\right)^2+\left(\frac{\delta r}{\delta z}\right)^2=N^2.\left\{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\};$$

which, by means of the preceding equation, becomes $1=N^2.\left\{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\}$, as in [21]. Hence $N=\frac{1}{\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}}$; consequently $\delta r=N.\delta u$ becomes

$$\delta r=\frac{\delta u}{\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}};\quad \text{and}$$

$$R.\delta r=\frac{R.\delta u}{\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}};$$

which, by using the value of λ, [22], becomes $R.\delta r=\lambda.\delta u$. This, substituted in [19], produces [23].

To illustrate the above method of finding the value of N, we shall give an example, in which the proposed surface is a sphere, corresponding to the equation [19*e*],

$$u=x^2+y^2+z^2-r^2=0;$$

in which the radius r is given, or constant; hence $\left(\frac{\delta u}{\delta x}\right)=2x$; $\left(\frac{\delta u}{\delta y}\right)=2y$; $\left(\frac{\delta u}{\delta z}\right)=2z$;

[25*a*] therefore $\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}=\sqrt{4x^2+4y^2+4z^2}=\sqrt{4r^2}=2r$; which, substituted in the above expression of N, gives $N=\frac{1}{2r}$; and this, substituted in the assumed value $\delta r=N.\delta u$, becomes $\delta r=\frac{\delta u}{2r}$. This might also be found in the following manner. The proposed surface being spherical, the line r drawn perpendicular to it must pass through the centre of the sphere, which was taken for the origin of the co-ordinates, and if we take this centre for the origin of the force R, and of the line r, we shall have $a=0$, $b=0$, $c=0$, $r=r'$, and the formula [20] will become $r=\sqrt{x^2+y^2+z^2}$; whence $\delta r=\frac{x\delta x+y\delta y+z\delta z}{\sqrt{x^2+y^2+z^2}}$; or $\delta r=\frac{x\delta x+y\delta y+z\delta z}{r}$; but the preceding expression of $u=x^2+y^2+z^2-r^2$ gives

therefore by putting

$$\lambda . = \frac{R}{\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}} ;$$ [22]

the term $R.\delta r$ will be changed into

$$R.\delta r = \lambda . \delta u ;$$ [22']

and the equation (c) [19] will become

$$0 = \Sigma . S . \delta s + \lambda . \delta u ;$$ [23]

in which equation we ought to put the coefficients of the variations δx, δy, δz, separately equal to nothing, which furnishes three equations; but they are only equivalent to two equations, between x, y, z, because of the indeterminate quantity λ, which they contain. Therefore, instead of exterminating one of the variations δx, δy, δz, from the equation (b) [18], by means of the differential equation of the surface, we may add to it this last equation multiplied by an indeterminate quantity λ, and then consider the variations of δx, δy, δz, as independent. This method, which results from the theory of extermination, combines the advantage of simplicity in calculation with that of showing the pressure $-R$ which the point M exerts against the surface.

Suppose this point to be contained in a canal of simple or double curvature,* it will suffer a reaction from the canal, which we shall denote by k, and this

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$\delta u = 2x\delta x + 2y\delta y + 2z\delta z$, or $\frac{\delta u}{2r} = \frac{x\delta x + y\delta y + z\delta z}{r}$, which is equal to the preceding expression of δr, hence $\delta r = \frac{\delta u}{2r}$, as above. The terms a, b, c, were put nothing for the sake of simplicity, otherwise we might have put $a\delta x + b\delta y + c\delta y = 0$, which equation is easily deduced from $\delta r = 0$, $\delta u = 0$.

* (15*b*) The intersection of a *plane* with any curve *surface* forms in general a curve line. Thus a plane cutting successively a cone in certain directions will produce the three conic sections, and in particular cases will also form the triangle and the circle. Curves thus produced will be wholly in the same plane. But if two *curve surfaces* intersect each other, their intersection may form a curve line whose points are not in the same plane, and then it is called a *curve of double curvature*. As a very simple example of this last species of curves, we may mention that formed by the intersection of two right cylinders with circular bases, whose axes intersect each other at right angles. The curve traced upon the surface of the greatest cylinder by the other is a curve of double curvature. As all the points of a curve of double curvature appertain to *both* the generating surfaces, whose equations are $u = 0$, $u' = 0$, the equations of this curve must be defined by the equations of those surfaces, as is very evident. [25*b*]

Curve of double curvature.

will be equal and directly opposite to the pressure which the point exerts against the canal, and its direction will be perpendicular to the side of the canal: now the curve formed by this canal is the intersection of two surfaces, whose equations express its nature; we may therefore suppose the force k to be the resultant of the two reactions R and R', which the point M suffers from each of these surfaces; for the directions of the three forces R, R' and k, being perpendicular to the side of the curve, they must be in the same plane. Putting therefore δr, $\delta r'$, for the elements of the directions of the forces R and R', which directions are perpendicular to the surfaces respectively, we must add to the equation (b) [18] the two terms $R.\delta r$, $R'.\delta r'$, which changes it into the following,

[24] $$0=\Sigma.S.\delta s+R.\delta r+R'.\delta r'. \qquad (d)$$

If we determine the variations δx, δy, δz, so that they shall appertain at the same time to the two surfaces, and consequently to the curve formed by the canal, δr and $\delta r'$ will vanish, and the preceding equation will be reduced to the equation (b) [18], which therefore still takes place in the case where the point M is forced to move in a canal; provided that two of the variations δx, δy, δz, are exterminated by means of the two equations which express the nature of this canal.

Suppose that $u=0$, and $u'=0$, are the equations of the two surfaces whose intersection forms the canal. If we put

[25] $$\lambda=\frac{R}{\sqrt{\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}};$$
$$\lambda'=\frac{R'}{\sqrt{\left(\frac{\delta u'}{\delta x}\right)^2+\left(\frac{\delta u'}{\delta y}\right)^2+\left(\frac{\delta u'}{\delta z}\right)^2}};$$

the equation (d) [24] will become[*]

[26] $$0=\Sigma.S.\delta s+\lambda.\delta u+\lambda'.\delta u';$$

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[*] (15c) The reasoning used in finding [23] may be used for [26]. For in the same manner in which $R.\delta r$ is introduced in [19], we may introduce $R.\delta r+R'.\delta r'$ in [24]; and for the same reason that δr was put $=0$ in note (14a), we may here put δr, $\delta r'$ equal to nothing, the lines r, r', being perpendicular to the canal. The assumed values of λ, λ' in
[26a] [25], being similar to [22], lead to the result [22'], $R.\delta r=\lambda.\delta u$, $R'.\delta r'=\lambda'.\delta u'$, as in note (15$a$). These, substituted in [24], give [26.]

in which the coefficients of each of the variations δx, δy, δz, ought to be put separately equal to nothing; we shall therefore have three equations, by [26]
means of which we may determine the values of λ, λ', which will give the reactions R and R' of the two surfaces; and by combining them we shall have the reaction k of the canal upon the point M, consequently the pressure which this point exerts against the canal. This reaction, resolved parallel to the axis of x, is equal to* $R.\left(\frac{\delta r}{\delta x}\right)+R'.\left(\frac{\delta r'}{\delta x}\right)$; or $\lambda.\left(\frac{\delta u}{\delta x}\right)+\lambda'.\left(\frac{\delta u'}{\delta x}\right)$: the equations of condition $u=0$, $u'=0$, to which the motion of the point M is subjected, express, therefore, by means of the partial differentials of the functions, which are equal to nothing because of these equations, the resistances which the point suffers, in consequence of the conditions of its motion.

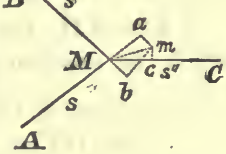
From what has been said, it follows that the equation of equilibrium (*b*) [18] is generally satisfied if the variations δx, δy, δz, are subjected to the conditions of equilibrium. This equation furnishes the following principle.

"If we vary by an infinitely small quantity the position of the particle M, in such a manner that it may remain always upon the surface or upon the curve along which it would move, if it were not wholly free; the sum of the forces which act upon it, each multiplied by the space which the particle describes according to the directions of the forces, is equal to nothing in the case of equilibrium."†

---

* (15*d*) The sum of the reactions R, R', multiplied by the elements of their directions, are $R.\delta r+R'.\delta r'$; hence by formula [17] this reaction resolved in a direction parallel to the axis of x, is $R.\left(\frac{\delta r}{\delta x}\right)+R'.\left(\frac{\delta r'}{\delta x}\right)$; and by substituting for $R.\delta r$, $R'.\delta r'$, their values, [26*a*], $\lambda.\delta u$, $\lambda'.\delta u'$, it becomes $\lambda.\left(\frac{\delta u}{\delta x}\right)+\lambda'.\left(\frac{\delta u'}{\delta x}\right)$, as above.

† (15*e*) The infinitely small space described in the direction of *any* one of the forces, must be considered as *positive*, if the motion tend to *increase* the distance of the body from the origin of that force, but *negative* if it tend to *decrease* it. As an example of the formula [18] $0=\Sigma.S.\delta s$, let there be three forces, S, S', S'', originating at the points A, B, C, at the distances $AM(=s)$, $BM(=s')$, $CM(=s'')$, from the moveable body M respectively, and acting upon the body in the directions of those lines. Then the preceding expression will become $0=S.\delta s+S'.\delta s'+S''.\delta s''$. Now



The variations δx, δy, δz, being supposed arbitrary and independent, we may, in the equation (a), [16], substitute for the co-ordinates x, y, z, three other quantities which are functions of them, and then put the coefficients of the variations of these quantities equal to nothing. Thus let ρ be the radius drawn from the origin of the co-ordinates to the projection of the point M, upon the plane of x, y, and ϖ the angle formed by ρ and the axis of x, we shall have*

[27] $$x = \rho \,.\, \cos.\, \varpi \,; \qquad y = \rho \,.\, \sin.\, \varpi \,;$$

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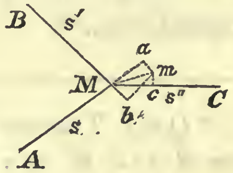
if the body be moved from M to any other point m, through the infinitely small space Mm, and we let fall from m the perpendiculars ma, mb, mc, upon the lines AM, BM, CM, continued if necessary, the variations of the lines AM, BM, CM, will be represented by $Am - AM$, $Bm - BM$, $Cm - CM$, which, by neglecting infinitely small quantities of the second order, become Ma, Mb, $-Mc$, respectively; the negative sign being prefixed to the last because CM decreases. These being substituted for δs, $\delta s'$, $\delta s''$, in the preceding expression of $\Sigma \,.\, S \,.\, \delta s$, it becomes $S \,.\, Ma + S' \,.\, Mb - S'' \,.\, Mc$, which by formula [18] ought to be equal to nothing in the case of equilibrium. And that this equation really takes place is easily perceived by geometrical considerations. For $Ma = Mm \,.\, \cos.\, aMm$; $Mb = Mm \,.\, \cos.\, bMm$; $Mc = Mm \,.\, \cos.\, CMm$; which being substituted in the preceding value of $\Sigma \,.\, S \,.\, \delta s$, it becomes

$$\Sigma \,.\, S \,.\, \delta s = Mm \,.\, \{ S \,.\, \cos.\, aMm + S' \,.\, \cos.\, bMm - S'' \,.\, \cos.\, CMm \}.$$

But by formula [11] the quantities $S \,.\, \cos.\, aMm$, $S' \,.\, \cos.\, bMm$, $-S'' \,.\, \cos.\, CMm$, represent the parts of the forces S, S', S'', resolved in the direction Mm; the latter having a different sign from the two former, because it has an opposite direction; now as the body is by hypothesis in equilibrium, the sum of these forces must be nothing; therefore

$$S \,.\, \cos.\, aMm + S' \,.\, \cos.\, bMm - S'' \,.\, \cos.\, CMm = 0,$$

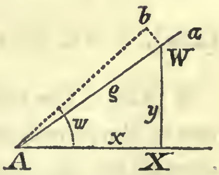
consequently $\Sigma \,.\, S \,.\, \delta s = 0$, as above.



* (16) Let AXW be the plane of x, y; W the projection of the place of the particle M upon that plane; the co-ordinates of the point W will be $AX = x$, $XW = y$. Draw the line $AW = \rho$, and put the angle $XAW = \varpi$, then we shall have

$$AX = AW \,.\, \cos.\, XAW \,; \qquad XW = AW \,.\, \sin.\, XAW \,;$$

which, by substituting the symbols x, y, ρ, ϖ, become as in [27].



Substitute these in (12) and we get $s = \sqrt{(\rho \,.\, \cos.\, \varpi - a)^2 + (\rho \,.\, \sin.\, \varpi - b)^2 + (z - c)^2}$, and the formula [16] may be considered as containing ρ, ϖ, instead of x, y. The partial differential of this equation, taken relative to ϖ, will then be as in [28]. Now it appears from the equations [14, 15,] that the force V resolved into three forces, in directions parallel to the axes x, y, z, by considering, therefore, in the equation (a) [16], u, s, s', &c., as functions of ρ, ϖ, and z; and comparing the coefficients of $\delta\varpi$, we shall have

$$V.\left(\frac{\delta u}{\delta\varpi}\right)=\Sigma.S.\left(\frac{\delta s}{\delta\varpi}\right). \qquad (e)$$ [28]

$\frac{V}{\rho}.\left(\frac{\delta u}{\delta\varpi}\right)$ is the expression of the force V resolved in the direction of the element $\rho\delta\varpi$. Let V' be the force V, resolved in a direction parallel to the
plane of x and y, and p the perpendicular let fall from the axis of z upon the [28']
direction of V' parallel to the same plane; $\frac{pV'}{\rho}$ will be a second expression*

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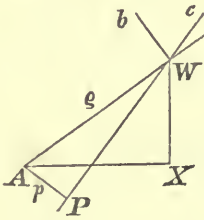
or parallel to the rectangular elements δx, δy, δz, are represented by $V.\left(\frac{\delta u}{\delta x}\right)$; $V.\left(\frac{\delta u}{\delta y}\right)$; [28a]
$V.\left(\frac{\delta u}{\delta z}\right)$; and as the axes of x, y, z, are arbitrary, we may put any other rectangular elements in place of δx, δy, δz. Thus instead of the rectangular elements δx, δy, parallel to the axes of x, y, we may take, in the plane of these two co-ordinates, the element $\delta\rho = Wa$, upon the continuation of the line AW; and the element $\rho\delta\varpi = Wb$, perpendicular to AW, corresponding to the variation of the angle ϖ, represented by $WAb=\delta\varpi$; and we may then use the elements $\delta\rho$, $\rho\delta\varpi$, δz, instead of δx, δy, δz, and the expression of V, resolved in directions parallel to the rectangular elements $\delta\rho$, $\rho\delta\varpi$, δz, will become $V.\left(\frac{\delta u}{\delta\rho}\right)$, $V.\left(\frac{\delta u}{\rho\delta\varpi}\right)$, $V.\left(\frac{\delta u}{\delta z}\right)$, as is evident, by changing δx, δy, into $\delta\rho$, $\rho\delta\varpi$, in the formulas [28a]. By bringing the term ρ from under the parenthesis, the force in the
direction of the element $\rho\delta\varpi$ becomes $\frac{V}{\rho}.\left(\frac{\delta u}{\delta\varpi}\right)$, as above. [28b]



* (16a) Suppose the figure to be similar to that in the last note, with the addition of the line PW, representing the projection of the direction of the force V, upon the plane of x, y, and let the line $AP=p$ be drawn perpendicular to PW. Then the force V' in the direction parallel to PW, may be resolved into two forces, in the directions parallel to Wa, Wb, of which the last, in the direction parallel to Wb, is equal to $V'.\cos.bWc$, [11], or $V'.\cos.WAP$; and as $\cos.WAP=\frac{AP}{AW}=\frac{p}{\rho}$, this force in the direction Wb will be equal to $\frac{pV'}{\rho}$ as above. Putting this equal to the expression of the same force found in [28b], we get $\frac{pV'}{\rho}=\frac{V}{\rho}.\left(\frac{\delta u}{\delta\varpi}\right)$. Multiplying by ρ, it becomes as in [29.]

of the force V, resolved in the direction of the element $\rho\,\delta\varpi$; therefore we shall have

[29] $$p\,V' = V.\left(\frac{\delta u}{\delta\varpi}\right) = \Sigma\,.\,S\,.\left(\frac{\delta s}{\delta\varpi}\right).$$

Momentum of any force about an axis.

If we suppose the force V' to be applied at the extremity of the perpendicular p, it will tend to make it turn about the axis of z; the product of this force, by the perpendicular, is what is called *the momentum of the force*

[29′] *V, about the axis of z*; this momentum is therefore equal to $V.\left(\frac{\delta u}{\delta\varpi}\right)$; and it follows from the equation (e) [28] that the momentum of the resultant of any number of forces, is equal to the sum of the momenta of these forces.*

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* (16b) The partial variation relative to ϖ being taken in the formula [16] gives

[29a] $V.\left(\frac{\delta u}{\delta\varpi}\right)=\Sigma.S.\left(\frac{\delta s}{\delta\varpi}\right)$, and by substitution in [29] we obtain $p\,V' = \Sigma\,.\,S\,.\left(\frac{\delta s}{\delta\varpi}\right)$, in which the first member $p\,V'$ represents the momentum of the resulting force V' about the axis of z, [29′], and the second member is the sum of the momenta of all the forces S, S', &c., about the same axis. For the force S resolved in a direction parallel to the element $\rho\,\delta\varpi$, is $\frac{S}{\rho}\,.\left(\frac{\delta s}{\delta\varpi}\right)$, [28$b$]. This multiplied by ρ, gives, as in [29′], the momentum of this force about the axis of z, equal to $S\,.\left(\frac{\delta s}{\delta\varpi}\right)$; in like manner the momenta of the forces S', S'', &c., are $S'\,.\left(\frac{\delta s'}{\delta\varpi}\right)$, $S''\,.\left(\frac{\delta s''}{\delta\varpi}\right)$, &c., and the sum of all these momenta is represented by $\Sigma\,.\,S\,.\left(\frac{\delta s}{\delta\varpi}\right)$; and this, for the sake of reference, is inserted in [29], though it is not so placed in the original.

CHAPTER II.

ON THE MOTION OF A MATERIAL POINT.

4. A POINT or particle at rest cannot give itself any motion, since there is no reason why it should move in one direction rather than in another. When it is acted upon by any force and afterwards left to itself, it will continue to move uniformly in the direction of that force, if not opposed by any resistance. This tendency of matter to continue in its state of motion or rest, is what is called its *inertia*. This is the first law of the motion of bodies. Inertia.

That the direction of motion is in a right line follows evidently from this, that there is no reason why the point should deviate to the right rather than to the left of its first direction; but the uniformity of its motion is not equally evident. The nature of the moving force being unknown, it is impossible to know, *a priori*, whether this force is constantly retained or not. However, as a body is incapable of giving to itself any motion, it seems equally incapable of altering the motion it has received, so that the law of inertia is at least the most natural and simple that can be imagined; it is also confirmed by experience; for we observe upon the earth that motions continue longer in proportion as the opposing obstacles are decreased; which leads us to suppose that the motion would always continue if these obstacles were removed.

But the inertia of matter is most remarkable in the motions of the heavenly bodies, which, during a great many ages, have not suffered any sensible alteration. We shall therefore consider the inertia of bodies as a law of nature; and when we shall observe any alteration in the motion of a body, we shall conclude that it has arisen from a different cause.

In uniform motions, the spaces passed over are proportional to the times; but the times employed in describing a given space are longer or shorter according to the magnitude of the moving force. This has given rise to the idea of *velocity*, which, in uniform motion, is the ratio of the space to the Velocity.

time employed in describing it; therefore, s representing the space, t the time, and v the velocity, we have

[29″]
$$v = \frac{s}{t}.$$

Unit of space, time or velocity.

Time and space being heterogeneous quantities, cannot be *directly* compared with each other; therefore an interval of time, as a *second*, is taken for the *unit* of time; and a given space, as a *metre*, is taken for the *unit* of space; then space and time are expressed by abstract numbers, denoting how many measures of their particular species each of them contains, and they may then be compared with each other. In this manner the velocity is expressed by the ratio of two abstract numbers, and its *unit* is the velocity of a body, which describes one metre in a second.

Force proportional to the velocity.

5. Force being known only by the space it causes a body to describe in a given time, it is natural to take this space for its measure; but this supposes that several forces acting in the same direction would make a body describe a space equal to the sum of the spaces that each of them would have caused it to describe separately, or, in other words, that the force is proportional to the velocity. We cannot be assured of this *a priori*, owing to our ignorance of the nature of the moving force: we must therefore again have recourse to experience upon this subject; for whatever is not a necessary consequence of the little which we know respecting the nature of things, must be the result of observation.

Let v be the velocity of the earth, which is common to all the bodies upon its surface; f the force by which one of these bodies M is urged in consequence of this velocity, and let us suppose that $v = f \cdot \varphi(f)$, expresses the relation between the velocity and the force; $\varphi(f)$ being a function of f, to be determined by observation. Put a, b, c, for the three partial forces, into which the force f is resolved, parallel to three rectangular axes. Let us then suppose that the body M is acted upon by another force f', which may be resolved into three others a', b', c', parallel to the same axes. The whole forces acting on the body in the directions of these axes will be $a+a'$, $b+b'$, $c+c'$; putting F for the single resulting force, we shall have, by what precedes,*

[30]
$$F = \sqrt{(a+a')^2 + (b+b')^2 + (c+c')^2}.$$

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* (16c) Using the figure as in (11a), the forces $AD = a+a'$, $Dd = b+b'$, $dc = c+c'$, the resultant Ac will, as in (11a), be equal to $\sqrt{(a+a')^2+(b+b')^2+(c+c')^2}$, or F.

If we put U for the velocity corresponding to F; $\frac{(a+a')\,U}{F}$ will be this velocity resolved in a direction parallel to the axis of a; hence the relative velocity of the body upon the earth, parallel to that axis, will be $\frac{(a+a')\,U}{F}-\frac{a\,v}{f}$, or $(a+a')\,.\,\varphi(F)-a\,\varphi(f)$. The greatest forces which we can impress on bodies upon the surface of the earth, being much smaller than those with which they are affected by the motion of the earth, we may consider a', b', c', as infinitely small in comparison with f; we shall therefore have*

$$F=f+\frac{a\,a'+b\,b'+c\,c'}{f};\ \text{and}\dagger\ \varphi(F)=\varphi(f)+\frac{a\,a'+b\,b'+c\,c'}{f}\,.\,\varphi'(f); \qquad [31]$$

$\varphi'(f)$ being the differential of $\varphi(f)$ divided by df. The relative velocity of M, in the direction of the axis a, will therefore become

$$a'\,.\,\varphi(f)+\frac{a}{f}\,.\,\{a\,a'+b\,b'+c\,c'\}\,.\,\varphi'(f). \qquad [32]$$

Its relative velocities in the directions of the axes b and c, will be

$$b'\,.\,\varphi(f)+\frac{b}{f}\,.\,\{a\,a'+b\,b'+c\,c'\}\,.\,\varphi'(f);$$

$$c'\,.\,\varphi(f)+\frac{c}{f}\,.\,\{a\,a'+b\,b'+c\,c'\}\,.\,\varphi'(f). \qquad [33]$$

The position of the axes a, b, c, being arbitrary, we may take the direction

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And we shall have $Ac:AD::$ velocity U in the direction Ac, to the corresponding velocity resolved in the direction AD, which is therefore equal to $\frac{(a+a')\,U}{F}$, as above. In like manner the velocity of the earth in the direction AD is $\frac{a\,v}{f}$. Hence their relative velocity is $\frac{(a+a')}{F}\,U-\frac{a\,v}{f}$; which, by putting $U=F\varphi(F)$, $v=f\,.\,\varphi(f)$, becomes $(a+a')\,.\,\varphi(F)-a\,\varphi(f)$, as above.

* (16d) The expression [30], neglecting a'^2, b'^2, c'^2, on account of their smallness, becomes $F=\sqrt{(a^2+b^2+c^2)+2\,(a\,a'+b\,b'+c\,c')}=\sqrt{f^2+2\,(a\,a'+b\,b'+c\,c')}$, extracting the square root, still neglecting a'^2, b'^2, c'^2, we obtain F [31].

† (16e) This expression of $\varphi(F)$ is easily deduced from the general development of $\varphi(t+\alpha)$, according to the power of α, by Taylor's theorem, [617], retaining only the two first terms $\varphi(t)+\alpha\,.\,\frac{d\,.\,\varphi(t)}{dt}$, or $\varphi(t)+\alpha\,.\,\varphi'(t)$, and putting $t=f$, $\alpha=\frac{a\,a'+b\,b'+c\,c'}{f}$.

of the impressed force, for the axis of a, and then b' and c' will vanish, and the preceding relative velocities will become

[34] $$a' \cdot \left\{ \varphi(f) + \frac{a^2}{f} \cdot \varphi'(f) \right\}; \quad \frac{ab}{f} \cdot a' \cdot \varphi'(f); \quad \frac{ac}{f} \cdot a' \cdot \varphi'(f).$$

If $\varphi'(f)$ does not vanish, the moving body, by means of the impressed force a', will have a relative velocity, perpendicular to the direction of that force, provided b and c do not vanish ;* that is, unless the direction of this force coincide with that of the motion of the earth. Therefore if we suppose a spherical ball at rest upon a very smooth horizontal plane, to be struck by the base of a right cylinder, moving horizontally in the direction of its axis ; the relative apparent motion of the ball would not be parallel to that axis, in all the positions of the axis with respect to the horizon : this furnishes therefore a simple method of discovering by experiment whether $\varphi'(f)$ has a sensible value upon the earth ; but, by the most exact experiments, the least deviation is not perceived in the apparent motion of the ball from the direction of the impressed force ; whence it follows that upon the earth, $\varphi'(f)$ is very nearly nothing. Its value, however small it might be, would be most easily perceived in the time of vibration of a pendulum, which would vary if the position of the plane of its motion should alter with respect to the direction of the motion of the earth. Now, since the most accurate observations do not indicate any such difference, we may infer that $\varphi'(f)$ is insensible, and it may be supposed equal to nothing upon the surface of the earth.

If the equation $\varphi'(f) = 0$ exists for all values of f, $\varphi(f)$ would be constant, and the velocity would be proportional to the force ; it might also be proportional to it if the function $\varphi(f)$ was composed of more than one term, since otherwise $\varphi'(f)$ could not vanish unless f was nothing ;† we must

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* (17) There is one case not noticed by the author, namely, when the motion of the earth is in a plane passing through the origin of the co-ordinates perpendicular to the axis of a ; for then $a = 0$, the relative velocities in the directions parallel to the axes b, c, will be 0, and in the direction parallel to a will be $a' \varphi(f)$. This omission does not however affect the general reasoning of the author, nor the correctness of the conclusion he has drawn.

† (17a) If $\varphi(f)$ was composed but of one term, as af^m, a being a constant quantity, it would give $\varphi'(f) = maf^{m-1}$, which would become 0 either when $m = 0$, or when $m > 1$ and $f = 0$. The first case gives $\varphi(f) = a$, whence $\varphi'(f) = 0$, for all values of f. If $\varphi(f)$ was composed of more than one term, as $af^m + a'f^{m'}$, it would give

$$\varphi'(f) = maf^{m-1} + m'a'f^{m'-1},$$

therefore, if the velocity is not proportional to the force, suppose that in nature the function of the velocity, which expresses the force, is composed of several terms, which is nowise probable; and that the velocity of the earth is exactly that which corresponds to the equation $\varphi'(f) = 0$, which is contrary to all probability. Moreover, the velocity of the earth varies at different seasons of the year: it is about a thirtieth part greater in winter than in summer. This variation is yet more considerable, if, as everything appears to indicate, the solar system itself has a motion in space; for according as this progressive motion conspires with that of the earth, or is opposed to it, there must result, in the course of the year, great variations in the absolute motion of the earth; which would alter the equation we are treating of, and the ratio of the impressed force to the absolute velocity which results from it, unless this equation and velocity are independent of the motion of the earth: however no sensible alteration is perceived by observation.

We have thus obtained from observation two laws of motion; namely, the law of inertia, and that of the force proportional to the velocity. They are the most natural and simple that can be imagined, and without doubt have their origin in the nature of matter itself; but this nature being unknown, they are, as it respects us, facts deduced from observation, and are the only ones which the science of mechanics derives from experience.

6. The velocity being proportional to the force, the one of these quantities may be represented by the other, and all we have previously established respecting the composition of forces may be applied to the composition of [34] velocities. Hence it follows, that the relative motions of a system of bodies acted upon by any forces, are the same, whatever may be their common motion; for this last motion resolved into three others parallel to the three fixed axes, increases by the same quantity, the partial velocities of each of the bodies, parallel to these axes; and as their relative velocity only depends upon the difference of these partial velocities, it must be the same, whatever be the common motion of all the bodies: it is therefore impossible to judge

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and this might be nothing if $m > 1$, $m' > 1$, either when $f = 0$, or $f = \left(-\frac{m'a'}{m\,a}\right)^{\frac{1}{m-m'}}$. Hence we see that the only case in which $\varphi'(f)$ is nothing and f indeterminate, is when $\varphi(f)$ is a constant quantity a, and $v = f \cdot \varphi(f) = af$.

of the absolute motion of the system of bodies of which we make a part, by the appearances we observe in them, and this is what characterizes the law of the proportionality of the force to the velocity.

Radius Vector.

It follows also from § 3, that if we project each force and its resultant upon a fixed plane; the sum of the momenta of the composing forces, thus projected about a fixed point taken in this plane, is equal to the momentum of the projection of the resultant:* now, if from this point, we draw to the moving body, a radius which we shall call the *radius vector*, this radius projected upon the fixed plane would describe upon it, by means of each force acting separately, an area equal to the product of the projection of the line which it would cause the moving body to describe, by half the perpendicular let fall from the fixed point upon this projection:† this area is therefore proportional

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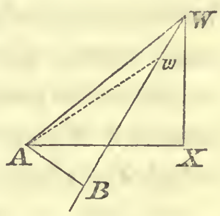
* (17*b*) This is proved in note (16*b*).



† (17*c*) Let $A\,X\,W$ be the plane of x, y; A the fixed point taken upon that plane, W the projection of the place of the body, $A\,W$ the projection of its radius vector, $W\,w$ the projection of the space it would describe in the time $d\,t$ by the force S if it acted alone upon the body, $d\,A =$ the element of the area $W\,A\,w$ described in the same time, $A\,B = \varpi$ the perpendicular let fall upon $W\,w\,B$, and σ the value of the force S resolved in a direction parallel to the line $W\,w$; then the force σ may be taken for the velocity in the direction $W\,w$ [34′], and this velocity multiplied by the time $d\,t$ gives the space $W\,w$ described in that time, hence $W\,w = \sigma\,.\,d\,t$. This multiplied by half the perpendicular $A\,B$ gives the area $A\,W\,w$, or $d\,A = \tfrac{1}{2}\,d\,t\,.\,\sigma\,\varpi$; whose integral taken relative to t gives the area described in that time $A = \tfrac{1}{2}\,t\,.\,\sigma\,\varpi$, supposing A to commence with t, and observing that in this integration the force σ and the perpendicular ϖ are constant. Now the quantity $\sigma\,\varpi$ is equal to the momentum of the force S about the fixed point A [29′], and if we put this momentum equal to m, we shall have $A = \tfrac{1}{2}\,t\,.\,m$. In like manner, if we put A', A'', &c., for the areas, and m', m'', &c., for the momenta corresponding to the forces S', S'', &c.; we shall have $A' = \tfrac{1}{2}\,t\,.\,m'$; $A'' = \tfrac{1}{2}\,t\,.\,m''$; &c., and the sum of all these is $\Sigma\,.\,A = \tfrac{1}{2}\,t\,.\,\Sigma\,.\,m$. In like manner, $A_{\prime}$ being put for the area which would be
[34*a*] described about the same axis in the time t by means of the single force V' [28′] which is the resultant of all the forces S, S', S'', &c., resolved in a direction parallel to the plane of x, y; we shall have as above $A_{\prime}$ equal to the product of $\tfrac{1}{2}\,t$ by the momentum $p\,V'$ [29] of the force V' about that axis, or $A_{\prime} = \tfrac{1}{2}\,t\,.\,p\,V'$. Now $p\,V'$ is equal to $\Sigma\,.\,S\,.\left(\frac{\delta s}{\delta \varpi}\right)$ [29], which last expression represents the sum of the momenta of all the forces S, S', &c. about that axis; and this momenta we have before put equal to $\Sigma\,.\,m$, or $p\,V' = \Sigma\,.\,m$, therefore

to the time. It is also, in a given time, proportional to the momentum of [34″] the projection of the force; hence the sum of the areas which the projection **Description of areas.** of the radius vector would describe, by means of each force acting separately, is equal to the area that the resultant would cause it to describe. Hence it follows that if a body is at first projected in a right line, and is afterwards acted upon by any forces directed towards the fixed point, its radius vector will always describe about this point, areas proportional to the times, since the areas which these last forces would cause the radius vector to describe would be nothing.* Inversely, we must conclude that if the moving body describes about the fixed point, areas proportional to the times; the resultant of the new forces acting upon it must be always directed towards that [34‴] point.†

7. Let us now consider the motion of a point acted upon by forces, which, like gravity, appear to act continually. The causes of this force, and of the similar forces which exist in nature, being unknown, it is impossible to discover whether they act without intermission, or their successive actions are separated by insensible intervals of time; but it is easy to prove that the phenomena ought to be very nearly the same in both hypotheses; for if we represent the velocity of a body upon which a force acts incessantly, [34 iv] by the ordinate of a curve whose absciss represents the time; this curve, in the second hypothesis, will be changed into a polygon of a very great number

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$A_{,} = \frac{1}{2} t \Sigma . m$, which, by substitution in the preceding value of $\Sigma . A$, gives $A_{,} = \Sigma . A$. [34b] Hence it follows that the sum of all the areas which would be described by each force acting separately is equal to the area $A_{,}$ which would be described by means of the resultant V', and as this area $A_{,}$ is equal to $\frac{1}{2} t . \Sigma . m$, it must be proportional to the time of description.

* (17d) If the direction Ww of any force σ of the last note passes through the point A, the perpendicular $AB = \varpi$ would become nothing, therefore the momentum m of this force, which is equal to $\varpi \sigma$, would also be nothing, consequently $\Sigma . m$ would not be affected by this force, and as the area $A_{,}$ described by the resulting force was shown in the last note to be equal to $\frac{1}{2} t . \Sigma . m$, that area will not be affected by any force passing through the point A.

† (17e) By [34b] we have $A_{,} = \frac{1}{2} t . \Sigma . m$, therefore, if A be proportional to t, the quantity $\Sigma . m$ must be constant, and the momentum of any new force must be nothing, which takes place only when the perpendicular AB is nothing, that is, when the direction of the force Ww passes through the fixed point A.

of sides, and, for that reason, it may be considered as coinciding with the curve.* We shall, with geometricians, adopt the second hypothesis, and suppose that the interval of time which separates two consecutive actions of [34 v] any force is equal to the element of time $d\,t$, the whole time being denoted by t. It is evident that we must suppose the action of the force to be greater in proportion as the interval which separates the successive actions is increased, in order that the velocity may be the same at the end of the same time t: the instantaneous action of a force ought therefore to be supposed in the ratio of its intensity and of the element of the time during which it is supposed to act. Therefore, if we denote this intensity by P, we ought to suppose, at the beginning of each instant $d\,t$, that the body is urged by a force $P\,.\,d\,t$, and moves uniformly during this instant. This being premised;

[34 vi] We may reduce all the forces, which act upon a point M, to three forces, P, Q, R, in directions parallel to the three rectangular co-ordinates x, y, z,

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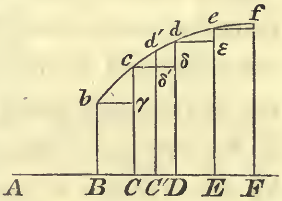

* (17f) Let the times be measured on the absciss $A\,B\,F$, in which are taken the equal intervals $B\,C$, $C\,D$, $D\,E$, &c. Suppose the velocities corresponding to the points B, C, D, &c., to be represented by the ordinates $B\,b$, $C\,c$, $D\,d$, &c. Complete the parallelograms $B\,C\,\gamma\,b$, $C\,D\,\delta\,c$, &c. Then if the forces act *incessantly* during the intervals $B\,C$, $C\,D$, &c., the velocities will gradually vary, and the general expression of the velocity will be represented by the ordinate of a regular curve $b\,c\,d\,e$ &c. drawn through the proposed points b, c, d, &c. On the contrary, if the force act *instantaneously* at the points corresponding to B, C, &c., the velocity through the interval $B\,C$ would be equal to $B\,b$; at the point C it would instantaneously become $C\,c$, and would remain the same during the interval $C\,D$, when it would become $D\,d$, &c., so that the irregular figure $b\,\gamma\,c\,\delta\,d\,\varepsilon\,e$, &c. would be the limit of the ordinates representing the velocities. In both hypotheses the acquired velocities $B\,b$, $C\,c$, &c., at the points B, C, &c., are equal, so that the velocity computed for any point, as E, by either hypothesis is the same; and if the intervals $B\,C$, $C\,D$, &c. are taken infinitely small, and equal to $d\,t$, the velocity corresponding to any portion of the line $A\,F$, computed in either way, cannot differ but by an infinitely small quantity of the order $c\,\gamma$, $d\,\delta$, &c.; therefore we may use either hypothesis at pleasure. Again, it is evident that if the intervals of time $B\,C$, $C\,D$, &c. should be decreased, the instantaneous forces acting at B, C, D, &c. must be decreased in the same ratio. For if the interval was $C\,C'$, the velocity corresponding to the point C' would be $C'\,\delta'$, and its increase at C' would be $\delta'\,d'$, which is to $\delta\,d$ as $C\,C'$ to $C\,D$, and the increment of the velocity would be as the intensity of the force P multiplied by the element of the time $d\,t$, or $P\,.\,d\,t$ as above.

which determine the position of this point; we shall suppose each of these forces to act in *a contrary direction to that of the origin of the co-ordinates,* [34vii] or, in other words, that *these forces tend to increase the co-ordinates.* At the beginning of the next instant $d\,t$, the body acquires in the direction of each of these co-ordinates, the increments of force, or of velocity, $P.d\,t$, $Q.d\,t$, $R.d\,t$. The velocities of the point M, parallel to these co-ordinates, are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$; because in an infinitely small moment of time, they may be supposed uniform, consequently equal to the elementary spaces divided by the element of the time. The velocities of the point at the beginning of the second instant of time, are therefore

$$\frac{dx}{dt}+P.dt; \qquad \frac{dy}{dt}+Q.dt; \qquad \frac{dz}{dt}+R.dt; \tag{35}$$

or

$$\begin{aligned}&\frac{dx}{dt}+d.\frac{dx}{dt}-d.\frac{dx}{dt}+P.dt;\\ &\frac{dy}{dt}+d.\frac{dy}{dt}-d.\frac{dy}{dt}+Q.dt;\\ &\frac{dz}{dt}+d.\frac{dz}{dt}-d.\frac{dz}{dt}+R.dt;\end{aligned} \tag{36}$$

but in this second instant, the actual velocities of the point parallel to the co-ordinates x, y, z, are evidently $\frac{dx}{dt}+d.\frac{dx}{dt}$; $\frac{dy}{dt}+d.\frac{dy}{dt}$; $\frac{dz}{dt}+d.\frac{dz}{dt}$; [36′] the forces $-d.\frac{dx}{dt}+P.dt$, $-d.\frac{dy}{dt}+Q.dt$, $-d.\frac{dz}{dt}+R.dt$, ought therefore to be destroyed, so that the point M would be in equilibrium, if acted upon by these forces only. Therefore, if we denote by δx, δy, δz, any [36″] variations of the three co-ordinates x, y, z, which must not be confounded with the differentials $d\,x$, $d\,y$, $d\,z$, representing the spaces described by the point parallel to the co-ordinates during the instant $d\,t$,* the equation (b) § 3 [18]

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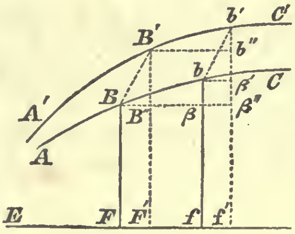
* (17g) We shall here explain in a geometrical manner the principles of the method of variations, so far as it may be necessary to understand the computations made in the present work. Let $A\,B\,b\,C$ be the orthographic projection of any curve upon the plane of x, y, so that the co-ordinates of any point B are $EF=x$, $FB=y$. The co-ordinates of the point b of the same projection of the curve, infinitely near to B, will be represented, according



Method of variations.

will become

[37] $$0=\delta x.\left\{d.\frac{dx}{dt}-P.dt\right\}+\delta y.\left\{d.\frac{dy}{dt}-Q.dt\right\}+\delta z.\left\{d.\frac{dz}{dt}-R.dt\right\}; \quad (f)$$

If the point M be free, we must put the coefficients of δx, δy, δz, separately

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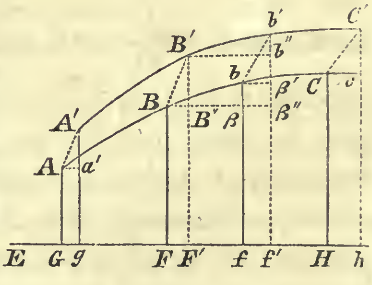
to the usual differential notation, by $Ef=x+dx$, $fb=y+dy$, and if we draw $B\beta$ parallel [36b] and equal to Ff, the line $B\beta=Ff=dx$, and $\beta b=dy$, will represent the differentials of the absciss EF, and of the ordinate FB respectively; these *differentials* being the differences of the co-ordinates of two consecutive points B, b, of *the same curve* $ABbC$. But if the *nature* of the proposed curve be changed in an *arbitrary* manner, so that its projection may become $A'B'b'C'$ *infinitely near* to $ABbC$, the points B, b, being changed into B', b', the co-ordinates of the point B will be then changed into EF', $F'B'$, corresponding to the point [36c] B', which co-ordinates are represented by $EF'=x+\delta x$, $F'B'=y+\delta y$, and the changes in the values of the co-ordinates are called the *variations*; thus $FF'=BB''=\delta x$, is the *variation* of the absciss EF, and $B'B''=\delta y$, is the *variation* of the ordinate FB. In like manner, if we draw the ordinate $b'f'$ parallel to bf, and let fall upon it the perpendiculars $b\beta'$, $B\beta''$, $B'b''$, the variations of the ordinates Ef, fb, will be represented by ff', and $b'\beta'$ respectively. From this explanation of the term *variation*, it is evident that the variation of any function of x, y, z, &c., is found by changing x, y, z, &c., into $x+\delta x$, $y+\delta y$, $z+\delta z$, &c., respectively, and subtracting the former value from the latter, neglecting as in the differential calculus, the powers and products of δx, δy, δz, &c., so that the variation of any function is found in the same manner as its differential, using the sign δ instead of d.



We may proceed from the point B to b' in two different ways. *First* from B to B' by the method of *variations*, then from B' to b' by the *differential* of the curve $A'B'$. *Secondly*, from B to b by the *differential* of the curve AB, then from b to b' by the method of *variations*. The comparison of these two methods furnishes a very important theorem in the doctrine of variations. To avoid a complication of letters we shall put $dx=\dot{x}$, $\delta x=x'$, then $EF=x$, $Ef=x+\dot{x}$, $EF'=x+x'$; now by proceeding as in the first method, we have $B'b''$, or $F'f'$ equal to the differential of EF', along the curve $A'B'$, therefore $B'b''=F'f'=dx+dx'$, this added to $EF'=x+\delta x$, gives $Ef'=x+\delta x+dx+dx'$. And by the second method we have $b\beta'$ or ff' equal to the variation of Ef, therefore $b\beta'=ff'=\delta x+\delta\dot{x}$, this added to $Ef=x+dx$ gives $Ef'=x+dx+\delta x+\delta\dot{x}$. Putting these two expressions of Ef' equal to each other, we get

$$x+\delta x+dx+dx'=x+dx+\delta x+\delta\dot{x},$$

Equations of the motion of a point.

equal to nothing; and by supposing the element of the time dt constant, we shall obtain the three differential equations

$$\frac{ddx}{dt^2} = P; \qquad \frac{ddy}{dt^2} = Q; \qquad \frac{ddz}{dt^2} = R; \qquad [38]$$

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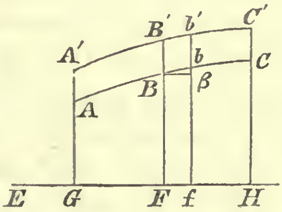
this, by neglecting the terms in each member which destroy each other, gives $dx' = \delta \dot{x}$, consequently the *differential of the variation* of x is equal to the *variation of the differential* of the same quantity, and if we substitute for x', $\dot{x}$ their values $\delta x, dx$, it becomes $d\delta x = \delta dx$, [36d] consequently the *characteristic $d\delta$ may be changed into δd, or the contrary.* What is here said relative to the axis of x may be easily proved in the same manner for that of y or z, which, however, is evident of itself, since the axis of x may be changed into that of y or z, the names being arbitrary, so that $d\delta y = \delta dy$; $d\delta z = \delta dz$. This theorem might be generalized, [36e] but it will not be necessary in the present work.

Another theorem of great importance is this. The *variation of the integral of any quantity U is equal to the integral of the variation of the same quantity*, or in symbols $\delta . \int U = \int . \delta U$. This is easily proved, for if we substitute $x + \delta x$, $y + \delta y$, $z + \delta z$, for x, y, z, respectively in U, and call the result U', we shall have $\delta . \int U = \int U' - \int U = \int (U' - U)$, and as [36f] $U' - U = \delta U$, this becomes $\delta . \int U = \int . \delta U$, consequently the characteristic $\delta . \int$ may be changed into $\int . \delta$, in the same manner as δd was changed into $d\delta$. This agrees with the geometrical consideration of the subject in the following paragraph.

It may not be amiss to explain in a geometrical manner the value of an expression of this form $\delta . \int M dx$, which frequently occurs. For greater simplicity we shall suppose $\delta x = 0$, $M = y$; then the lines BB', bb', will fall on the continuations of the ordinates FB, fb, so that EF being $= x$, and $FB = y$, we shall have $Ff = dx$, $BB' = \delta y$, $\beta b = dy$, $d\delta x = 0$; and [36g] the element of the area $BFfb = y dx$, that of $BB'b'b = \delta y . dx$. Taking the integrals of these expressions, supposing them to be limited by the ordinates GAA', HCC', they will give the areas $GACH$, $AA'C'C$, namely $GACH = \int y dx$; $AA'C'C = \int \delta y . dx$; now it is evident that the latter area is the variation of the former, and as the variation of $\int y dx$ is denoted by $\delta . \int y dx$, we shall have the area



$$AA'C'C = \delta . \int y dx = \int \delta y . dx.$$

The identity of these two expressions is also a consequence of the preceding theorem that the characteristic $\delta \int$ may be changed into $\int \delta$, for by that means the first expression $\delta . \int y dx$ becomes $\int . \delta (y dx)$, and as $\delta dx = 0$, this is evidently equal to $\int \delta y . dx$. From this simple example we may obtain a better idea of the import of such expressions as $\delta . \int y dx$, $\int \delta y . dx$, &c., than could be done without considering the subject geometrically.

If the point M be not free, but forced to move upon a surface or a curve line, we must, by means of the equations of the surface or curve, exterminate from the equation (f) [37], as many of the variations δx, δy, δz, as there are equations in this surface or curve, and then put the coefficients of the remaining equations separately equal to nothing.

8. It is possible to suppose in the equation (f) [37], that the variations δx, δy, δz, are equal to the differentials dx, dy, dz, respectively, since these differentials are necessarily subjected to the conditions of the motion of the

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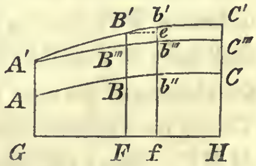
An expression of the form $\int M.\delta dx$ may be reduced so as to contain δx without its differential. For, by putting $\delta dx = d\delta x = dx'$, ($x'$ being as above $=\delta x$), it becomes $\int M.dx'$, which, by integrating by parts, is equal to $Mx' - \int dM.x'$, as is easily proved by [36h] differentiation. Resubstituting for x' its value δx, it becomes $\int M.\delta dx = M.\delta x - \int dM.\delta x$. We might add a constant quantity to the second member to complete the integral, so as to render it nothing at the *first* point A, (Fig. page 35), where it commences, the co-ordinates of which point we shall call $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, those of the *last* point C of the integral being $x_{\prime\prime}$, $y_{\prime\prime}$, $z_{\prime\prime}$; the values of M corresponding to the points A, C, being respectively $M_{\prime}$, $M_{\prime\prime}$. Hence the [36i] complete integral $\int M.d\delta x = M.\delta x - M_{\prime}.\delta x_{\prime} + \int dM.\delta x$, and the whole integral comprised between the points A, C, is $\int M.d\delta x = M_{\prime\prime}.\delta x_{\prime\prime} - M_{\prime}.\delta x_{\prime} + \int dM.\delta x$, the term affected with the sign $\int$ being taken within the same limits. If M is constant, $dM = 0$, and $\int M.d\delta x = M.\delta x_{\prime\prime} - M.\delta x_{\prime}$. In like manner $\int M.d\delta y = M_{\prime\prime}.\delta y_{\prime\prime} - M_{\prime}.\delta y_{\prime} + \int dM.\delta y$, and if $M = 1$, and $dM = 0$, $\int .d\delta y = \delta y_{\prime\prime} - \delta y_{\prime}$. The import of this integral may be explained geometrically, supposing $\delta x = 0$. Then $BB' = \delta y$, and if we draw $B'e$ parallel and equal to Bb'', and suppose the curve $A'B'''b'''C'''$ to be such that the intercepted parts of the ordinates BB''', $b''b'''$, CC''', may be equal to AA', we shall evidently have $d.\delta y = b''b' - BB' = b'e$, and $\int .d\delta y$ is equal to the sum of all the lines $b'e$ comprised between the points A', C', and this sum is evidently equal to $C'C''' = CC' - CC''' = CC' - AA' = \delta y_{\prime\prime} - \delta y_{\prime}$, as above. We might extend these remarks to a much greater length, but what we have said will suffice for all the purposes of the present work, and we shall conclude by observing that the calculus of variations is of great importance in finding the form of functions like $\int y\delta x$, having the property of a maximum or minimum; which is obtained by the usual principles of the maximum or [36k] minimum, by putting its variation $\delta .\int y dx = 0$, or the area $AA'B'C'C$ equal to nothing.



moving particle *M*.* Making this supposition, and then integrating the equation (*f*) [37], we shall have

$$\frac{dx^2+dy^2+dz^2}{dt^2}=c+2.\int(P.dx+Q.dy+R.dz)\,; \qquad [39]$$

c being an arbitrary constant quantity. $\frac{dx^2+dy^2+dz^2}{dt^2}$ is the square of the velocity of *M*,† which velocity we shall denote by v; supposing therefore that $P.dx+Q.dy+R.dz$ is an exact differential of a function φ, we shall have [39′] [39″]

$$v^2=c+2\varphi. \qquad (g) \qquad [40]$$

This case takes place when the forces acting upon the particle *M* are functions of the respective distances from the origins of these forces to this

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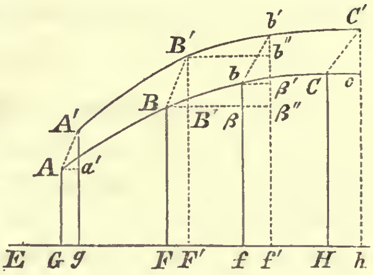

* (17*h*) If the point *M* was compelled to move in a curve whose projection is *A B C*, the curve *A′ B′ C′*, depending on the variations, might be supposed to coincide with *A B C*, and we might take the arbitrary variations δx, δy, such that the point B' would fall in b, and then we should have $\delta x=dx$, $\delta y=dy$, and the projection upon the plane of x, y, would, upon similar principles, give $\delta z=dz$. Substitute these in [37], multiply by $\frac{2}{dt}$, and transpose the terms depending on P, Q, R, it becomes

$$\frac{2dx}{dt}.d.\frac{dx}{dt}+\frac{2dy}{dt}.d.\frac{dy}{dt}+\frac{2dz}{dt}.d.\frac{dz}{dt}=2(P.dx+Q.dy+R.dz),$$

whose integral gives [39], because $d.\left(\frac{dx}{dt}\right)^2=\frac{2dx}{dt}.d.\frac{dx}{dt}$; $d.\left(\frac{dy}{dt}\right)^2=\frac{2dy}{dt}.d.\frac{dy}{dt}$; $d.\left(\frac{dz}{dt}\right)^2=\frac{2dz}{dt}.d.\frac{dz}{dt}$.

† (17*i*) If, in the figure of note (13*b*), we suppose *A c* to be infinitely small, and the points *A*, *c*, to represent two consecutive points or places of the body; the ordinates of the point *A* being $KG=x$, $GE=y$, $EA=z$; those of the point *c*, $KH=x+dx$, $Hf=y+dy$, $fc=z+dz$, we shall evidently have $AD=dx$, $Dd=dy$, $dc=dz$, $Ac=ds$, and $Ac^2=AD^2+Dd^2+dc^2=dx^2+dy^2+dz^2$, [11*a*], and since $\frac{Ac}{dt}=$ velocity v, we shall [40*a*] have as above $\frac{dx^2+dy^2+dz^2}{dt^2}=v^2$, which, substituted in [39], gives [40]. Moreover it appears from the figure, that the cosines of the angles which the element of the curve ds makes with lines drawn parallel to the axes x, y, z, are represented by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, respectively. [40*b*]

particle, which comprises almost all the forces in nature. For S, S', &c., representing these forces, and s, s', &c., being the distances of the particle M from their origins; the resultant of all these forces, multiplied by the variation of its direction, will be, by § 2 [16], equal to $\Sigma . S . \delta s$; it is also equal to $P . \delta x + Q . \delta y + R . \delta z$; therefore we shall have*

[41] $$P . \delta x + Q . \delta y + R . \delta z = \Sigma . S . \delta s;$$

and as the second member of this equation is an exact differential, the first member must be so.

From the equation (g) [40] it follows, 1st. That if the particle M is not
[41′] acted upon by any forces, its velocity will be constant, because then $\varphi = 0$.†
This is easy to prove in another way, by observing that a body moved upon a surface or a curve, loses at the contact with each infinitely small plane of the surface, or each infinitely small side of the curve, but an infinitely small
[41″] part of its velocity of the second order.‡ 2d. That the particle M, in passing

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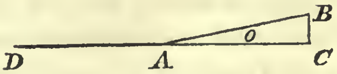
* (17k) By hypothesis the forces S, S', S'', &c., acting in the directions s, s', s'', &c., are equivalent to the three forces P, Q, R, acting in directions parallel to the axes x, y, z, respectively. Now supposing, as in § 2, V to be the resultant of the first named forces, and u its direction, we have $V . \delta u = \Sigma . S . \delta s$ [16]; and as V is also by hypothesis equivalent to the forces P, Q, R, we have by the same formula [16] $V . \delta u = P . \delta x + Q . \delta y + R . \delta z$, hence $P . \delta x + Q . \delta y + R . \delta z = \Sigma . S . \delta s$, the second member of which is by note (13f) an exact variation of a function, consequently $P . \delta x + Q . \delta y + R . \delta z$ is an exact variation of a function φ, or

[40c] $$\delta \varphi = P . \delta x + Q . \delta y + R . \delta z.$$

† (17l) If the particle is not acted upon by any forces, we shall have $P = 0$, $Q = 0$, $R = 0$, [34^{vi}], and $d\varphi = P . dx + Q . dy + R . dz$ [39″] would then become $d\varphi = 0$, whose integral $\varphi =$ constant, may be put $\varphi = 0$, including this constant term in the quantity c [40].

‡ (18a) Thus if a body move with the velocity z in the direction BA, and at the point A be compelled to change its direction to the line AD, and we continue DA to C, and put the infinitely small angle $BAC = \theta$, we shall have, (by 11, 34′) the velocity in the direction $AD = z . \cos . \theta = z (1 - \frac{1}{2} \theta^2 + \text{\&c.})$, by 44 Int.; this differs from the original velocity z by the quantity $\frac{1}{2} \theta^2 . z$ of the second order, as is stated above; and the loss of velocity on an infinite number of such lines, or on the whole curve, would be an infinitely small quantity of the first order only.



from a given point, with a given velocity, towards another given point, will
have acquired, upon arriving at this last point, the same velocity, whatever [41'']
be the curve which it may have described.*

But if the particle is not forced to move upon a determinate curve, the
curve which it describes possesses a singular property, which had been
discovered by metaphysical considerations; but which is in fact nothing
more than a remarkable result of the preceding differential equations. It
consists in this, that the integral $\int v\,ds$, comprised between the two extreme
points of the described curve, is less than on every other curve, if the body
be free; or less than on every other curve described on the surface upon [41''']
which the particle is forced to move, if it be not wholly free.

Principle of the least action.

To prove this, we shall observe that $P.dx+Q\cdot dy+R.dz$ being supposed an exact differential, the equation (*g*) [40] gives†

$$v\,\delta v=P.\delta x+Q.\delta y+R.\delta z\,;$$ [42]

the equation (*f*) [37] of the preceding article therefore becomes

$$0=\delta x.d.\frac{dx}{dt}+\delta y.d.\frac{dy}{dt}+\delta z.d.\frac{dz}{dt}-v\,dt.\delta v.$$ [43]

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* (18*b*) Suppose the values of v, φ, corresponding to the first point of the curve, to be v', φ', those to the last point v'', φ''. The equation [40] at the first point will become $v'^2=c+2\varphi'$, whence $c=v'^2-2\varphi'$, which being substituted in [40] gives generally $v^2=v'^2-2\varphi'+2\varphi$; hence at the last point of the curve we have $v''^2=v'^2-2\varphi'+2\varphi''$. [40*d*]
Now φ [39''] is a function of S, S', &c., s, s', &c., which quantities are given at the first and last points of the curve, consequently φ', φ'', must be given, and v' is also given, by hypothesis, therefore the value of v'' must be the same whatever be the curve described. That is, we can determine the difference of the squares of the velocities at two points without knowing the curve described by the body, and this curve might become a right line in the case where a body should fall freely towards a point to which it is attracted by a force varying as any function of the distance. We must however always observe that the theorem [40*d*] would not hold true if $P.dx+Q.dy+R.dz$ was not an exact differential $d\varphi$, and it would not generally be an exact differential if it contained terms depending upon the *particular* curve described, as might, for example, be the case if the curve produced a particular resistance or friction.

† (18*c*) Taking the variation of [40], dividing it by 2, and substituting for $\delta\varphi$ its value $P.\delta x+Q.\delta y+R.\delta z$ [40*c*], we get [42]. Substituting now in [37] the value of $P.\delta x+Q.\delta y+R.\delta z$ [42], we obtain [43].

Put ds for the element of the curve described by the particle, we shall have*

[44] $$v\,dt=ds; \qquad ds=\sqrt{dx^2+dy^2+dz^2};$$

therefore

[45] $$0=\delta x\,.\,d\,.\frac{dx}{dt}+\delta y\,.\,d\,.\frac{dy}{dt}+\delta z\,.\,d\,.\frac{dz}{dt}-ds\,.\,\delta v; \qquad (h)$$

Taking the differential of the expression ds relative to the characteristic δ, we shall have

[46] $$\frac{ds}{dt}\,.\,\delta\,.\,ds=\frac{dx}{dt}\,.\,\delta\,.\,dx+\frac{dy}{dt}\,.\,\delta\,.\,dy+\frac{dz}{dt}\,.\,\delta\,.\,dz.$$

The characteristics d and δ being independent, we may place them at pleasure the one before the other; we may therefore give to the preceding equation the following form,†

[47] $$v\,.\,\delta ds=\frac{d\,.\,\{dx.\delta x+dy.\delta y+dz.\delta z\}}{dt}-\delta x\,.\,d\,.\frac{dx}{dt}-\delta y\,.\,d\,.\frac{dy}{dt}-\delta z\,.\,d\,.\frac{dz}{dt};$$

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* (18d) The expression $v\,dt=ds$, follows from the equation $v=\frac{s}{t}$ [29″] by changing t into dt, s into ds, the velocity v being esteemed uniform during the time dt. The equation $ds=\sqrt{dx^2+dy^2+dz^2}$, is deduced, by putting $\mathcal{A}c=ds$ in [40a]. The substitution of $v\,dt=ds$ in [43] gives [45]. Taking the variation of $ds^2=dx^2+dy^2+dz^2$ [44], and dividing it by $2\,dt$, gives [46].

† (19) Substituting $\frac{ds}{dt}=v$ in the first member of [46], it becomes like the first of [47]. The second member of [46] may be transformed by observing that $\frac{dx}{dt}\,.\,\delta dx=\frac{d.(dx.\delta x)}{dt}-\delta x\,.\,d\,.\frac{dx}{dt}$; $\frac{dy}{dt}\,.\,\delta dy=\frac{d.(dy.\delta y)}{dt}-\delta y\,.\,d\,.\frac{dy}{dt}$; $\frac{dz}{dt}\,.\,\delta dz=\frac{d.(dz.\delta z)}{dt}-\delta z\,.\,d\,.\frac{dz}{dt}$; as is easily proved by developing the first terms of the second members, these, substituted in [46], give [47]. Again, by the equation [45], we have $\delta x.d\,.\frac{dx}{dt}+\delta y\,.\,d\,.\frac{dy}{dt}+\delta z\,.\,d\,.\frac{dz}{dt}=ds\,.\,\delta v$, this being substituted in [47], it becomes $v\,.\,\delta ds=\frac{d\,.\,\{dx.\delta x+dy.\delta y+dz.\delta z\}}{dt}-ds\,.\,\delta v.$ Transposing the term $ds\,.\,\delta v$, the first member becomes $v\,.\,\delta ds+ds\,.\,\delta v$, which is evidently the variation of $v\,ds$, or $\delta.(v\,ds)$, hence we obtain [48], whose integral, changing the characteristic $\int\delta$ into $\delta\int$, as in [36f], gives [49]. If we suppose as in [36h] that $x_{\prime}, y_{\prime}, z_{\prime}$, are the co-ordinates of the first point of

Subtracting from the first member of this equation the second member of the equation (*h*) [45], we shall have

$$\delta.(v\,d\,s)=\frac{d.(d\,x.\delta x+d\,y.\delta y+d\,z.\delta z)}{d\,t}; \qquad [48]$$

This, being integrated with respect to the characteristic *d*, gives

$$\delta.\int v\,d\,s=\text{constant}+\frac{d\,x.\delta x+d\,y.\delta y+d\,z.\delta z}{d\,t}. \qquad [49]$$

If we extend the integral to the whole curve described by the particle, and suppose the extreme points of the curve to be invariable, we shall have $\delta.\int v\,d\,s=0$; that is, of all the curves which a body could describe in passing from one given point to another, when subjected to the forces *P*, *Q*, *R*, it will describe that in which the variation of the integral $\int v\,d\,s$ is nothing, consequently that in which this integral is a minimum. [49']

If the particle moves along a curve surface, without being acted upon by any force, its velocity will be constant,* and the integral $\int v\,d\,s$ will become $v\int d\,s$; therefore, the curve described by the particle is then the shortest that can be traced upon the surface, from the point of departure to the point arrived at. [49'']

Pressure of a particle upon a surface.

9. We shall now investigate the pressure of a particle upon a surface on which it moves. Instead of exterminating from the equation (*f*) [37] of § 7, one of the variations δx, δy, δz, by means of the equation of the surface,

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the curve, and $x_{,,}$, $y_{,,}$, $z_{,,}$, those of the last point, and take the integral [49] so as to be nothing at the first point, it will be generally expressed by

$\delta.\int(v\,d\,s)=\frac{d\,x.\delta x+dy.\delta y+d\,z.\delta z}{d\,t}-\frac{(d\,x_{,}.\delta x_{,}+d\,y_{,}.\delta y_{,}+d\,z_{,}.\delta z_{,})}{dt}$, and the whole integral becomes $\delta.\int(v\,d\,s)=\frac{(d\,x_{,,}.\delta x_{,,}+d\,y_{,,}.\delta y_{,,}+d\,z_{,,}.\delta z_{,,})}{d\,t}-\frac{(d\,x_{,}.\delta x_{,}+d\,y_{,}.\delta y_{,}+d\,z_{,}.\delta z_{,})}{d\,t}$. If the extreme points of the curve corresponding to *A*, *C*, [Fig. page 32], are *fixed* or given, the variations $\delta\,x_{,}$, $\delta\,y_{,}$, $\delta\,z_{,}$; $\delta\,x_{,,}$, $\delta\,y_{,,}$, $\delta\,z_{,,}$, will be nothing, therefore the second member of the preceding equation will be nothing, consequently $\delta.\int v\,d\,s=0$, which corresponds to its minimum, as is observed [36*k*].

* (19*a*) This is shown in the remarks which follow the formula [41']. When v is constant $\delta.\int v\,d\,s=0$, becomes $v.\delta\int d\,s=0$, or $\delta\int d\,s=0$, and as $\int d\,s=s$, this becomes $\delta s=0$, corresponding to the minimum value of s.

As an example of the application of the principle of the least action, we may mention the manner of deducing from it the laws of reflection and refraction of light. Thus if a ray of we may, by § 3, add to that equation the differential equation of the surface, multiplied by an indeterminate quantity,* $-\lambda dt$, and we may then consider the variations δx, δy, δz, as independent. Let $u=0$ be the equation of the surface; we must add to the equation (f) the term $-\lambda.\delta u.dt$, and the pressure of the point against the surface will be by § 3, equal to

[50] $$\lambda.\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2};$$

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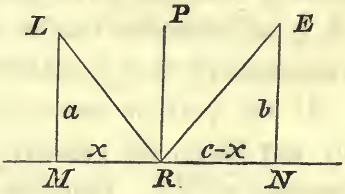
light proceed from a luminous object to the eye of the observer, its path is a straight line in conformity with the principles of the least action [49″]. Again, if the ray proceed from the luminous point L, and be reflected from the plane surface MRN at R, in the direction RE, to the eye of the spectator at E; the space passed over, $s=LR+RE$, ought by this principle [49″] to be a minimum, the velocity v being supposed constant. From this we may easily deduce the equality of the angles of incidence and reflection, PRL, PRE; the line PR, as well as LM, EN, being perpendicular to MN. For, if we put $LM=a$, $EN=b$, $MN=c$, $MR=x$, $RN=c-x$, we shall get $s=\sqrt{x^2+a^2}+\sqrt{(c-x)^2+b^2}$, whose variation, relative to x, being put equal to nothing, gives $\frac{x}{\sqrt{x^2+a^2}}=\frac{c-x}{\sqrt{(c-x)^2+b^2}}$, or $\frac{MR}{LR}=\frac{RN}{RE}$; whence $\sin.MLR=\sin.REN$, or $\sin.PRL=\sin.PRE$; therefore in the case of reflection, the principle of the least action would make the angles PRL, PRE, equal.



If the velocity on the line LR is equal to m, and on the line RE is equal to n; the principle of the least action [49′], $\delta.\int v\,ds=0$, would require that the variation of $m\sqrt{x^2+a^2}+n\sqrt{(c-x)^2+b^2}$ should be nothing, corresponding to the minimum. Proceeding with the calculation as above, we should get $m.\sin.PRL=n.\sin.PRE$, and this is the same as the usual law of refraction, supposing the point E to fall below MN upon the continuation of the line EN of the present figure, and that the ray of light enters the refracting medium at the point R situated in the line MN, which separates the two mediums, where the velocity of the ray is changed from m to n. In this case the sine of the angle of incidence is to the sine of refraction in the constant ratio n to m.

* (19b) The second member [37] corresponds to $\Sigma.S.\delta s$ of [18] and [23] and to this term is added $\lambda.\delta u$ in [23] on account of the equation of the surface, and it is shown in the remarks following [23] that the pressure the particle M exerts upon the surface is $-R$. If we had changed the sign of λ and added in the equation [23] the term $-\lambda\delta u$, the sign of λ

Suppose at first that the particle is not acted upon by any force, its velocity v will be constant;* then as $v\,dt = ds$, the element of the time dt being supposed constant, ds must also be constant, and the equation (*f*) [37], augmented by the term $-\lambda.\delta u.dt$, will give the three following equations;†

$$0=v^2.\frac{ddx}{ds^2}-\lambda.\left(\frac{du}{dx}\right);\quad 0=v^2.\frac{ddy}{ds^2}-\lambda.\left(\frac{du}{dy}\right);\quad 0=v^2.\frac{ddz}{ds^2}-\lambda.\left(\frac{du}{dz}\right); \qquad [51]$$

whence we deduce

$$\lambda.\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}=\frac{v^2.\sqrt{(ddx)^2+(ddy)^2+(ddz)^2}}{ds^2}; \qquad [52]$$

but ds being constant, the radius of curvature of the curve described by the particle is equal to‡ Radius of Curvature.

$$\frac{ds^2}{\sqrt{(ddx)^2+(ddy)^2+(ddz)^2}}; \qquad [53]$$

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in [22] would be changed, and that expression [22] would give for the pressure $-R$ the quantity $\lambda.\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}$ as in [50].

* (19*c*) As is shown in [41′].

† (19*d*) By hypothesis the forces P, Q, R, are nothing; substituting these in [37] and adding, as above, the term $-\lambda.\delta u.dt$, which by [14*a*] is equal to

$$-\lambda dt\left\{\left(\frac{du}{dx}\right).\delta x+\left(\frac{du}{dy}\right).\delta y+\left(\frac{du}{dz}\right).\delta z\right\},$$

it becomes, by dividing by dt,

$$0=\delta x.\left\{\frac{ddx}{dt^2}-\lambda.\left(\frac{du}{dx}\right)\right\}+\delta y.\left\{\frac{ddy}{dt^2}-\lambda.\left(\frac{du}{dy}\right)\right\}+\delta z.\left\{\frac{ddz}{dt^2}-\lambda.\left(\frac{du}{dz}\right)\right\}.$$

Substituting for dt^2 its value deduced from [44] $\frac{ds^2}{v^2}$, and putting the coefficients of δx, δy, δz, equal to nothing, gives the equations [51], whence $\lambda.\left(\frac{du}{dx}\right)=\frac{v^2}{ds^2}.ddx$; $\lambda.\left(\frac{du}{dy}\right)=\frac{v^2}{ds^2}.ddy$; $\lambda.\left(\frac{du}{dz}\right)=\frac{v^2}{ds^2}.ddz$. Squaring each of these equations, adding them together, and taking the square root of the sum, we get the equation [52].

‡ (19*e*) Let $FabcE$ (see figure on page 42) be the proposed curve, ab, bc, two infinitely small and equal parts of it, considered as right lines, whose centre of curvature is C,

and by putting this radius equal to r we shall have

[54] $$\lambda.\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2}=\frac{v^2}{r};$$

Pressure. that is, the pressure of the particle against the surface is equal to the squares of the velocity, divided by the radius of curvature of the curve it describes.

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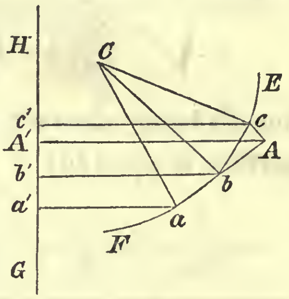
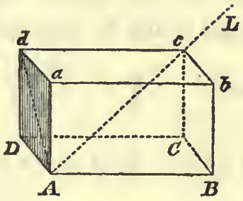
making $Ca=Cb=Cc$. Continue the line ab to A, making $bA=ab=bc$, equal to the element of the curve ds supposed constant. Then, by construction, the angle cbA is equal to aCb, and as the triangles aCb, cbA, are isosceles, they must be similar. Hence $Ac:bc::ab:Cb$, or in symbols $Ac:ds::ds:r$, whence $r=\frac{ds^2}{Ac}$. Let $Ga'b'A'c'H$ be the axis of x, which in general is in a different plane from Cac. Upon this let fall the perpendiculars aa', bb', AA', cc', then $a'b'=dx$, $b'c'=dx+ddx$, and since by construction $ab=bA$, we have $b'A'=a'b'=dx$, therefore $A'c'=ddx$. That is to say, the projection of the line Ac upon the axis of x is equal to ddx. In the same way we may prove that the projection of the line Ac upon the axis of y is ddy, and its projection upon the axis of z is ddz. If we therefore, upon Ac as a diagonal, describe a rectangular parallelopiped, whose sides AD, Dd, dc, are respectively parallel to the axes of x, y, z, we shall have $AD=ddx$, $Dd=ddy$, $dc=ddz$. The lower figure was drawn separately from the other to enlarge it, so as to avoid confusion in the lines. Now we have as in [11a] $Ac=\sqrt{AD^2+Dd^2+dc^2}$, and by substituting the preceding values of AD, Dd, dc,





$$Ac=\sqrt{(ddx)^2+(ddy)^2+(ddz)^2}.$$

This, substituted in $r=\frac{ds^2}{Ac}$, gives

[53a] $$r=\frac{ds^2}{\sqrt{(ddx)^2+(ddy)^2+(ddz)^2}},$$

as in [53]. This value of $\frac{ds^2}{\sqrt{(ddx)^2+(ddy)^2+(ddz)^2}}$ being substituted in the second member of [52], produces the expression [54]. The first member of which is by [50], equal to the pressure of the point against the surface, consequently that pressure is also equal to the second member $\frac{v^2}{r}$ of the same expression.

If the particle move upon a spherical surface, it will describe the circumference of a great circle of the sphere, which passes by the primitive direction of its motion; since there is no reason why it should deviate to the right rather than to the left of the plane of this great circle: its pressure against the surface, or in other words, against the circumference it describes, is therefore equal to the square of its velocity, divided by the radius of this circumference.

Suppose the particle to be attached to the extremity of an infinitely thin thread, void of gravity, whose other extremity is fixed at the centre of the sphere; it is evident that the pressure exerted by the particle against the circumference will be equal to the tension of the thread, if the particle were wholly supported by it. The effort of the particle to stretch the thread, and to move from the centre to the circumference, is what is called the *centrifugal force*; therefore the centrifugal force is equal to the square of the velocity, divided by the radius. [54′]

Centrifugal force.

In the motion of a particle upon any curve whatever, the centrifugal force is equal to the square of the velocity, divided by the radius of curvature of the curve; since the infinitely small arch of this curve coincides with the circumference of the circle of curvature; we shall therefore have the pressure of a particle upon the surface it describes, by adding to the square of the [54″] velocity, divided by the radius of curvature, the pressure arising from the forces which act upon the particle.

If the curve is situated in a plane surface, that surface may be taken for the plane of two of the co-ordinates, as y, z, and then x, dx, ddx, may be neglected in [44, 53*a*], which will become

$$ds^2 = dy^2 + dz^2, \qquad r = \frac{ds^2}{\sqrt{(ddy)^2+(ddz)^2}}. \qquad [53b]$$

The differential of ds^2, supposing always ds constant, gives $0 = dy.ddy + dz.ddz$, whence

$$(ddy)^2 + (ddz)^2 = \frac{dz^2.(ddz)^2}{dy^2} + (ddz)^2 = \frac{dz^2+dy^2}{dy^2}.(ddz)^2 = \frac{ds^2}{dy^2}.(ddz)^2;$$

whence $\sqrt{(ddy)^2+(ddz)^2} = \frac{ds.ddz}{dy}$. Substituting this in r [53*b*], it becomes

$$r = \frac{ds.dy}{ddz}, \qquad [53c]$$

which will hereafter be used.

In the motion of a particle upon a surface, the pressure arising from the centrifugal force, is equal to the square of the velocity, divided by the radius of curvature of the curve described by the particle, and multiplied by the [54‴] sine of the inclination of this circle of curvature to the tangential plane of the surface :* by adding to this pressure what arises from the action of the other forces acting upon this particle, we shall have the whole pressure of the particle against the surface.

We have proved [54] that if the particle is not acted upon by any force, its pressure against the surface is equal to the square of its velocity, divided by the radius of curvature of the described curve ; the plane of the circle of [54^iv] curvature, that is, the plane which passes through two consecutive points of the curve described by the particle, is in this case perpendicular to the surface.† This curve, relative to the surface of the earth, is what is

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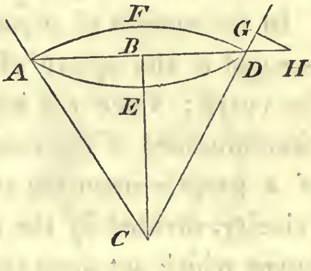

* (20) To illustrate this, suppose the particle to move upon the surface $AEDF$ of a right cone, whose axis is CB, and to describe, with the velocity v, the circumference of the circle $AEDF$, whose radius $r=BD$ is perpendicular to CB. Continue BD to H, making $DH=\frac{v^2}{r}$, draw HG perpendicular to CD. Then, by [54], while the particle is at D, moving towards F, its centrifugal force f in the direction BD, will be represented by $f=DH=\frac{v^2}{r}$, and this may be resolved into two forces DG, GH, of which the former, being parallel to the side of the cone, produces no pressure, the latter, GH, represents the actual pressure $P=GH$. Now if we put I equal to the inclination of the circle $AEDF$ upon the plane which is a tangent to the conical surface at D, it is evident that $I=$ angle $BDC=$ angle GDH. But by trigonometry [54b] $GH=DH.\sin.GDH$, or in symbols $P=f.\sin.I=\frac{v^2}{r}.\sin.I$, as above. What is here said of a cone will evidently apply to any other surface, supposing always the axis BC, to be drawn through the centre of the circle of curvature perpendicular to that circle till it meet in C the plane which is the tangent of the proposed surface at the point D.

† (20a) Comparing the expression of the pressure [54] $\frac{v^2}{r}$ with that in [54b], we get $\frac{v^2}{r}=\frac{v^2}{r}.\sin.I$, whence $1=\sin.I$; therefore the inclination I must be a right angle.

called the *perpendicular to the meridian*,* and we have proved in § 8 [49″], that it is the shortest which can be drawn between two points upon the surface.

10. Of all the forces which we observe upon the earth, the most remarkable is that of gravity; it penetrates the inmost parts of bodies, and were it not for the resistance of the air, it would cause all bodies to fall with an equal velocity. Gravity is nearly the same at the greatest heights to which we can ascend, and at the lowest depths to which we can descend: its direction is perpendicular to the horizon; but in the motions of projectiles, we may [54^{v}] suppose, without sensible error, that it is constant, and that it acts in parallel directions, on account of the shortness of the curves which they describe, in comparison with the circumference of the earth. These bodies being moved in a resisting fluid, we shall call ϵ the resistance which they suffer. This resistance is in the direction of the element ds of the described curve. We shall also put g for the force of gravity. This being premised,

Let us resume the equation (*f*) [37] § 7, and suppose that the plane of x, y, is horizontal, and the origin of z at the highest point; the force ϵ will produce, in the directions of the co-ordinates x, y, z, the three forces†

$$-\epsilon.\frac{dx}{ds}, \qquad -\epsilon.\frac{dy}{ds}, \qquad -\epsilon.\frac{dz}{ds}; \qquad [54^{vi}]$$

therefore we shall have, by § 7 [34^{vi}],

$$P=-\epsilon.\frac{dx}{ds}, \qquad Q=-\epsilon.\frac{dy}{ds}, \qquad R=-\epsilon.\frac{dz}{ds}+g; \qquad [55]$$

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* (20*b*) The properties of this curve are shown in [1897] &c.

† (21) This follows from the principle of the decomposition of forces [11]. Thus, in the figure page 7, a force ϵ in the direction Ac, may be resolved into three rectangular forces, $\epsilon.\frac{AD}{Ac}$, $\epsilon.\frac{Dd}{Ac}$, $\epsilon.\frac{dc}{Ac}$, which, by putting, as in [40*a*], $AD=dx$, $Dd=dy$, $dc=dz$, $Ac=ds$, becomes as in [54^{vi}], the negative sign being prefixed, because the resistance tends to decrease the co-ordinates. Adding the gravity g to the last of these forces, we get P, Q, R, [34^{vi}], as in [55]. Substituting these in [37], it becomes as in [56]; and if the body be wholly free, it will not be necessary to introduce terms like $\lambda\,\delta u$, $\lambda'\,\delta u'$, [26], but we must put the coefficients of δx, δy, δz, separately equal to nothing, and by that means we shall obtain the three equations [57].

and the equation (f) [37] will become

[56] $$0=\delta x.\left\{d.\frac{dx}{dt}+\epsilon.\frac{dx}{ds}.dt\right\}+\delta y.\left\{d.\frac{dy}{dt}+\epsilon.\frac{dy}{ds}.dt\right\}+\delta z.\left\{d.\frac{dz}{dt}+\epsilon.\frac{dz}{ds}.dt-gdt\right\}.$$

If the body is wholly free, we shall have the three equations

[57] $$0=d.\frac{dx}{dt}+\epsilon.\frac{dx}{ds}.dt;\qquad 0=d.\frac{dy}{dt}+\epsilon.\frac{dy}{ds}.dt;\qquad 0=d.\frac{dz}{dt}+\epsilon.\frac{dz}{ds}-gdt.$$

The two first give*

[58] $$\frac{dy}{dt}.d.\frac{dx}{dt}-\frac{dx}{dt}.d.\frac{dy}{dt}=0;$$

whence by integration, $dx=fdy$, f being an arbitrary constant quantity. This is the equation of a horizontal line; therefore the body moves in a vertical plane. Taking this plane for that of x, z, we shall have $y=0$; the two equations

[59] $$0=d.\frac{dx}{dt}+\epsilon.\frac{dx}{ds}.dt;\qquad 0=d.\frac{dz}{dt}+\epsilon.\frac{dz}{ds}.dt-gdt$$

will give, by supposing dx constant,†

[60] $$\epsilon=\frac{ds.d^2t}{dt^3};\qquad 0=\frac{d^2z}{dt}-\frac{dz.d^2t}{dt^2}+\epsilon.\frac{dz}{ds}.dt-gdt;$$

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* (21b) Multiplying the first of the equations [57] by $\frac{dy}{dt}$, the second by $-\frac{dx}{dt}$, and adding these products, we get [58]. If, for brevity, we put $\frac{dx}{dt}=x'$, $\frac{dy}{dt}=y'$, it becomes $y'dx'-x'dy'=0$; dividing by y'^2, we obtain $\frac{y'dx'-x'dy'}{y'^2}=0$, whose integral is $\frac{x'}{y'}=f$, f being the constant quantity required to complete the integral. Replacing the values of x', y', it becomes $\frac{dx}{dy}=f$, or $dx=fdy$, as above. The integral of this equation is $x=fy+f'$, f' being another constant quantity. This represents the equation of a horizontal right line, since the vertical ordinate z does not occur in it, and it corresponds to the first of the equations [19b''], by putting $\mathcal{A}=\frac{1}{f}$, $\mathcal{A}'=-\frac{f'}{f}$. We may consider this line as the vertical projection of the path of the body upon the plane of x, y; and, as this projection is a right line, the body must evidently move in a vertical plane.

† (22) Developing the terms $d.\frac{dx}{dt}$, $d.\frac{dz}{dt}$, in [59], we get [60] by a very small reduction. The value of ϵ [60] substituted in the second equation [60] gives $g\,dt^2=d^2z$ [60'].

whence we deduce

$$g\,dt^2 = d^2 z\,; \qquad [60']$$

taking its differential, we get $2g\,dt\,.\,d^2t = d^3z$; by substituting in it for d^2t its value $\frac{\epsilon\,dt^3}{ds}$, [60], and for dt^2 its value $\frac{d^2z}{g}$, we shall have

$$\frac{\epsilon}{g} = \frac{ds\,.\,d^3z}{2\,.\,(d^2z)^2}. \qquad [61]$$

This equation gives the law of the resistance ϵ, which would be necessary to make the projectile describe any given curve.

If the resistance be proportional to the square of the velocity, ϵ will be proportional to $h\,.\,\frac{ds^2}{dt^2}$, h being constant in case the density of the medium is uniform. We shall then have [60′]

$$\frac{\epsilon}{g} = \frac{h\,.\,ds^2}{g\,dt^2} = \frac{h\,.\,ds^2}{d^2z}\,; \qquad [62]$$

therefore* $h\,.\,ds = \frac{d^3z}{2\,d^2z}$, which gives by integration

$$\frac{d^2z}{dx^2} = 2a\,.\,c^{2hs}, \qquad [63]$$

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* (23) Comparing the two values of $\frac{\epsilon}{g}$, [61, 62], gives $h\,ds = \frac{d^3z}{2d^2z}$, or $2h\,ds = \frac{d^3z}{d^2z}$, whose integral is, (by 59 Int.), $2hs = \log.\,d^2z$, to which must be added a constant quantity, which, for the sake of homogeneity, may be put $-\log.\,2a\,.\,dx^2$, and we shall then have $2hs = \log.\,\frac{ddz}{2a\,.\,dx^2}$. Multiplying the first member by $\log.\,c = 1$, it becomes $2hs\log.\,c$, or $\log.\,c^{2hs}$, hence $\log.\,c^{2hs} = \log.\,\frac{ddz}{2a\,.\,dx^2}$, or $c^{2hs} = \frac{ddz}{2a\,.\,dx^2}$. Multiplying by $2a$, we get [63]. The integral of this equation has been obtained by putting $dz = p\,dx$, whose differential is $d^2z = dp\,.\,dx$, also $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+p^2}$. Substitute these in [63] multiplied by $dx\,.\sqrt{1+p^2} = ds$, it becomes $dp\,.\sqrt{1+p^2} = 2a\,.\,c^{2hs}\,ds$, whose [63a] integral is $\tfrac{1}{2}.\{p\sqrt{1+p^2} + \log.\,(p + \sqrt{1+p^2})\} = \frac{a}{h}\,.\,c^{2hs} + \frac{b}{2h}$, as may be proved by taking the differential and reducing, b being a constant quantity added to complete the integral. Substitute in this the value of $\frac{a}{h}\,.\,c^{2hs}$ deduced from the preceding equation [63a],

a being an arbitrary constant quantity, and c the number whose hyperbolic logarithm is unity. If the resistance of the medium be supposed nothing, or $h=0$; we shall have, by integration, the equation of the parabola*

[64] $$z=ax^2+bx+e;$$

b, e, being arbitrary constant quantities.

The differential equation $d^2z=g\,dt^2$ will give† $dt^2=\frac{2a}{g}.dx^2$, whence we deduce $t=x.\sqrt{\frac{2a}{g}}+f'$. Suppose that x, z, and t, commence together, we shall have $e=0$, $f'=0$, conseqently

[65] $$t=x.\sqrt{\frac{2a}{g}}; \qquad z=ax^2+bx;$$

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which gives successively, $\frac{a}{h}c^{2hs}=\frac{dp.\sqrt{1+p^2}}{2h.ds}=\frac{dp.\sqrt{1+p^2}}{2hdx.\sqrt{1+p^2}}=\frac{dp}{2h.dx}$, we get

$$dx=\frac{dp}{h.\{p\sqrt{1+p^2}+\log.(p.+\sqrt{1+p^2})\}-b},$$

in which the variable quantities are separated, and we may, by the usual methods, obtain its integral, and we shall thus have x in terms of p. This value of dx being substituted in $dz=p\,dx$, $ds=dx.\sqrt{1+p^2}$, will also, by integration, give z and s in terms of p, and by means of the quantity p, the path of the trajectory may be determined.

[64a] * (24) Put $h=0$ in [63] and it becomes $d^2z=2a.dx^2$, or $d.\frac{dz}{dx}=2a.dx$, whose integral is $\frac{dz}{dx}=2ax+b$, whence $dz=2ax\,dx+b\,dx$. Again integrating we obtain [64]. If we alter the origin of the co-ordinates, putting $z=z'+e-\frac{1}{4}.\frac{bb}{a}$, and $x=x'-\frac{b}{2a}$, the equation [64] becomes $z'+e-\frac{1}{4}.\frac{bb}{a}=a\left(x'-\frac{b}{2a}\right)^2+b\left(x'-\frac{b}{2a}\right)+e$,
[64b] which, by developing and reducing, becomes $z'=ax'^2$, the well known equation of a parabola.

† (25) Putting the values of d^2z [60′, 64a] equal to each other, we get $g\,dt^2=2a.dx^2$, or $dt=dx.\sqrt{\frac{2a}{g}}$, whose integral is $t=x.\sqrt{\frac{2a}{g}}+f'$, or, as in [65],

$$t=x.\sqrt{\frac{2a}{g}}, \text{ whence } x=t.\sqrt{\frac{g}{2a}}.$$

This value of x, substituted in z [65], gives [66].

whence

$$z=\frac{gt^2}{2}+bt\,.\sqrt{\frac{g}{2a}}.$$ [66]

These three equations contain the whole theory of projectiles in a vacuum; and it follows from them that the velocity in a horizontal direction is uniform, and the vertical velocity is the same as would be acquired by the body falling down the vertical.* [66′]

If the body fall from a state of rest, b will vanish, and we shall have

$$\frac{dz}{dt}=gt; \qquad z=\tfrac{1}{2}gt^2;$$ [67]

Motion of bodies falling from rest by Gravity.

the acquired velocity therefore increases in proportion to the time, and the space increases as the square of the time.

Centrifugal Force compared with Gravity.

It is easy, by means of these formulas, to compare the centrifugal force with that of gravity. We have shown before, [54′], that v being the velocity of a body moving in the circumference of a circle whose radius is r, the centrifugal force will be $\frac{v^2}{r}$. Let h be the height from which the body ought to fall to acquire the velocity v; we shall have, by what [67′]

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* (25a) The horizontal velocity is evidently denoted by $\frac{dx}{dt}$; and the differential of the first of the equations [65] gives $\frac{dx}{dt}=\sqrt{\frac{g}{2a}}$, which is constant because g, $2a$ are given quantities. The vertical velocity is evidently $\frac{dz}{dt}$, which we shall call v; and the differential of [66], gives $\frac{dz}{dt}(=v)=gt+b\,.\sqrt{\frac{g}{2a}}$. If the projected velocity, resolved in a vertical direction, be v', the preceding equation, when $t=0$, will become $v'=b\sqrt{\frac{g}{2a}}$, consequently $v=gt+v'$; and if this projected velocity v' be given, the vertical velocity v will be the same at the end of a given time t, whether the body be projected obliquely or vertically, supposing the vertical velocity of projection v' to be the same in both cases. If $v'=0$, the preceding value of $v'=b\,.\sqrt{\frac{g}{2a}}$, gives $b=0$, a, g, being finite. This value of b, being substituted in [66, 67a], gives [67.] [67a] [67b]

precedes,* $v^2 = 2gh$, hence $\frac{v^2}{r} = g \cdot \frac{2h}{r}$. If $h = \frac{1}{2} r$, the centrifugal force becomes equal to gravity g; therefore if a heavy body be attached to one end of a thread, and the other end be fixed to a horizontal plane, the tension of the thread will be the same as if it were suspended vertically, provided that the body be made to move upon the plane, with the velocity it would [67''] have acquired in falling from a height equal to half the length of the thread.

Motion of a body upon a spherical surface.

11. Let us now consider the motion of a heavy body upon a spherical surface. Putting r for the radius of the spherical surface, and fixing the origin of the co-ordinates x, y, z, at its centre, we shall have [19*e*]

[67''']
$$r^2 - x^2 - y^2 - z^2 = 0;$$

this equation, compared with $u = 0$, gives $u = r^2 - x^2 - y^2 - z^2$; adding therefore to the equation (*f*) § 7 [37] the function δu, multiplied by the indeterminate quantity† $-\lambda\, dt$, we shall have

[68]
$$0 = \delta x \cdot \left\{ d \cdot \frac{dx}{dt} + 2\lambda x . dt \right\} + \delta y \cdot \left\{ d \cdot \frac{dy}{dt} + 2\lambda y . dt \right\} + \delta z \cdot \left\{ d \cdot \frac{dz}{dt} + 2\lambda z . dt - g dt \right\},$$

in which equation we may put the coefficients of δx, δy, δz, separately equal to nothing [26'], which will give the three following equations:

[69]
$$\left.\begin{aligned} 0 &= d \cdot \frac{dx}{dt} + 2\lambda x . dt; \\ 0 &= d \cdot \frac{dy}{dt} + 2\lambda y . dt; \\ 0 &= d \cdot \frac{dz}{dt} + 2\lambda z . dt - g dt. \end{aligned}\right\} \qquad (A)$$

The indeterminate quantity λ shows the pressure which the moving body exerts against the surface. This pressure is by § 9 [50], equal to

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* (25*b*) When $v' = 0$, we shall have $v = gt$ [67*b*], whence $t = \frac{v}{g}$, which, being [67*c*] substituted in z [67], gives $z = \frac{1}{2} \cdot \frac{v^2}{g}$, or $2gz = v^2$, and by changing z into h, it becomes as above.

† (25*c*) In the same manner as in note (19*b*); observing also that $P = 0$, $Q = 0$, and $R = g$.

$$\lambda . \sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}\,;$$

consequently it is equal to $2\lambda r$.[*] Now by § 8 we have[†]

$$c + 2gz = \frac{dx^2 + dy^2 + dz^2}{dt^2}, \qquad [70]$$

c being an arbitrary constant quantity; by adding this equation, to the equations (A) divided by dt, and multiplied respectively by x, y, z; then observing that the differential equation of the surface, is $0 = xdx + ydy + zdz$, we shall have[‡]

$$0 = x\,ddx + y\,ddy + z\,ddz + dx^2 + dy^2 + dz^2\,; \qquad [71]$$

we shall find[§]

$$2\lambda r = \frac{c + 3gz}{r}. \qquad [72]$$

If we multiply the first of the equations (A) [69] by $-y$, and the

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[*] (26) The pressure is $\lambda . \sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}$, [50], and by [25a], the terms under the radical are equal to $2r$, therefore the pressure is $\lambda . 2r$, or $2 . \lambda r$, as above.

[†] (27) This equation is the same as [39], putting $P = 0$, $Q = 0$, $R = g$, whence $2 . \int R\,dz = 2gz$.

[‡] (27a) The differential of the equations of the surface $0 = x^2 + y^2 + z^2 - r^2$ [19e] gives $0 = xdx + ydy + zdz$, and the differential of this is as in [71]; dividing by dt^2, it becomes $0 = x . \frac{ddx}{dt^2} + y . \frac{ddy}{dt^2} + z . \frac{ddz}{dt^2} + \frac{dx^2 + dy^2 + dz^2}{dt^2}$, which is used in the next note.

[§] (27b) The sum, found by adding [70] to the equations [69] multiplied respectively by $\frac{x}{dt}$, $\frac{y}{dt}$, $\frac{z}{dt}$, is

$$c + 2gz = x . \frac{ddx}{dt^2} + y . \frac{ddy}{dt^2} + z . \frac{ddz}{dt^2} + 2\lambda . (x^2 + y^2 + z^2) - gz + \frac{dx^2 + dy^2 + dz^2}{dt^2}.$$

From which, subtracting the equation at the end of the last note, it becomes

$$c + 2gz = 2\lambda\,(x^2 + y^2 + z^2) - gz, \qquad [72a]$$

or $c + 3gz = 2\lambda r^2$, because $x^2 + y^2 + z^2 = r^2$, [67‴]. This, divided by r, gives [72.]

second by x, we shall have, by adding these products, and integrating their sum,*

[73] $$\frac{x\,d\,y - y\,d\,x}{d\,t} = c'\,;$$

c' being another arbitrary constant quantity.

The motion of the point is therefore reduced to these three differential equations of the first order,

[74] $$x\,d\,x + y\,d\,y = -z\,d\,z\,;$$
$$x\,d\,y - y\,d\,x = c'\,d\,t\,;$$
$$\frac{d\,x^2 + d\,y^2 + d\,z^2}{d\,t^2} = c + 2\,g\,z.$$

Squaring each of the two first equations, and adding them together, we get

[75] $$(x^2 + y^2)\,.\,(d\,x^2 + d\,y^2) = c'^2\,d\,t^2 + z^2\,d\,z^2\,;$$

if we substitute, instead of $x^2 + y^2$, its value $r^2 - z^2$ [67‴], and, instead of $\frac{d\,x^2 + d\,y^2}{d\,t^2}$, its value $c + 2\,g\,z - \frac{d\,z^2}{d\,t^2}$, [70], we shall have, by supposing that the body recedes from the vertical,†

[76] $$d\,t = \frac{-r\,d\,z}{\sqrt{(r^2 - z^2)\,.\,(c + 2\,g\,z) - c'^2}}$$

the function under the radical, may be put under the form

[76′] $$(a - z)\,.\,(z - b)\,.\,(2\,g\,z + f),$$

a, b, f, being determined by the equations‡

[77] $$f = \frac{2\,g\,.\,(r^2 + a\,b)}{a + b}\,;$$
$$c = \frac{2\,g\,.\,(r^2 - a^2 - a\,b - b^2)}{a + b}\,;$$
$$c'^2 = \frac{2\,g\,.\,(r^2 - a^2)\,.\,(r^2 - b^2)}{a + b}.$$

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* (27c) This sum is $0 = x\,.\,d\,.\,\frac{dy}{dt} - y\,.\,d\,.\,\frac{dx}{dt}$, or $0 = \frac{x\,d\,d\,y - y\,d\,d\,x}{d\,t}$, whose integral is $c' = \frac{x\,dy - y\,dx}{dt}$, as in [73].

† (27d) This requires that z should *decrease* while t *increases*, consequently, $d\,t$ being supposed positive, $d\,z$ must be supposed negative, as in [76].

‡ (28) The expression [76] gives the velocity in the direction of the vertical z, or

We may therefore substitute for the constant quantities c and c', other constant quantities a and b, of which the first is the greatest value of z, and [77'] the second is its least value.* Then putting†

$$\sin.\theta = \sqrt{\frac{a-z}{a-b}}, \qquad [78]$$

$\frac{dz}{dt} = -\frac{\sqrt{(r^2-z^2).(c+2gz)-c'^2}}{r}$. Now this velocity must evidently be nothing at the *highest* and *lowest* points of the curve, at which points we shall suppose the values of z to be b and a respectively: therefore $z=b$, and $z=a$, ought to make $\frac{dz}{dt}$ equal to nothing, consequently these values ought to make the numerator of that value of $\frac{dz}{dt}$ equal to nothing; so that a, b, must be roots of the equation $(r^2-z^2).(c+2gz)-c'^2=0$; therefore, $a-z=0$, and $z-b=0$, must be factors of that equation, and by division we shall find, that the other factor must be of the form $2gz+f=0$, so that

$$(r^2-z^2).(c+2gz)-c'^2=(a-z).(z-b).(2gz+f).$$

Developing both sides of this equation, and putting the coefficients of z^2, z and the constant terms equal to each other, we obtain the three following equations: $-c=2g.(a+b)-f$; $2r^2g=f.(a+b)-2gab$; $r^2c-c'^2=-fab$. The value of f, deduced from the second of these, gives the first of the equations [77]. Substituting this in the first of the preceding equations, and changing the signs of all the terms we obtain

$$c=2g.\left\{-(a+b)+\frac{r^2+ab}{a+b}\right\};$$

which, by reducing to the common denominator $a+b$, gives the second of the equations [77]. Lastly, by substituting the values of c, f, in the last of the preceding equations

$$r^2c-c'^2=-fab, \qquad \text{or} \qquad c'^2=r^2c+fab,$$

it becomes $c'^2=\frac{2g}{a+b}.\left\{(r^2-a^2-ab-b^2).r^2+(r^2+ab).ab\right\}$, which, by a small reduction, is easily put under the same form as the third equation [77].

* (28a) This is proved in the preceding note.

† (29) This expression squared and reduced, gives

$$z=a-(a-b)\sin.^2\theta=a(1-\sin.^2\theta)+b\sin.^2\theta=a.\cos.^2\theta+b\sin.^2\theta \qquad [78a]$$

as in [80]. The differential of the first value of z gives $-dz=2(a-b).\sin.\theta.\cos.\theta.d\theta$. The same equation also gives

$$a-z=(a-b).\sin.^2\theta; \qquad z-b=(a-b).(1-\sin.^2\theta)=(a-b).\cos.^2\theta;$$

$$2gz+f=2g.\{a-(a-b)\sin.^2\theta\}+f.$$

the preceding differential will become

[79] $$d\,t = \frac{r\,.\sqrt{2\,.(a+b)}}{\sqrt{g\,.\,\{(a+b)^2+r^2-b^2\}}}\,.\,\frac{d\,\theta}{\sqrt{1-\gamma^2\,\text{sin.}\,\theta^2}},$$

by putting

[79′] $$\gamma^2 = \frac{a^2-b^2}{(a+b)^2+r^2-b^2}.$$

The angle θ gives the ordinate z by means of the equation [78*a*],

[80] $$z = a\,.\,\text{cos.}^2\,\theta + b\,\text{sin.}^2\,\theta,$$

and this ordinate z, divided by r, gives the cosine of the angle which the radius r makes with the vertical. Let ϖ be the angle which the vertical plane passing through the radius r, makes with the vertical plane passing through the axis of x; we shall have*

[81] $$x = \sqrt{r^2-z^2}\,.\,\text{cos.}\,\varpi\,; \qquad y = \sqrt{r^2-z^2}\,.\,\text{sin.}\,\varpi\,;$$

from which we get†

[82] $$x\,d\,y - y\,d\,x = (r^2-z^2)\,.\,d\,\varpi\,;$$

Substituting in this last, the value of f [77] and reducing, it becomes

$$2\,g\,z+f = 2\,g\,.\,\frac{(a+b)^2+r^2-b^2-(a^2-b^2).\,\text{sin.}^2\,\theta}{a+b} = 2\,g\,.\,\frac{(a+b)^2+r^2-b^2}{a+b}\,.\,(1-\gamma^2\,\text{sin.}^2\,\theta).$$

Hence the factor of $-r\,d\,z$ [76] which was assumed, in [76′], equal to

$$\frac{1}{\sqrt{(a-z)\,.\,(z-b)\,.\,(2\,g\,z+f)}},$$

becomes

$$\frac{\sqrt{2\,.(a+b)}}{2\,.\,(a-b)\,.\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta\,.\,\sqrt{g.\{(a+b)^2+r^2-b^2\}}\,.\,\sqrt{1-\gamma^2\,\text{sin.}^2\,\theta}},$$

this multiplied by the preceding value of $-d\,z$ and by r gives $d\,t$ [79].

* (29*a*) In the figure page 7, let A be the centre of the sphere and c a point of its circumference, whose co-ordinates are $A\,D = x$, $D\,d = y$, $d\,c = z$, we shall evidently have in the rectangular triangle $A\,d\,c$, $A\,d^2 = A\,c^2 - d\,c^2$, or $A\,d = \sqrt{r^2-z^2}$; this is the quantity called ρ in [27], and by substituting this for ρ it gives [81], observing that in this case ϖ = angle $D\,A\,d$.

† (30) This expression is found most easily by dividing the value of y [81] by x, which gives $\frac{y}{x} = \text{tang.}\,\varpi$, whose differential is $\frac{x\,d\,y-y\,d\,x}{x^2} = \frac{d\,\varpi}{\text{cos.}^2\,\varpi}$. This multiplied by the square of x [81], that is, by $x^2 = (r^2-z^2)\,.\,\text{cos.}^2\,\varpi$, produces the expression [82].

the equation $x\,dy - y\,dx = c'\,dt$ [74], will therefore give

$$d\varpi = \frac{c'\,dt}{r^2 - z^2}\,; \qquad [83]$$

by substituting for z and dt, their preceding values in θ, we shall have the angle ϖ in a function of θ; and we may thence obtain, at any time, the two angles θ, ϖ, which are sufficient to determine the position of the moving body.*

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* (30a) These angles θ and ϖ may be obtained, very easily, by means of the Tables of elliptical integrals, computed by Le Gendre, and published in the third volume of his *Exercices de calcul intégral,* in which he uses the following abridged symbols.

$$b = \sqrt{1 - c^2}\,; \qquad \Delta(c, \varphi) = \sqrt{1 - c^2 . \sin.^2 \varphi}\,;$$

$$\Delta(b, \varphi) = \sqrt{1 - b^2 . \sin.^2 \varphi} \qquad \mathrm{F}(c, \varphi) = \int \frac{d\varphi}{\Delta(c, \varphi)}\,; \qquad [82a]$$

$$\mathrm{E}(c, \varphi) = \int d\varphi . \Delta(c, \varphi)\,; \qquad \Pi(n, c, \varphi) = \int \frac{d\varphi . \Delta(c, \varphi)}{1 + n . \sin.^2 \varphi}.$$

The functions F, E, Π, being called by him elliptical functions of the *first, second,* and *third* species, and when these integrals are taken between the limits $\varphi = 0$ and $\varphi = 90^d$, they are denoted by $\mathrm{F}'(c)$, $\mathrm{E}'(c)$, $\Pi'(n, c)$ respectively. The values of the functions F, E, have [82b] been computed for each degree of the arch φ, from 0 to 90^d, and for the various values of the modulus c, from 0 to 1. By means of these tables all calculations of these integrals are much facilitated.

Putting for brevity $\dfrac{r . \sqrt{2 . (a + b)}}{\sqrt{g} . \{(a+b)^2 + r^2 - b^2\}} = h$, changing also θ into φ, and γ into c, to conform to Le Gendre's notation; the formula [79] will become, $dt = h \dfrac{d\varphi}{\Delta(c, \varphi)}$, whose integral is

$$t = h . \mathrm{F}(c . \varphi), \qquad [82c]$$

therefore, when t is given, we may obtain φ, (or θ of La Place), by means of the tables of the integrals of the first species.

Substituting this value of dt in [83], and observing that $\dfrac{1}{r^2 - z^2}$ may be put under the form $\dfrac{1}{2r} . \left\{ \dfrac{1}{r+z} + \dfrac{1}{r-z} \right\}$, we get, $d\varpi = \dfrac{c'}{2r} \left\{ \dfrac{1}{r+z} + \dfrac{1}{r-z} \right\} . \dfrac{h\,d\varphi}{\Delta(c, \varphi)}$; now $z = a - (a - b) . \sin.^2 \varphi$ [78a], hence

$$d\varpi = \frac{c'}{2r} . \left\{ \frac{1}{r + a - (a - b) . \sin.^2 \varphi} + \frac{1}{r - a + (a - b) . \sin.^2 \varphi} \right\} . \frac{h\,d\varphi}{\Delta(c, \varphi)}$$

Time of oscillation in a conical surface.

We shall call a *half-oscillation* of the body the time it takes to pass from the greatest to the least value of z; which time we shall put equal to $\frac{1}{2}T$. To compute it, we must integrate the preceding value of dt from $\theta=0$ to

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and if we put $n=-\frac{(a-b)}{r+a}$, $n'=\frac{a-b}{r-a}$, $m=\frac{c'h}{2r.(r+a)}$, $m'=\frac{c'h}{2r.(r-a)}$, it becomes

$$d\varpi=m.\frac{d\varphi}{(1+n.\sin.^2\varphi).\Delta(c,\varphi)}+m'.\frac{d\varphi}{(1+n'.\sin.^2\varphi).\Delta(c,\varphi)};$$

whose integral is

[82d] $$\varpi=m.\Pi(n,c,\varphi)+m'.\Pi(n',c,\varphi).$$

Therefore ϖ may be found by means of *two* integrals of the third species.

In a semi-vibration of the pendulum φ varies from 0 to 90^d [83a], and if we suppose ϖ' to be the value of ϖ, corresponding to $\varphi=90^d$, the integral commencing with $\varphi=0$ we shall have

[82e] $$\varpi'=m.\Pi'(n,c)+m'.\Pi'(n',c).$$

These definite integrals of the *third* species may be reduced to those of the *first* and *second* species, by the formulas in pages 137, 141, Vol. I, of Le Gendre's work.

Thus, by putting $n=-1+b^2.\sin.\theta'^2$, we have by formula (m') page 141,

[82f] $$\Pi'(n,c)=F'(c)+\frac{\Delta(b,\theta')}{b^2.\sin.\theta'.\cos.\theta'}.\left\{\frac{\pi}{2}+F'(c).F(b,\theta')-E'(c).F(b\theta')-F'(c).F(b,\theta')\right\},$$

and by putting $n'=\cot.^2\theta$ in formula (k') page 137 of the same work, we shall have

[82g] $$\Pi(n',c)=\frac{\sin.\theta.\cos.\theta}{\Delta(b,\theta)}.\left\{\frac{\pi}{2}+\frac{\sin.\theta}{\cos.\theta}.\Delta(b,\theta).F'(c)+F'(c).F(b,\theta)-F'(c).E(b,\theta)-E'(c).F(b,\theta)\right\}.$$

These, being substituted in ϖ', give its value in functions of the first and second species, which may be found from the Tables.

John Bernoulli, in Vol. III, page 171 of his works, remarks that the motion of a pendulum of this kind, in which a and b differ but little from each other, and neither of them vary much from the whole length of the thread, by which the body is suspended, may be made to represent, beautifully, the progressive motion of the moon's apsides. For the projection of the path of the body upon the horizontal plane will be nearly an ellipsis; and in the time $2T$ of a double oscillation, the longer axis of this ellipsis will have passed over an arch, which is equal to four times the excess of the angle ϖ' above a right angle; so that in every successive vibration of the pendulum at the arrival of the body at its highest point, corresponding to the extremities of the longer axis, this axis will have moved about the centre of the ellipsis, in a manner wholly similar to the progressive motion, observed in the moon's apsides, in the successive revolutions of that body about the earth.

$\theta = \frac{1}{2}\pi$,* π being the semi-circumference of a circle whose radius is unity, we shall by this means find‡

$$T = \pi . \sqrt{\frac{r}{g}} . \sqrt{\frac{2r.(a+b)}{(a+b)^2+r^2-b^2}} . \left\{ 1 + \left(\frac{1}{2}\right)^2 . \gamma^2 + \left(\frac{1.3}{2.4}\right)^2 . \gamma^4 . + \left(\frac{1.3.5}{2.4.6}\right)^2 . \gamma^6 + \&c. \right\} . \qquad [84]$$

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* (31) These limits are easily deduced from [78], by substituting in it the greatest and least values of z, which are a, b, [77']; for the values of $\sin. \theta$ [78] corresponding are $\sqrt{\frac{a-a}{a-b}} = 0$, and $\sqrt{\frac{a-b}{a-b}} = 1$, which give, at the limits, $\theta = 0$, and $\theta = \frac{1}{2}\pi$. [83a]

† (32) By developing the radical $\frac{1}{\sqrt{1-\gamma^2 . \sin.^2 \theta}}$, [79], by means of the binomial theorem, we shall have

$$dt = \frac{r\sqrt{2.(a+b)}}{\sqrt{g} . \sqrt{(a+b)^2+r^2-b^2}} . \left\{ d\theta + \frac{1}{2} . \gamma^2 . d\theta . \sin.^2 \theta + \frac{1.3}{2.4} \gamma^4 . d\theta . \sin.^4 \theta + \frac{1.3.5}{2.4.6} . \gamma^6 . d\theta . \sin.^6 \theta + \&c. \right\} .$$

The integral of this, taken between the limits $\theta = 0$, $\theta = \frac{1}{2}\pi$, gives $\frac{1}{2} T$. This integral may be found by substituting the values of $\sin.^2 \theta$, $\sin.^4 \theta$, &c. (Int. Form. 1, 2, &c.) or by the following formula, Integral Formulas.

$$\int d\theta . \sin.^n \theta = -\frac{1}{n} . \cos. \theta . \sin.^{n-1} \theta + \frac{n-1}{n} . \int d\theta . \sin.^{n-2} \theta ; \qquad [84a]$$

which is easily proved to be correct by taking the differential of the whole, and reducing by means of $\cos.^2 \theta = 1 - \sin.^2 \theta$, n being any number whatever. Now at the limits when $\theta = 0$, or $\theta = \frac{1}{2}\pi$, the term without the sign $\int$ generally becomes nothing, and if we take the integrals with those limits we shall have, n being an integer greater than 1,

$$\int d\theta . \sin.^n \theta = \frac{n-1}{n} \int d\theta . \sin.^{n-2} \theta ; \qquad [84b]$$

and as $\int d\theta = \frac{\pi}{2}$, we shall have, when $n = 2$,

$$\int d\theta . \sin.^2 \theta = \tfrac{1}{2} \int d\theta = \frac{1}{2} . \frac{\pi}{2} \qquad [84c]$$

If $n = 4$, the preceding formula gives $\int d\theta . \sin.^4 \theta = \frac{3}{4} . \int d\theta . \sin.^2 \theta$; substituting the value of $\int d\theta . \sin.^2 \theta$, it becomes

$$\int d\theta . \sin.^4 \theta = \frac{1.3}{2.4} . \frac{\pi}{2} . \qquad [84d]$$

Putting $n = 6$, the formula gives $\int d\theta . \sin.^6 \theta = \frac{5}{6} . \int d\theta . \sin.^4 \theta$, and, by using the value of $\int d\theta . \sin.^4 \theta$, it becomes

$$\int d\theta . \sin.^6 \theta = \frac{1.3.5}{2.4.6} . \frac{\pi}{2}, \ \&c. \qquad [84e]$$

Suppose the point to be suspended from the extremity of a thread without mass, the other extremity being fixed. If the length of the thread is r, the point will be moved exactly as if it was upon the interior of a spherical surface; Pendulum. and it will in this manner form a pendulum, in which the greatest deviation from the vertical will be measured by an angle whose cosine is $\frac{b}{r}$. If we suppose that when it is in this situation, the velocity of the point is nothing; it will oscillate in a vertical plane, and we shall have, in this case, $a=r$, $\gamma^2=\frac{r-b}{2r}$.* The fraction $\frac{r-b}{2r}$ is the square of the sine of half the greatest angle which the pendulum makes with the vertical; the whole duration T of the oscillation of the pendulum will therefore be

[85] $$T=\pi.\sqrt{\frac{r}{g}}.\left\{1+\left(\frac{1}{2}\right)^2.\left(\frac{r-b}{2r}\right)+\left(\frac{1.3}{2.4}\right)^2.\left(\frac{r-b}{2r}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2.\left(\frac{r-b}{2r}\right)^3+\&c.\right\}.$$

Time of oscillation of a simple pendulum, whose length is r, Gravity being g.

If the oscillation is very small, $\frac{r-b}{2r}$ will be a very small fraction, which may be neglected, and then we shall have

[86] $$T=\pi.\sqrt{\frac{r}{g}};$$

The law of continuation being manifest. Substituting these in the preceding integral of dt representing $\frac{1}{2}T$, it becomes

$$\tfrac{1}{2}T=\frac{r\sqrt{2.(a+b)}}{\sqrt{g}.\sqrt{(a+b)^2+r^2-b^2}}.\left\{\frac{\pi}{2}+\left(\frac{1}{2}\right)^2.\gamma^2.\frac{\pi}{2}+\left(\frac{1.3}{2.4}\right)^2.\gamma^4.\frac{\pi}{2}+\left(\frac{1.3.5}{2.4.6}\right)^2.\gamma^6.\frac{\pi}{2}+\&c.\right\},$$

whence we can easily deduce the value of T [84].

* (33) The value $a=r$ being substituted in the expression of γ^2 [79′], it becomes $\frac{r^2-b^2}{(r+b)^2+r^2-b^2}$, or by developing, $\frac{r^2-b^2}{2r^2+2rb}=\frac{r^2-b^2}{2r.(r+b)}=\frac{r-b}{2r}$, as above. Now the cosine of the greatest angle which the pendulum makes with the vertical being $\frac{b}{r}$, its versed sine is $\frac{r-b}{r}$, and (by 1 Int.) the square of the sine of half this angle is equal to the half of this versed sine or $\frac{r-b}{2r}$. Substituting these values of a and γ in [84] we obtain [85], observing that the factor $\sqrt{\frac{2r.(a+b)}{(a+b)^2+r^2-b^2}}$ becomes $\sqrt{\frac{2rr+2rb}{2rr+2rb}}$, which is equal to 1.

very small oscillations are therefore isochronal, or of the same duration, whatever be the length of the described arch; and we may, by means of this duration, and of the corresponding length of the pendulum, determine the variations of the intensity of gravity, in different places of the earth.

Let z be the height through which a body would fall by the force of gravity in the time T; we shall have, by § 10 [67],* $2z = gT^2$, consequently $z = \frac{1}{2}\pi^2 . r$; we shall therefore have with great precision, by means [86'] of the length of a pendulum vibrating in a second, the space through which gravity would cause a body to fall in the first second of its descent. It has been found, by very exact experiments, that the length of such a pendulum, vibrating in a second, is the same, whatever be the substance of the oscillating body; hence it follows that gravity acts equally upon all bodies, and that in the same place, it tends to impress upon them the same velocity, in the same time.

12. The oscillations of a pendulum not being perfectly isochronal, it is interesting to know the curve upon which a heavy body ought to move, to arrive in the same time at the point where its motion ceases, whatever may be the length of the arch which it shall describe from its lowest point. To solve this problem in the most general manner, we shall suppose, conformably to what really takes place in nature, that the body moves in a resisting medium. Let s be the arch described from the lowest point of the curve; z the vertical absciss counted from that point; dt the element of the time, and g the force of gravity. The retarding forces along the arch of the curve will be; *First*, gravity resolved in the direction of the arch ds, and which is therefore equal to† $g . \frac{dz}{ds}$; *Second*, the resistance of the medium, which we [86'']

Tauto-chronous Curve.

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* (34) This is deduced from [67], by writing T for t. Substituting the value of T [86] we obtain $z = \frac{1}{2}\pi^2 r$.

† (34*a*) The cosine of the angle formed by the elements ds, dz, [40*b*] is

$$\frac{dz}{ds} = \text{cos. } cAB, \text{ (Fig. page 7.)}$$

and the force of gravity g, acting in the vertical dz, is resolved in a direction along the arc ds, by multiplying it by this cosine, hence it becomes $g . \frac{dz}{ds}$ [86''].

shall denote by $\varphi\left(\frac{ds}{dt}\right)$, $\frac{ds}{dt}$ being the velocity of the body, and $\varphi\left(\frac{ds}{dt}\right)$, any function whatever of this velocity. The differential of this velocity* will be by § 7 equal to $-g\,.\frac{dz}{ds}.dt-\varphi\left(\frac{ds}{dt}\right).dt$; we shall therefore have, supposing dt constant,

[87] $$0=\frac{dds}{dt^2}+g\,.\,\frac{dz}{ds}+\varphi\left(\frac{ds}{dt}\right). \qquad (i)$$

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* (35) The forces of gravity and resistance, computed as above, give the whole force P', acting in the direction of the *element* ds of the curve, hence $P'=-g\,.\frac{dz}{ds}-\varphi\left(\frac{ds}{dt}\right)$; the *negative* sign being given to the terms, because the forces tend to *decrease* that element. Now from the formula [38] we have $\frac{ddx}{dt^2}=P$, or $d\,.\frac{dx}{dt}=P\,dt$, in which $\frac{dx}{dt}$ is the velocity in the direction of the element dx, and P is the force in that direction. This equation would take place, whatever be the direction of the arbitrary axis x, and we may assume generally, that the increment of velocity in any direction is equal to the force in that direction, multiplied by the element of the time; and as $\frac{ds}{dt}$ is the velocity in the direction ds, we shall have $d\,.\frac{ds}{dt}=P'\,dt$, or $0=\frac{dds}{dt^2}-P'$, which, by substituting the preceding value of P', becomes as in [87].

(35*a*) We may apply formula [87] to the computation of the velocity of a body projected directly upwards, along an inclined plane, which will be wanted in note 39. In this case the curve becomes a right line inclined to the horizon by a given angle I, and we shall evidently have $\frac{dz}{ds}=\sin.\,I$, so that if we, for brevity, put $g'=g\,.\sin.\,I$, we shall get

$$g\,.\frac{dz}{ds}=g\,.\sin.\,I=g', \qquad \text{therefore} \qquad 0=\frac{dds}{dt^2}+g'+\varphi\left(\frac{ds}{dt}\right),\ [87].$$

If we suppose the resistance to be nothing, it will become $0=\frac{dds}{dt^2}+g'$, or $\frac{dds}{dt}=-g'\,dt$, whose integral is $\frac{ds}{dt}=v'-g'\,t$; v' being the constant quantity added to complete the integral, and as $\frac{ds}{dt}$ represents the velocity v of the body, we shall have $v=v'-g'\,t$, hence it is evident that v' will be the initial velocity of the body when $t=0$. If it be required to find the time T, in which the whole motion v will cease, we must put, in the preceding equation, $t=T$, and $v=0$, and we shall have, $0=v'-g'\,T$, hence $T=\frac{v'}{g'}$. Consequently
[87*a*] the time T, in which the whole motion would be destroyed, is directly proportional to the initial velocity v'.

Suppose

$$\varphi\left(\frac{ds}{dt}\right) = m \,.\, \frac{ds}{dt} + n \,.\, \frac{ds^2}{dt^2} ;^* \qquad s = \psi(s') ;$$ [87′]

put also $\psi'(s')$ for the differential of $\psi(s')$ divided by ds', and $\psi''(s')$ for the differential of $\psi'(s')$ divided by ds'; then we shall have

$$\frac{ds}{dt} = \frac{ds'}{dt} \,.\, \psi'(s') ;$$
$$\frac{dds}{dt^2} = \frac{dds'}{dt^2} \,.\, \psi'(s') + \frac{ds'^2}{dt^2} \,.\, \psi''(s') ;$$ [88]

and the equations (i) [87] will become†

$$0 = \frac{dds'}{dt^2} + m \,.\, \frac{ds'}{dt} + \frac{ds'^2}{dt^2} \,.\, \left\{ \frac{\psi''(s') + n \,.\, \{\psi'(s')\}^2}{\psi'(s')} \right\} + \frac{g \,.\, dz}{ds' \,.\, \{\psi'(s')\}^2} ; \qquad (l)$$ [89]

the term multiplied by $\frac{ds'^2}{dt^2}$ may be made to disappear by means of the following equation

$$0 = \psi''(s') + n \,.\, [\psi'(s')]^2 ;$$ [90]

from which by integration we obtain‡

$$\psi(s') = \log. \left\{ h \,.\, (s' + q)^{\frac{1}{n}} \right\} = s,$$ [91]

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* (36) The solution is given for a much more general form of resistance in [103].

† (37) Take the first and second differentials of the assumed value $s = \psi(s')$ [87′], and divide them respectively by dt, dt^2, we shall obtain $\frac{ds}{dt}$, $\frac{dds}{dt^2}$, [88]. The assumed form of $\varphi \frac{ds}{dt}$ [87′] changes the equation [87] into $0 = \frac{dds}{dt^2} + g \,.\, \frac{dz}{ds} + m \,.\, \frac{ds}{dt} + n \,.\, \frac{ds^2}{dt^2}$. Substitute in this the values of $\frac{ds}{dt}$, $\frac{dds}{dt^2}$ [88], and divide by $\psi'(s')$, we shall obtain the formula [89].

‡ (38) Transposing the second term of the assumed formula [90], and multiplying by $\frac{ds'}{\psi'(s')}$ it becomes $\frac{\psi''(s') \,.\, ds'}{\psi'(s')} = - n \,.\, \psi'(s') \,.\, ds'$, whose integral is (59 Int.)

$$\log. \psi'(s') = - n \,.\, \psi(s') + \text{constant}.$$

Multiplying the first term of the second member by $\log. c = 1$, and putting the constant term equal to $\log. \frac{h^n}{n}$ we shall get

$$\psi'(s') = \frac{h^n}{n} \,.\, c^{-n\psi(s')}$$ [87b]

h and q being arbitrary constant quantities. If we suppose s and s' to
[91′] commence together, we shall have $h\,q^{\frac{1}{n}}=1$; and if, for greater simplicity, we put $h=1$, we shall have

[92] $$s'=c^{ns}-1,$$

c being the number whose hyperbolic logarithm is unity: the differential equation (l) [89] then becomes

[93] $$0=\frac{dds'}{dt^2}+m\,.\frac{ds'}{dt}+n^2g\,.\frac{dz}{ds'}\,.\,(1+s')^2.$$

Supposing s' to be very small, we can develop the last term of this equation, in a series ascending according to the powers of s', which will be
[93′] of this form* $k\,s'+l\,s'^i+$ &c.; i being greater than unity; and then this

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Multiply this by $n\,c^{n\psi(s')}\,ds'$, and it becomes $c^{n\psi(s')}\,.\,n\psi'(s')\,.\,ds'=h^n ds'$, whose integral is $c^{n\psi(s')}=h^n(s'+q)$; $h^n q$ being the constant quantity, added to complete the integral. Extracting the root n, we obtain $c^{\psi(s')}=h\,(s'+q)^{\frac{1}{n}}$, whose logarithm gives $\psi\,(s')$ [91]. If for $\psi\,(s')$ we substitute its assumed value s [87′], the preceding expression will become $c^s=h\,(s'+q)^{\frac{1}{n}}$. Now by hypothesis [91′], $s'=0$ when $s=0$; these values being substituted in this equation, we get $1=h\,q^{\frac{1}{n}}$, and if we suppose $h=1$, it will make $q=1$, therefore the preceding equation will be $c^s=(s'+1)^{\frac{1}{n}}$ or $c^{ns}=s'+1$, whence $s'=c^{ns}-1$, [92]. The same value of $h=1$ makes [87b] become $\psi'(s')=\frac{1}{n}\,.\,c^{-n\psi(s')}$, and by substituting in the second member for $\psi\,(s')$ its value s [91], it becomes

[87c] $$\psi'(s')=\frac{1}{n}\,.\,c^{-ns}=\frac{1}{n\,c^{ns}}=\frac{1}{n\,.\,(1+s')}\quad[92],$$

and this, being substituted in the last term of [89], changes it into [93], neglecting the coefficient of $\frac{ds^2}{dt^2}$; which vanishes by reason of the assumed equation [90].

* (39) The slightest attention will make it evident that the tautochronous curve, whose horizontal ordinate is y, and vertical ordinate z, both commencing at the lowest point of the curve, as their origin, must, at that point, be parallel to the horizon; or in symbols, that $\frac{dz}{ds}=0$, when $z=0$, $y=0$, $s=0$. Moreover, the radius of curvature $r=\frac{ds\,.\,dy}{ddz}$ [53c] must at that

equation becomes

$$0=\frac{dds'}{dt^2}+m\,.\,\frac{ds'}{dt}+k\,s'+l\,s'^{i}+\&c. \qquad [94]$$

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point be a finite quantity. For if the curve, at its lowest point, be inclined to the axis of y, the body would at first move on an inclined plane, and if projected successively with infinitely small but different velocities, the times would, by [87a], be as the initial velocities nearly, and therefore not equal as the problem requires. But at the lowest point of the curve, where $\frac{dz}{ds}=0$, we have by [53b] $ds^2=dy^2+dz^2=dy^2$ or $ds=dy$, hence $\frac{1}{r}=\frac{ddz}{ds^2}$, which must at the lowest point be a finite quantity; since, if r were infinite, the curve would become a horizontal line at its lowest part, and if $r=0$, the curvature would be infinite, which could not be the case for this curve. Suppose now the general value of $\frac{dz}{ds}$ for *any* point whatever of the curve to be denoted by $\frac{dz}{ds}=\alpha\,.\,s^a+\beta\,.\,s^b+\gamma\,.\,s^c+\&c.$ in which the terms are arranged according to the magnitudes of the exponents a, b, &c. a being less than b, b less than c, &c. Then since at the lowest point $\frac{dz}{ds}=0$ and $s=0$; a, b, c, &c. must in general be positive. Again, this assumed value of $\frac{dz}{ds}$ gives, by taking its differential, (ds being constant), $\frac{ddz}{ds^2}=\alpha\,.\,a\,.\,s^{a-1}+\beta\,.\,b\,.\,s^{b-1}+\&c.$ Now to make this finite when $s=0$, as is required above [94a] it is necessary to put $a=1$. For if $a>1$, it becomes 0, and if $a<1$, it becomes $\frac{1}{0}$, when $s=0$. Therefore we shall have [94b], [94a] [94b]

$$\frac{dz}{ds}=\alpha\,.\,s+\beta\,.\,s^b+\gamma\,.\,s^c+\&c. \qquad [94c]$$

b, c, &c. being greater than unity.

Again, $c^{n\,s}=1+s'$ [92], its logarithm, divided by n, is $s=\frac{1}{n}\,.\,\left\{s'-\tfrac{1}{2}s'^2+\&c.\right\}$ (55 Int.), its differential gives $ds=\frac{ds'}{n}(1-s'+\&c.)$ Substitute these values of s and ds in [94c] and we shall obtain an expression of the form $\frac{dz}{ds'}=\alpha'\,.\,s'+\beta'\,.\,s'^e$ &c. in which the exponent of the first term s' of the second member is unity, and the other exponents exceed unity. Lastly, if we multiply the expression by $n^2 g\,(1+s')^2$ we shall obtain, for

$$n^2\,.\,g\,.\,\frac{dz}{ds'}(1+s')^2,$$

a similar expression, of the form $k\,s'+l\,s'^{i}+$ &c. i being greater than unity, which is the form assumed in [93'].

This equation multiplied by $c^{\frac{mt}{2}}.\left\{\cos.\gamma t+\sqrt{-1}\,.\sin.\gamma t\right\}$, and then integrated, becomes, by supposing $\gamma=\sqrt{k-\frac{m^2}{4}}$,* [94′]

[95]

$$c^{\frac{mt}{2}}.\left\{\cos.\gamma t+\sqrt{-1}\,.\sin.\gamma t\right\}.\left\{\frac{ds'}{dt}+\left(\frac{m}{2}-\gamma\,.\sqrt{-1}\right).s'\right\}$$

$$=-l\int s'^{i}.dt.c^{\frac{mt}{2}}.\left\{\cos.\gamma t+\sqrt{-1}\,.\sin.\gamma t\right\}-\&c.$$

By comparing together the real and the imaginary parts of this equation,

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* (40) Having an equation of the form

[95a]

$$0=\frac{dds'}{dt^2}+m.\frac{ds'}{dt}+ks'+Q,$$

if we multiply it by $c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}\,dt$, it will become,

$$c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.\left\{\frac{dds'}{dt}+m.ds'+ks'dt\right\}=-Q\,dt.c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}$$

The integral of the first member is

$$c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.\left\{\frac{ds'}{dt}+\left(\frac{m}{2}-\sqrt{\frac{m^2}{4}-k}\right).s'\right\},$$

as is easily proved by taking its differential, which is

$$c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.dt.\left(\frac{m}{2}+\sqrt{\frac{m^2}{4}-k}\right)\left\{\frac{ds'}{dt}+\left(\frac{m}{2}-\sqrt{\frac{m^2}{4}-k}\right)s'\right\}$$

$$+c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.\left\{\frac{dds'}{dt}+\left(\frac{m}{2}-\sqrt{\frac{m^2}{4}-k}\right)ds'\right\},$$

or, by reduction,

$$c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.dt.\left\{\frac{dds'}{dt^2}+m.\frac{ds'}{dt}+k.s'\right\}.$$

Hence we have the integral of [95a]

[95b]

$$c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.\left\{\frac{ds'}{dt}+\left(\frac{m}{2}-\sqrt{\frac{m^2}{4}-k}\right)s'\right\}=-\int Q\,dt.c^{\frac{mt}{2}+t\sqrt{\frac{m^2}{4}-k}}.$$

Now if for $\sqrt{\frac{m^2}{4}-k}$ we substitute its value $\sqrt{-1}\,.\sqrt{k-\frac{m^2}{4}}=\gamma\,.\sqrt{-1}$, [94′] and for $c^{\gamma t\sqrt{-1}}$ its value $\cos.\gamma t+\sqrt{-1}\,\sin.\gamma t$, [13 Int.], we shall obtain [95], observing that Q includes the terms l, &c.

we shall have two equations, by means of which we may exterminate $\frac{ds'}{dt}$; but it will suffice here to consider the following:*

$$c^{\frac{mt}{2}} \cdot \frac{ds'}{dt} \cdot \sin.\gamma t + c^{\frac{mt}{2}} \cdot s' \cdot \left\{ \frac{m}{2} \cdot \sin.\gamma t - \gamma . \cos.\gamma t \right\} = -l . \int s'^{i} dt . c^{\frac{mt}{2}} \sin.\gamma t - \&c. \qquad [96]$$

the integrals of the second member being supposed to commence with t. Put T equal to the value of t, at the instant the motion ceases, when $\frac{ds}{dt} = 0$; we shall have, at that time,

$$c^{\frac{mT}{2}} . s' . \left\{ \frac{m}{2} \sin.\gamma T - \gamma . \cos.\gamma T \right\} = -l . \int s'^{i} dt . c^{\frac{mt}{2}} . \sin.\gamma t - \&c. \qquad [97]$$

If we suppose s' to be infinitely small, the second member of this equation will vanish, in comparison with the first,† and we shall have

$$0 = \frac{m}{2} . \sin.\gamma T - \gamma . \cos.\gamma T; \qquad [98]$$

whence we deduce

$$\text{tang}.\gamma T = \frac{2\gamma}{m}; \qquad [99]$$

and as the time T is, by supposition, independent of the arch passed over, this value of tang. γT takes place for any arch whatever; we shall therefore have for all values of s',

$$0 = l . \int s'^{i} . dt . c^{\frac{mt}{2}} . \sin.\gamma t + \&c. \qquad [100]$$

the integral being taken from $t = 0$, to $t = T$. Supposing s' to be very

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* (41) This expression is deduced from that part of [95] connected with $\sqrt{-1}$, by putting it equal to nothing and dividing it by $\sqrt{-1}$.

† (42) The exponent i being greater than unity, [93'], the term s'^{i} will be infinitely less than s', if s' be supposed infinitely small; therefore, the second member of [97] must be nothing in comparison with the first; hence

$$c^{\frac{mT}{2}} s' . \left\{ \frac{m}{2} . \sin.\gamma T - \gamma . \cos.\gamma T \right\} = 0.$$

Dividing this by the factor $c^{\frac{mT}{2}} s'$, we get [98], and this, divided by $\frac{m}{2} \cos.\gamma T$, gives $0 = \text{tang}.\gamma T - \frac{2\gamma}{m}$, [99].

small, this equation will be reduced to its first term, which cannot be satisfied except by putting $l=0$; for the factor $c^{\frac{mt}{2}}.\sin.\gamma t$ being always positive from $t=0$ to $t=T$, the preceding equation is necessarily positive* in that interval. The curve cannot therefore be tautochronous except we have

[101] Equation of the Tauto-chronous Curve.

$$n^2 g.\frac{dz}{ds'}.(1+s')^2=ks';$$

which gives for the equation of the tautochronous curve†

[102]

$$g\,dz=\frac{k\,ds}{n}.(1-c^{-ns}).$$

In a vacuum, and when the resistance is proportional to the simple power of the velocity, n is nothing, and the preceding expression becomes the same as the equation of a cycloid,‡

Cycloid.

[102']

$$g\,dz=ks\,ds.$$

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* (43) The part $c^{\frac{mt}{2}}$ is evidently positive for all real values of $\frac{mt}{2}$, we have therefore only to examine the sign of the term $\sin.\gamma t$. Now from [99, 94'] we have

[100a]

$$\text{tang}.\gamma T=\frac{2\gamma}{m}=\sqrt{\frac{4k}{m^2}-1}.$$

To render this expression of $\text{tang}.\gamma T$ possible, it is necessary that the *unknown* quantity $4k$ should be positive and equal to, or exceed, the *known* quantity m^2; and if we suppose $4k$ to be increased from m^2 to ∞, its sign would always remain positive, so that γt would never exceed a right angle. Hence we easily perceive that γt must be less than a right angle, and its sign must therefore be positive, consequently $c^{\frac{mt}{2}}.\sin.\gamma t$ must be positive. Therefore the equation $0=l\int s'^i.dt.c^{\frac{mt}{2}}.\sin.\gamma t+$&c. [100] cannot be satisfied except by putting $l=0$. In the same manner we may prove any other following term of the series $ks'+l.s^i+$&c., assumed in [93'] for $n^2g.\frac{dz}{ds'}.(1+s')^2$ to be nothing, so that we shall have, as in [101], $n^2g.\frac{dz}{ds'}.(1+s')^2=ks'$.

† (44) Substituting in [101] the value of s' [92] we get

$$n^2g.\frac{dz}{n\,c^{ns}\,ds}.c^{2ns}=k.(c^{ns}-1).$$

Multiplying by $\frac{ds.c^{-ns}}{n}$, and reducing, it becomes $g\,dz=\frac{k\,ds}{n}(1-c^{-ns})$ as in [102].

‡ (45) The general expression of the resistance assumed in [87'] is $m.\frac{ds}{dt}+n.\frac{ds^2}{dt^2}$. If we suppose the term depending on the second power of the velocity to be nothing, it will

It is remarkable that the coefficient n of the part of the resistance, proportional to the square of the velocity, does not enter into the expression of the time T,* and it is evident, by the preceding analysis, that this

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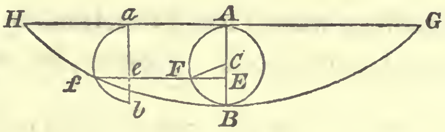
make $n=0$. Now for all values of n we have $c^{-ns}=1-ns+\frac{1}{2}.n^2s^2+\&c.$ (56 Int.), hence $\frac{1-c^{-ns}}{n}=s-\frac{1}{2}ns^2+\&c.$; the second member of which becomes s when $n=0$, and the formula [102] becomes, in this case, $g\,dz=k\,s\,ds$, whose integral, supposing z and s to begin together, is $gz=\frac{1}{2}ks^2$. This, as we shall soon show, is the equation of a cycloid, the diameter of whose generating circle is $\frac{g}{2k}$; and by putting $\frac{g}{2k}=2r$, this equation becomes

$$8rz=s^2. \qquad [102a]$$



A Cycloid is a curve $GBfH$, formed by the motion of a point of the circumference of a circle bfa, while it rolls on the straight line GAH as a base. This moving point falls on the base at H and G, and is at its greatest height at B. The perpendicular BA being equal to the diameter $2r$ of the generating circle BFA or bfa. Through f draw the ordinate $feFE$ parallel to the base. Put

$$Ef=y,\ BE=bE=z,\ \text{arch } BF=\text{arch } bf=Aa=Ff=p,$$

and $FE=\sqrt{2rz-zz}$. Then, since by construction $fE=Ff+FE$, we get, for the equation of the curve, $y=p+\sqrt{2rz-zz}$. Its differential is

$$dy=dp+\frac{rdz-zdz}{\sqrt{2rz-zz}}, \quad \text{but} \quad dp=\frac{rdz}{\sqrt{2rz-zz}} \qquad [102b]$$

hence $dy=\frac{2rdz-zdz}{\sqrt{2rz-zz}}$, and as the numerator and denominator can be divided by $\sqrt{2r-z}$, it becomes $dy=dz\sqrt{\frac{2r-z}{z}}$. Susbtituting this in $ds'=\sqrt{dy^2+dz^2}$ [53b] we get $ds=dz\sqrt{\frac{2r}{z}}=dz.z^{-\frac{1}{2}}\sqrt{2r}$, whose integral is $s=2.z^{\frac{1}{2}}\sqrt{2r}$, z and s commencing together. The square of this is $s^2=8rz$, which agrees with the equation of the tautochronous curve before found [102a].

* (46) The time T is deduced from the formula [100a]

$$\text{tang. }\gamma\, T=\frac{2\gamma}{m}=\sqrt{\frac{4k}{m^2}-1}$$

which does not contain n.

expression would be the same, if we should add to the preceding law of resistance, the terms* $p \cdot \frac{ds^3}{dt^3} + q \cdot \frac{ds^4}{dt^4} + \&c.$ [102'']

In general let R be the retarding force in the direction of the curve; we shall have†

[103]
$$0 = \frac{dds}{dt^2} + R.$$

s is a function of the time t, and of the whole arch passed over, therefore that arch is a function of t and s. Taking the differential of this last function, we shall have a differential equation of this form,

[104]
$$\frac{ds}{dt} = V;$$

V being a function of t and s, which ought to be nothing by the condition of the problem, when t has a determinate value, whatever be the length of the whole arch passed over.‡ Suppose, for example, that $V = S.T$; S being a function [104'] of s only, and T a function of t only, we shall have*

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* (46a) If we suppose part of the resistance to be as the third power of the velocity and to be represented by $p \cdot \frac{ds^3}{dt^3}$, this, by substituting the value of $\frac{ds}{dt} = \frac{ds'}{dt} \cdot \psi'(s')$ [88] would have introduced into the equations [89, 93, 94] the term $p \cdot \frac{ds'^3}{dt^3} \cdot \left\{ \psi'(s') \right\}^2$; but $\psi'(s') = \frac{1}{n(1+s')}$ [87c] and $\frac{ds'}{dt}$ will, as in [95, 96], be represented by quantities depending on the first or higher powers of s', therefore the preceding quantity by which p is multiplied, will depend on s'^3 or higher powers of s'. But such terms produce nothing in the equations [98], &c., for the same reason that the term depending on s'^i produced nothing. Therefore the term $p \cdot \frac{ds^3}{dt^3}$ produces no alteration in the expression of the time T [98], and the same would be the case with terms like $q \cdot \frac{ds^4}{dt^4}$, &c. as is observed above.

† (47) This retarding force R is supposed to be the combined effect of the resistance of the medium and the force of gravity.

‡ (48) Let the whole arch described be a', the time of description $= t'$. This time does not vary from any change in the value of a', by the conditions of the problem, so that if any part of that arch described in the time t be represented by s, this arch s will be a function of a' and t, consequently a' must be a function of s, t, which we shall denote by $a' = \psi(s, t)$. To determine the velocity v of the body at the end of the time t, we may take the differential

$$\frac{d\,d\,s}{d\,t^2} = T.\frac{dS}{ds}.\frac{ds}{dt} + S.\frac{d\,T}{d\,t} = \frac{d\,S}{S\,d\,s}.\frac{d\,s^2}{d\,t^2} + S.\frac{d\,T}{d\,t};$$ [105]

but the equation $\frac{d\,s}{dt} = S\,T$, gives T, therefore $\frac{d\,T}{d\,t}$ is equal to a function of $\frac{d\,s}{S\,d\,t}$, which function we shall denote by $\frac{d\,s^2}{S^2.\,d\,t^2}.\psi\left(\frac{d\,s}{S\,d\,t}\right)$; we shall [105'] therefore have

$$\frac{d\,d\,s}{d\,t^2} = \frac{d\,s^2}{S.\,d\,t^2}.\left\{\frac{d\,S}{d\,s} + \psi\left(\frac{d\,s}{S\,d\,t}\right)\right\} = -R;$$ [106]

Such is the expression of the resistance which corresponds to the differential equation $\frac{d\,s}{dt} = S\,T$; and it is easy to see that it comprises the case of the

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of the preceding expression of a', supposing it to be constant, and we shall get,

$$0 = \left(\frac{d.\psi(s,t)}{d\,s}\right).\,d\,s + \left(\frac{d.\psi(s,t)}{d\,t}\right).\,d\,t;$$

whence $\frac{d\,s}{d\,t} = -\frac{\left(\frac{d\,.\,\psi(s,\,t)}{dt}\right)}{\left(\frac{d\,.\,\psi(s,\,t)}{d\,s}\right)}$. The first member $\frac{d\,s}{d\,t}$ is evidently equal to v, and the second member is a function of s, t, denoted above by V, therefore $\frac{d\,s}{d\,t} = V$, or $v = V$ At the end of the time t', when the body has described the whole arch a', the velocity v will be nothing, and in this case $V = 0$, as above.

* (49) The value of $V = S\,T$, substituted in $\frac{d\,s}{d\,t} = V$, gives $\frac{d\,s}{d\,t} = S\,T$, whose differential is $\frac{d\,d\,s}{d\,t^2} = \frac{d.(S\,T)}{d\,t}$, and by considering S as a function of s, and s as a function of t, it becomes $\frac{d\,d\,s}{d\,t^2} = T.\frac{d\,S}{d\,s}.\frac{d\,s}{d\,t} + S.\frac{d\,T}{d\,t}$, but $T = \frac{V}{S} = \frac{d\,s}{S\,d\,t}$, [104, 104'] hence $\frac{d\,d\,s}{d\,t^2} = \frac{d\,S}{S\,d\,s}.\frac{d\,s^2}{d\,t^2} + S.\frac{d\,T}{d\,t}$ as in [105]. Again, since $T = \frac{d\,s}{S\,d\,t}$ is a function of t, we shall have $t =$ function of $\left(\frac{d\,s}{S\,d\,t}\right)$, and as $\frac{d\,T}{d\,t} =$ function of t, we shall also have $\frac{d\,T}{d\,t} =$ function of $\left(\frac{d\,s}{S\,d\,t}\right)$, which, being assumed as above equal to $\frac{d\,s^2}{S^2\,d\,t^2}.\psi\left(\frac{d\,s}{S\,d\,t}\right)$, and substituted in the preceding value of $\frac{d\,d\,s}{d\,t^2}$, or $-R$ [103], becomes as in [106].

resistance proportional to the two first powers of the velocity, multiplied respectively by constant coefficients.* Other differential equations would give different laws of resistance.

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* (49*a*) If we put $S = c^{as}$, $\psi\left(\frac{ds}{S\,dt}\right) = b \cdot \left(\frac{ds}{S\,dt}\right)^{-1}$, it gives $\frac{dS}{ds} = a\,c^{as}$ or $\frac{dS}{ds} = a\,S$, which, substituted in [106], makes

$$-R = \frac{ds^2}{S\,dt^2}\left\{a\,S + b \cdot \left(\frac{ds}{S\,dt}\right)^{-1}\right\} \quad \text{or} \quad -R = a \cdot \frac{ds^2}{dt^2} + b \cdot \frac{ds}{dt},$$

which contains the two first powers of the velocity $\frac{ds}{dt}$.

(49*b*) It may be observed that John Bernoulli published in the Memoirs of the Academy of Sciences of Paris, for 1730, and afterwards in the third volume of his works, a curious paper on the Tautochronous curve, both in the ascending and descending branches. He justly remarks that there is a limit to the *whole length* of the curve, and mentions as a proof of it, the familiar instance of the cycloid, which is the tautochronous curve when the resistance is nothing [102′]. For if a body, falling freely by gravity along the whole cycloidal arch, would, in the time T, arrive at its lowest point, with the acquired velocity V; we might project the body upwards, along the arch, from its lowest point, with any velocity less than V, and the force of gravity would destroy the whole velocity, in the time T, whatever might be the length of the part of the arch described. But if the projected velocity should exceed V, the whole cycloidal arch would be passed over in a time less than T; so that this particular cycloid would cease to be tautochronous with such a projected velocity. In the preceding calculations it is not necessary to suppose the whole curve to be in the same plane; for, without altering the investigation, we may suppose the curve to be of double curvature, taking care not to have any abrupt bend, and keeping every point of the curve in the same horizontal plane, in which it was originally placed, when the whole curve was extended in a vertical plane. For example, the curve might be bent round a vertical cylinder, keeping the extremities at their former height. It being evident that, in this case, the spaces passed over, the times, the velocities, and the resistance would not be varied, for the reasons mentioned in note 18*a*.

CHAPTER III.

ON THE EQUILIBRIUM OF A SYSTEM OF BODIES.

13. THE most simple case of the equilibrium of several bodies, is that of two material points or particles which impinge against each other, in opposite directions, with equal velocities; their mutual impenetrability evidently destroys their velocities, and reduces them to a state of rest.

Suppose now that a number m of similar contiguous particles, are arranged in a right line, each having the velocity u, in the direction of this right line; and that a number m' of similar contiguous particles are arranged on the same right line, each having the velocity u' directly opposite to u, so that the two systems shall strike directly against each other. It is required to determine the ratio of u to u', that there may be an equilibrium at the instant of impact.

For this purpose we shall observe that the system m, having the velocity u, would be in equilibrium with a single particle having the velocity $m\,u$, in an opposite direction; for each particle of the system would destroy in this last particle, a velocity equal to u, consequently the whole number of particles m would destroy the whole velocity $m u$. Therefore we may substitute, for this system, a single particle, moving with the velocity $m u$. In like manner we may substitute for the system m', a single particle having the velocity $m' u'$; now the two systems being supposed in equilibrium, the two particles which have been substituted for them, ought also to be in equilibrium, which requires that their velocities should be equal; we have therefore, for the condition of the equilibrium of the two systems, $m\,u = m'\,u'$. [106′]

Quantity of motion.

The mass of a body is the number of its material particles, and the product of the mass by the velocity is called the *quantity of motion*, this is also what is understood by the force of a body in motion. To maintain the equilibrium between two bodies, or two systems of points, impinging against each other in opposite directions; the quantities of motion, or the opposite forces, ought to be equal; consequently the velocities ought to be inversely proportional to the masses.

Mass, Density, Magnitude, or Volume. The density of a body depends upon the number of material points or particles contained in a given space. To obtain the absolute density, it would be necessary to compare the mass with that of a substance without pores; but as no such substance is known, we can obtain only the relative density of a body; that is, the ratio of its density, to that of a given substance. It is evident that the mass is in a ratio compounded of that of the magnitude and density; putting therefore M for the mass of a body, U its magnitude, and D its density, we shall have, in general,

[106″] $$M = D\,U;$$

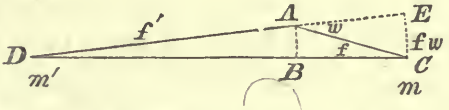
in which we ought to observe that the quantities M, D, U, express the ratios to the unity of each species, taken as a measure of those quantities.

In what has been said, it is supposed that bodies are composed of similar material particles, and that they differ only by the relative positions of these particles. But the nature of bodies being unknown, this hypothesis is at least precarious, and it is possible that there may be essential differences in the ultimate particles. Fortunately the uncertainty of this hypothesis does not affect the science of mechanics, and we may use it without fear of error, provided we understand, by *similar material points or particles*, such as would [106‴] be in equilibrium, if they impinged against each other with equal velocities, in opposite directions, whatever might be their nature.

14. Two material particles, whose masses are m and m', cannot act upon each other, but in the direction of the right line which connects them together It is true, that if the two particles are connected by a line passing over a fixed pulley, their reciprocal action cannot be in the direction of this line. But we may suppose the fixed pulley to have, at its centre, a mass of an [106^{iv}] infinite density, which reacts upon the two bodies m and m', whose action on each other may thus be considered as indirect.

Let p be the action which m exerts upon m' by means of an inflexible right line without mass, which is supposed to connect them. Conceiving this line to be affected by two equal and opposite forces p and $-p$; the force $-p$ will destroy in the body m, a force equal to p, and the force p of the right line will be wholly communicated to the body m'. This loss of force in m, occasioned by its action on m', is what is called the *reaction* of m'; thus, in **Reaction.** the communication of motion, *the reaction is always equal and opposite to the action.* [106v] It appears by observation, that this principle exists in all the operations of nature.

Suppose two heavy bodies m and m' to be attached to the extremities of an horizontal line, inflexible and without mass, which can turn freely about one **Lever, or Balance.** of its points. To conceive of the action of these bodies upon each other, when they are in equilibrium, we must suppose the right line to be bent, at its fixed point, through a very small angle, so as to form two right lines, making at that point an angle which differs from two right angles but by an infinitely small quantity ω. Let f, f', be the distances of m and m' from the fixed point; by resolving the gravity of m, into two forces, the one acting upon the fixed point, the other directed towards m', this last force will be*



* (50) To illustrate this, let DAC be the bent lever, A its point of suspension, C, D the extremities, to which m, m' are attached; the line CD being horizontal. Draw the vertical lines AB, CE, meeting DC, and DA (continued), in B and E. Then $AC = f$, $AD = f'$, $CAE = \omega$. Supposing the angle ω to be infinitely small, and neglecting its second and higher powers, we shall have $CB = f$, $DB = f'$, $DC = f + f'$, $CE = f\omega$; this last line being nearly equal to the arch of a circle, described about A as a centre, with the radius AC. The similar triangles DCE, DBA give $DC : DB :: CE : AB$; hence in symbols, $AB = \frac{ff'\omega}{f+f'}$. Now the weight m acts at C, in the direction EC, parallel to AB, with the force of its gravity mg, which may be represented by AB. This may be resolved into two forces AC, CB; of which the first is destroyed by the reaction of the point of support A; the other, in the direction CB, is equal to $mg \cdot \frac{CB}{AB}$; and, by substituting the above values of AB, CB it becomes, $\frac{mg\cdot(f+f')}{\omega f'}$. In a similar way, by changing f into f', m into m', and the contrary, we obtain the force of the weight m', acting at D, resolved in the direction DB, $\frac{m'g\cdot(f+f')}{\omega f}$, which agrees with the above. Putting these two expressions equal to each other and dividing by $\frac{g\cdot(f+f')}{\omega f' f}$ we get [106vi].

$\frac{mg\,.\,(f+f')}{\omega f'}$, g being the force of gravity. The action of m' upon m will likewise be $\frac{m'g\,.\,(f+f')}{\omega f}$; putting these forces equal to each other, on account of the equilibrium, we shall have

[106vi] $$mf=m'f';$$

which gives the known law of the equilibrium of a lever, and shows also, how we may conceive of the reciprocal action of parallel forces.

We shall now consider the equilibrium of a system of particles m, m', m'', &c., impelled by any forces, and reacting on each other. Let f be the distance of m from m'; f' the distance from m to m''; f'' the distance from m' to m'', &c.; p the reciprocal action of m on m'; p' that of m on m''; p'' that of m' on m'', &c. Moreover, let $m\,S$, $m'\,S'$, $m''\,S''$, &c., be the forces which act upon m, m', m'', &c.; and s, s', s'', &c., the right lines drawn from their origins to the bodies m, m', m'', &c.* This being supposed, the particle m may be considered as perfectly free, and in equilibrium, by means of the force $m\,S$, and the forces which the particles m', m'', &c., impart to it; but if the particle m be forced to move upon a surface, or a curve, we must add to these forces, the reaction of the surface or curve. Let δs be the whole variation of s; $\delta_{,}f$ the variation of f, supposing m' to be at rest; $\delta_{,}f'$ the variation of f', supposing m'' to be at rest, &c.; R, R' the reactions of the two surfaces, which by their intersection form the curve upon which the particle m is forced to move; and δr, $\delta r'$ the variations of the directions of these last forces. The equation (d) § 3 [24], will give

[107] $$0=mS.\delta s+p.\delta_{,}f+p'.\delta_{,}f'+\text{\&c.}+R.\delta r+R'.\delta r'+\text{\&c.}$$

In like manner m' may be supposed perfectly free, and in equilibrium by means of the force $m'\,S'$, the actions of the bodies m, m'', &c., and the reactions of the surfaces upon which it is forced to move, which reactions we

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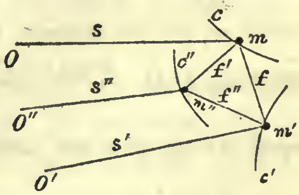
* (51) To illustrate this, let m, m', m'', be the particles; $c\,m$, $c'\,m'$, $c''\,m''$, the curves upon which they are forced to move; o, o', o'', the origins of the forces S, S', S''. Then $o\,m=s$, $o'\,m'=s'$, $o''\,m''=s''$; $m\,m'=f$, $m\,m''=f'$, $m'\,m''=f''$. The forces R, R', act at m; and R'', R''', at m', in directions perpendicular to the surfaces whose intersections form $c\,m$, $c'\,m'$.



shall denote by R'' and R'''. Let $\delta s'$ be the variation of s'; $\delta_{,,}f$ the variation of f supposing m to be at rest; $\delta_{,}f''$ the variation of f'' supposing m'' to be at rest, &c.; and $\delta r''$, $\delta r'''$ the variations of the directions of R'', R'''; the equilibrium of m' will give

$$0 = m'S'.\delta s' + p.\delta_{,,}f + p''.\delta_{,}f'' + \&c. + R''.\delta r'' + R'''.\delta r'''. \qquad [108]$$

We can form similar equations relative to the equilibrium of m'', m''', &c.; and by adding them together, observing that*

$$\delta f = \delta_{,}f + \delta_{,,}f; \qquad \delta f' = \delta_{,}f' + \delta_{,,}f'; \qquad \&c. \qquad [109]$$

δf, $\delta f'$, &c., being the whole variations of f, f', &c.; we shall have

$$0 = \Sigma.m.S.\delta s + \Sigma.p.\delta f + \Sigma.R.\delta r; \qquad (k) \qquad [110]$$

in which equation the variations of the co-ordinates of the different bodies of the system, are wholly arbitrary. It should be observed, that by the equation (*a*) § 2 [16], we may substitute, instead of $m\,S.\delta s$, the sum of the products of all the partial forces acting on m, by the variations of their respective directions. The same may be observed of the products $m'S'.\delta s'$, $m''S''.\delta s''$, &c.

If the bodies m, m', m'', &c., are firmly connected together in an invariable manner; the distances f, f', f'', &c. will remain constant; and we shall have for the conditions of the connexion of the parts of the system $\delta f = 0$; $\delta f' = 0$; $\delta f'' = 0$, &c. The variations of the co-ordinates in the equation (*k*) [110] being arbitrary, we may make them satisfy these last equations, and then the forces p, p', p'', &c., which depend upon the reciprocal action of the bodies of the system, will disappear from that equation. We may also [110′]

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* (51*a*) This follows from the known principle, that the complete differential, or variation, is equal to the sum of all the partial differentials, found by supposing each quantity separately to vary. Thus if x, y, z are the rectangular co-ordinates of m, x', y', z' those of m', we shall have their distance [12 or 118],

$$f = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}. \qquad [109a]$$

Now if for brevity we put $A = \frac{x'-x}{f}$, $B = \frac{y'-y}{f}$, $C = \frac{z'-z}{f}$; its complete variation will be, [109b]

$$\delta f = A\,(\delta x' - \delta x) + B\,(\delta y' - \delta y) + C\,(\delta z' - \delta z).$$

Its partial variation, supposing only the body m to vary, is $\delta_{,}f = -A\,\delta x - B\,\delta y - C\,\delta z$; and, if m' only vary, we get $\delta_{,,}f = A\,\delta x' + B\,\delta y' + C\,\delta z'$; whence we get $\delta f = \delta_{,}f + \delta_{,,}f$, as above. In the same way we get $\delta f'$ [109], &c.

make the terms $R\delta r$, $R'\delta r'$, &c., disappear, by subjecting the variations of the co-ordinates to the conditions necessary to satisfy the equations of the surfaces, upon which these bodies are forced to move; the equation (*k*) [110], by this means, will become

[110″] $$0=\Sigma.mS.\delta s\,; \qquad (l)$$

whence it follows, that in the case of equilibrium, the sum of the variations of the products of the forces, by the elements of their directions, will be nothing, in whatever manner we may vary the position of the system, provided that the conditions of the connexion of its parts be observed.

This theorem, which we have obtained in the particular case in which the bodies are connected together in an invariable manner, is true, whatever be the conditions of the connexion of the parts of the system with each other.* To prove this, it is sufficient to show that by subjecting the variations of the co-ordinates to these conditions, we shall have in the equation (*k*) [110].

[111] $$0=\Sigma.p.\delta f+\Sigma.R.\delta r\,;$$

now it is evident that δr, $\delta r'$, &c., are nothing in consequence of these conditions [19*a*]; it therefore only remains to prove, that by subjecting the variations of the co-ordinates to the same conditions, we shall have $0=\Sigma.p.\delta f$.

Suppose the system to be acted upon only by the forces p, p', p'', &c., and [111′] that the bodies are made to move upon the curves, that they would describe by means of these conditions. Then, these forces may be resolved into the following, namely, one part, q, q', q'', &c.,† directed along the lines f, f', f'',

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* (52) The meaning of this, in an analytical point of view, is that the equation [110″], $\Sigma.m\,S.\delta s=0$, takes place in all cases of equilibrium, provided as many of the variations are exterminated as there are conditions in the proposed system. For as La Grange has observed, in his *Mécanique Analytique*, "Each equation of condition is equivalent to one or more forces, applied to the system, according to given directions; so that the state of equilibrium will be the same, whether we employ the consideration of these forces, or that of the equations of condition." We may observe that the equation [110″] is used, in the rest of the work, in the case where we actually have $\delta f=0$, $\delta f'=0$, &c. The equation [116] being used in other cases.

† (52*a*) In the reasoning [110′] the forces p, p', &c. represent the reaction of the bodies upon each other, which are supposed mutually to destroy each other. Here p, p', &c.

&c., which mutually destroy each other, without producing any effect on the [111'] described curves; another part T, T', T'', &c. perpendicular to these curves; and lastly, the remaining part in the directions of the tangents of these curves, by means of which the system would be moved. But it is easy to perceive that these last forces ought to be nothing; for the system being supposed to submit to them freely,* they could neither produce a pressure upon the described curves, nor a reaction of the bodies upon each other; they could not therefore produce an equilibrium with the forces $-p$, $-p'$, $-p''$, &c.; q, q', q'', &c.; T, T', &c.; and they must therefore be nothing; [111''] consequently the system must be in equilibrium by means of the forces $-p$, $-p'$, $-p''$, &c.; q, q', q'', &c.; T, T', &c. Let δi, $\delta i'$, &c. be the variations of the directions of the forces T, T', &c.; we shall have, by means of the equation (k) [110],

$$0=\Sigma.(q-p).\delta f+\Sigma.T.\delta i; \qquad [112]$$

but the system being supposed to be in equilibrium by the forces q, q', &c., without producing any action on the described curves,† the equation (k) [110]

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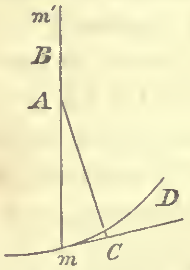
represent the *total* forces exerted upon each body, exclusive of the reaction or pressure of the curves; and the object of the author is to show that the decomposition of these *total* forces, produces forces, which are equivalent to the reciprocal action of the bodies above treated of; also that these and the remaining parts of the forces will balance each other. Thus let the bodies m, m' be situated at the points m, m', and suppose the total force p acts upon m, in the direction $m\,m'$, while m is subjected to move upon the curve $m\,D$, whose tangent is $m\,C$. Upon $m\,m'$ take $m\,B=p$, $A\,B=q$; and since, by hypothesis, this last force is destroyed, by the reaction of the other bodies; the remaining force will be $m\,A$, which may be resolved into the forces $A\,C=T$, perpendicular to the tangent; and $m\,C$ in the direction of the tangent. This last force must be nothing, for reasons stated by the author [111'']. Lastly, we may remark that if any of the forces q, q', &c., were in any particular instance equal to nothing, it would not affect the above demonstration.



* (52*b*) The system being at liberty to move in the respective directions of these tangential forces, would do so, unless it were held in equilibrium, by equal and opposite forces; so that the sums of the opposite forces acting upon these bodies must be equal, and these forces will vanish for all the bodies.

† (52*c*) Because, by hypothesis [111'], the forces q, q', &c. mutually destroy each other in the system.

will also give, $0=\Sigma.q.\delta f$; therefore the above expression [112] becomes

$$0=\Sigma.p.\delta f-\Sigma.T.\delta i. \qquad [113]$$

If we take the variations of the co-ordinates, so as to satisfy the conditions of the described curves, we shall have $\delta i=0$, $\delta i'=0$, &c. [19*a*]; and then we shall have

$$0=\Sigma.p.\delta f; \qquad [114]$$

and as the described curves are themselves arbitrary, being only subjected to the conditions arising from the connexion of the parts of the system; the preceding equation will take place, provided these conditions are satisfied; and then the equation (*k*) [110], will be changed into the equation (*l*) [110″]. This equation is the analytical expression of the following principle, known by the name of the *principle of virtual velocities.*

[114′] Virtual Velocities. "*If we vary, by an infinitely small quantity, the position of a system of bodies, subjecting the system to the conditions which ought to be satisfied, the sum of the forces which act upon the several bodies, multiplied each by the space that the body to which the force is applied describes in the direction of that force, must be equal to nothing, when the system is in equilibrium.*"*

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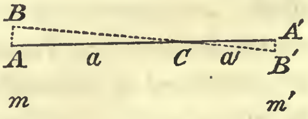
* (52*d*) For the purpose of illustration, and to show the manner of using the principle of virtual velocities [110″], we shall apply it to the investigation of some of the elementary propositions in mechanics.

First. Let $A\,C\,A'$ be an inflexible straight line, void of gravity, situated in a horizontal position, with the weights m, m', attached to the points A, A'; the rod being fixed at C, as a centre of suspension, so that it can move about that centre in a vertical direction, as in the common balance or steelyard. Put $C\,A=a$, $C\,A'=a'$, then if the weights m, m' be an equilibrium, and the rod be made to revolve about C, through an infinitely small angle $B\,C\,A=B'\,C\,A'=\omega$, so that the weight m may ascend through the vertical space $A\,B=a.\omega$; the distance of this weight from the centre of the force, which is in this case the centre of the earth, will be *increased* by the quantity $\delta s=a.\omega$. In like manner the distance of the weight m' from the centre of the earth, will be *decreased* by the quantity $A'\,B'=a'.\omega$, therefore $\delta s'=-a'.\omega$; the negative sign being prefixed, because the distance of the body m', from that centre, is decreased by this motion. The principle of virtual velocities [110″], becomes in this case,



$$m.S.\delta s+m'.S'.\delta s'=0, \qquad [114a]$$

This principle not only takes place in the case of equilibrium, but it assures the existence of the equilibrium. For, suppose the equation (l) [110″] to be satisfied, and that the particles m, m', &c., acquire the velocities

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and as the force of gravity, acting upon both bodies, is the same, we shall have $S = S'$, therefore,

$$m \,.\, \delta s + m' \,.\, \delta s' = 0.$$ [114b]

Substituting the above values of δs, $\delta s'$, it becomes, $m \,.\, a \,.\, \omega - m' \,.\, a' \,.\, \omega = 0$, hence $m \,.\, a = m' \,.\, a'$, which is the usual formula of the balance [106^{vi}].

It is evident that what is here stated, relative to the action of gravity, may be applied to the consideration of any other forces, acting at the extremities of a straight lever $A\,C\,A'$ in directions perpendicular to the arm of the lever, and in the same plane. In this case, instead of the forces $S\,m$, $S'\,m'$, representing the gravity, S, S', acting upon the bodies m, m', we may take the equivalent forces P, P', acting upon the extremities of the lever, and we may put the formula [114a], under the following form,

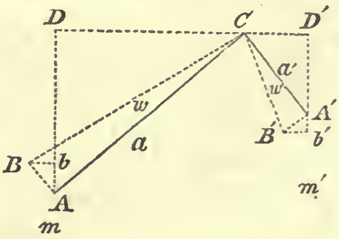
$$P \,.\, \delta s + P' \,.\, \delta s' = 0.$$ [114c]

Second. In the preceding calculation, the line $A\,C\,A'$ was supposed to be horizontal; but if it be inclined to the horizon, by an angle ε, the lines $A\,B$, $A'\,B'$ would be inclined to the vertical, by the same angle ε. In this case, the *vertical* ascent of the body m, in moving from A to B, would be $\delta s = A\,B \,.\, \cos.\, \varepsilon = a \,.\, \omega \,.\, \cos.\, \varepsilon$. In like manner

$$\delta s' = - A'\,B' \,.\, \cos.\, \varepsilon = - a' \,.\, \omega \,.\, \cos.\, \varepsilon.$$ [114d]

Substitute these in [114b], and reject the common factor $\omega \,.\, \cos.\, \varepsilon$, we obtain, as above, $m \,.\, a = m' \,.\, a'$. It is easy to apply the same principles to the action of any forces, applied to the extremities of the lever $A\,A'$, in any directions.

Third. Instead of supposing the balance to be a straight line, as in the two preceding examples, let it be bent at C, so as to form the oblique angle $A\,C\,A'$. Using the same notation as before, we shall have $A\,C\,B = A'\,C\,B' = \omega$, $C\,A = a$, $C\,A' = a'$, $A\,B = a \,.\, \omega$; $A'\,B' = a' \,.\, \omega$; these two last lines being perpendicular to $C\,A$, $C\,A'$, respectively. Draw the horizontal line $D\,C\,D'$, and upon it let fall the perpendiculars $A\,D$, $A'\,D'$, meeting the horizontal lines $B\,b$, $B'\,b'$ in b, b', respectively. Put the angles $D\,C\,A = B\,A\,b = C$, $D'\,C\,A' = B'\,A'\,b' = C'$. Then the *vertical* space passed over by the body m, while moving from A to B, will be



$$A\,b = A\,B \,.\, \cos.\, C = a \,.\, \omega \,.\, \cos.\, C,$$

hence $\delta s = a \,.\, \omega \,.\, \cos.\, C$. In like manner, the *vertical* space passed over by the body m', in moving from A' to B', will be $A'\,b' = A'\,B' \,.\, \cos.\, C' = a' \,.\, \omega \,.\, \cos.\, C'$, hence

$$\delta s' = - a' \,.\, \omega \,.\, \cos.\, C';$$

v, v', &c., by means of the forces $m\,S$, $m'\,S'$, &c., acting upon them. The system will be in equilibrium, by means of these forces and the forces $-m\,v$, $-m'\,v'$, $-m''\,v''$, &c.; put δv, $\delta v'$, &c., for the variations of the

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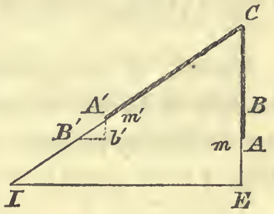
the sign — being prefixed, because the distance from the earth's centre is decreased by the motion. Substituting these in [114b], and dividing by ω, we get

$$m\,.\,a\,.\,\cos.\,C = m'\,.\,a'\,.\,\cos.\,C';$$

but $a\,.\cos.\,C = C\,D$, $a'.\cos.\,C' = C\,D'$, therefore, $m\,.\,C\,D = m'\,.\,C\,D'$, which is the well known principle of the bent balance.

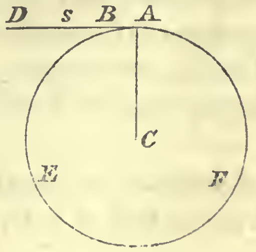
Fourth. Let $C\,I$ be an inclined plane. Draw the horizontal line $I\,E$, and the vertical line $C\,E$. Put the angle of inclination of the plane to the horizon $C\,I\,E = I$. Suppose two bodies m, m', to be connected together, by the flexible thread $A\,C\,A'$, void of gravity, passing over the vertex of the triangle $I\,C\,E$; so that the body m' may be at liberty to move along $C\,I$, while the body m moves in the vertical $C\,E$. It is very easy, in this case, to apply the principle of virtual velocities, when the bodies m, m', are in equilibrium. For, if we suppose the body m to move through the infinitely small space $A\,B$, in a vertical direction, its distance from the centre of the earth will be increased by $A\,B = \delta s$; and, during this motion, the body m' will slide down the line $C\,I$, through an equal space, $A'\,B' = \delta s$. To find the corresponding vertical distance $A'\,b'$, passed over by this body m', we may draw $B'\,b'$, $A'\,b'$, parallel to $I\,E$, $C\,E$, respectively; and we shall have $A'\,b' = A'\,B'\,.\,\sin.\,A'\,B'\,b' = A'\,B'\,.\,\sin.\,I = \delta s\,.\,\sin.\,I$, therefore $\delta s' = -\,\delta s\,.\,\sin.\,I$; the negative sign being prefixed, because the distance of the body m', from the earth's centre, is decreased. Substitute these values of δs, $\delta s'$, in [114b], it becomes,



$$m\,.\,\delta s - m'\,.\,\delta s\,.\,\sin.\,I = 0,$$

whence, $m = m'\,.\,\sin.\,I$; which agrees with the usual rule for the equilibrium of bodies upon an inclined plane.

Fifth. In the motion of a screw, let the power P be supposed to act at the extremity A of the horizontal lever $C\,A$; the direction of this force being horizontal and perpendicular to the arm of the lever; the screw turning about a vertical axis, passing through the point C, perpendicular to the plane of the figure; and raising the weight P', so that the extremity of the lever may describe the circumference of the circle $A\,E\,F\,A = c$, in the same time that the weight P' is raised, through a vertical height h, equal to the distance of the threads of the screw. Now the tangent $A\,D$ being drawn perpendicular to $C\,A$, we may take upon it any point D, as the origin of the force P, so that



directions of these last forces; we shall have, by the principle of virtual velocities,

$$0=\Sigma.mS.\delta s-\Sigma.m.v\delta v\,; \qquad [115]$$

but, by hypothesis, we have, $0=\Sigma.m.S.\delta s$; therefore $0=\Sigma.m.v\delta v$. As the variations δv, $\delta v'$, &c., ought to be subjected to the conditions of the system, we may suppose them to be equal to $v\,dt$, $v'\,dt$, &c., and we shall then have $0=\Sigma.mv^2$, which equation gives* $v=0$, $v'=0$, &c.; therefore the system will be in equilibrium by means of the forces mS, $m'S'$, &c.

The conditions of the connexion of the parts of a system may always be reduced to certain equations between the co-ordinates of its different bodies. Let $u=0$, $u'=0$, $u''=0$, &c. be these equations; we may, by § 3 [26], add to the equation (l) [110″], the function $\lambda\delta u+\lambda'\delta u'+$&c., or $\Sigma.\lambda.\delta u$; λ, λ', &c., being indeterminate functions of the co-ordinates of the bodies; this equation will thus become

$$0=\Sigma.mS.\delta s+\Sigma.\lambda.\delta u\,; \qquad (k') \qquad [116]$$

in this case, the variations of all the co-ordinates will be arbitrary, and we

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$AD=s$; and we may consider the infinitely small part of it AB, which is common to this tangent and to the circle, to represent the variation of s, therefore $\delta s=-AB$; the negative sign being prefixed, because, while the extremity of the lever moves, from A to B, the distance, from the origin of the force D, is decreased. It is evident, that during the motion from A to B, the weight P' will be raised through the space $\frac{b.AB}{c}=\delta s'$. Substitute these values of δs, $\delta s'$, in [114c], and reject the common factor AB, it becomes $-P+P'.\frac{b}{c}=0$, or $P=P'.\frac{b}{c}$, which is the usual formula for the screw.

Sixth. In the case of a compound pulley, in which a power P is applied, to raise a weight P' vertically; if we suppose the power to act at the end of the cord, while the weight is supported by n parts of the same cord, each bearing an equal part of the weight, so that while the weight P' is raised through the vertical height $\delta s'$, the power P is depressed by n times that quantity, we shall have $\delta s=-n.\delta s'$. These values being substituted in [114c], give $-Pn.\delta s'+P'.\delta s'=0$, hence $P'=Pn$, which is the usual rule for computing the force of a pulley.

* (53) Each term mv^2, $m'v'^2$, &c. of the equation $\Sigma.mv^2=0$, is *positive*, and to render their sum nothing, we must necessarily have $mv^2=0$, $m'v'^2=0$, &c. whence $v=0$, $v'=0$, &c. m, m', &c. being positive.

may put their coefficients equal to nothing, which will give an equal number of equations, by means of which we may determine the functions λ, λ', &c. If we then compare this equation with the equation (k) [110], we shall have

[117] $$\Sigma.\lambda.\delta u=\Sigma.p.\delta f+\Sigma.R.\delta r;$$

whence it will be easy to deduce the reciprocal actions of the bodies m, m', &c., and the pressures $-R$, $-R'$, &c., which they exert against the surfaces, upon which they are forced to remain.

Equation of a system of particles firmly connected with each other.

15. If all the bodies of a system are firmly attached together, its position may be determined by any three of its points, which are not in the same right line. The position of each of these points depends on three co-ordinates, which produce nine indeterminate quantities; but the mutual distances of the three points being given and invariable, we may, by means of them, reduce these quantities to six others, which, substituted in the equation (l) [110″], will introduce six arbitrary variations; putting their coefficients equal to nothing, we shall have six equations, which will contain all the conditions of the equilibrium of the system; we shall now develop these equations.

[117′] Let x, y, z be the co-ordinates of m; x', y', z' those of m'; x'', y'', z'' those of m'', &c.; we shall have*

[118]
$$f=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2};$$
$$f'=\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2};$$
$$f''=\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2};$$
&c.

If we suppose

[119]
$$\delta x=\delta x'=\delta x''=\text{\&c.};$$
$$\delta y=\delta y'=\delta y''=\text{\&c.};$$
$$\delta z=\delta z'=\delta z''=\text{\&c.};$$

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* (53a) The value f is the same as in [109a]; f', f'', &c. are of the same form, changing x', y', z' into x'', y'', z'', &c. The variation of f [109b] is

$$\delta f=A\,(\delta x'-\delta x)+B\,(\delta y'-\delta y)+C\,(\delta z'-\delta z),$$

which, by means of the equations [119], becomes $\delta f=0$. In like manner we get $\delta f'=0$, $\delta f''=0$, &c. as above.

we shall have

$$\delta f=0, \qquad \delta f'=0, \qquad \delta f''=0, \quad \&c.; \qquad [119']$$

the requisite conditions will therefore be satisfied, and we shall have, by means of the equation (l) [110″], the following:*

$$0=\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta x}\right); \qquad 0=\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta y}\right); \qquad 0=\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta z}\right); \quad (m) \qquad [119'']$$

we shall thus have three of the six equations, which contain the conditions of the equilibrium of the system. The second members of these equations are the sums of the forces of the system, resolved in directions parallel to the three axes x, y, z, [13]; each of these sums ought therefore to be nothing in the case of equilibrium. [119‴]

The equations $\delta f=0$, $\delta f'=0$, $\delta f''=0$, &c., will also be satisfied if we suppose z, z', z''. &c. to be invariable, and put† [119^{iv}]

$$\begin{aligned} \delta x&=y\,.\,\delta\varpi; & \delta y&=-x\,.\,\delta\varpi; \\ \delta x'&=y'\,.\,\delta\varpi; & \delta y'&=-x'\,.\,\delta\varpi; \\ &\&c. \end{aligned} \qquad [120]$$

$\delta\varpi$ being any variation whatever. Substituting these values in the equation (l) [110″], we shall have‡

$$0=\Sigma\,.\,m\,.\,S\,.\left\{y\,.\left(\frac{\delta s}{\delta x}\right)-x\,.\left(\frac{\delta s}{\delta y}\right)\right\}. \qquad [121]$$

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* (53b) The values [119] being substituted in [110″], developed as in [14a], it becomes

$$0=\delta x\,.\,\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta x}\right)+\delta y\,.\,\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta y}\right)+\delta z\,.\,\Sigma\,.\,m\,.\,S\,.\left(\frac{\delta s}{\delta z}\right).$$

Putting as in [26′] the coefficients of δx, δy, δz, separately equal to nothing, we obtain [119″].

† (54) Substitute in δf [109b], the values [120], also $\delta z=0$, $\delta z'=0$, &c. [119^{iv}], it becomes $\delta f=\delta\varpi\,.\,\{A\,.\,(y'-y)-B\,.\,(x'-x)\}$, and since, by [109$a,b$], $x'-x=f\,.\,A$, $y'-y=f\,.\,B$, it may be changed into $\delta f=f\,.\,\delta\varpi\{AB-AB\}=0$. The same takes place with $\delta f'$, $\delta f''$, &c. because all these expressions are symmetrical.

‡ (55) The co-ordinates z, z', &c. being invariable, the part of the equation [110″] depending upon the body m, will be $m\,.\,S\,.\left(\frac{\delta s}{\delta x}\right)\delta x+m\,.\,S\,.\left(\frac{\delta s}{\delta y}\right).\,\delta y$; or, by substituting the values of $\delta x, \delta y$, [120] $\delta\varpi\,.\left\{m\,.\,S\,.\,y\,.\left(\frac{\delta s}{\delta x}\right)-m\,.\,S\,.\,x\,.\left(\frac{\delta s}{\delta y}\right)\right\}$. In a similar manner the terms depending upon m' are,

$$\delta\varpi\,.\left\{m'\,.\,S'\,.\,y'\,.\left(\frac{\delta s'}{\delta x'}\right)-m'\,.\,S'\,.\,x'\,.\left(\frac{\delta s'}{\delta y'}\right)\right\}.$$

It is evident that we may change, in this equation, either x, x', x'', &c., or [121′] y, y', y'', &c., into z, z', z'', &c.; this will give two other equations, which, connected with the preceding, will furnish the following system:

[122]
$$\left.\begin{aligned} 0&=\Sigma.mS.\left\{y.\left(\frac{\delta s}{\delta x}\right)-x.\left(\frac{\delta s}{\delta y}\right)\right\}\\ 0&=\Sigma.mS.\left\{z.\left(\frac{\delta s}{\delta x}\right)-x.\left(\frac{\delta s}{\delta z}\right)\right\}\\ 0&=\Sigma.mS.\left\{y.\left(\frac{\delta s}{\delta z}\right)-z.\left(\frac{\delta s}{\delta y}\right)\right\}\end{aligned}\right\};\qquad (n)$$

[122′] The function $\Sigma.m.S.y.\left(\frac{\delta s}{\delta x}\right)$ is, by § 3 [29′], the sum of the momenta of all the forces, parallel to the axis of x, to make the system turn about the axis of z. Likewise the function $\Sigma.mS.x.\left(\frac{\delta s}{\delta y}\right)$ is the sum of the momenta of all the forces parallel to the axis of y, to make the system turn about the axis of z, but in a contrary direction to the first forces; the first of [122″] the equations (n) [122], therefore indicates, that the sum of the momenta of the forces, relative to the axis of z, is nothing. The second and third of these equations indicate, in like manner, that the sum of the momenta of the forces is nothing with respect to the axis of y, or x. Uniting these three conditions to the former [119‴], namely, that the sum of the forces, parallel to these axes, is nothing relative to each of them; we shall have the six conditions of the equilibrium of a system of bodies, invariably connected together.

If the origin of the co-ordinates is at rest, and invariably attached to the system, it will destroy the forces parallel to the three axes, and the conditions of the equilibrium of the system, about this origin, will be reduced to the following, that the sum of the momenta of the forces, to make the [122‴] system turn about these three axes, is nothing relative to each of them.

We shall suppose the bodies m, m', m'', &c., to be acted upon by no other force than gravity. Its action is the same upon all the bodies, and we may [122$^{\text{iv}}$] suppose its direction to be the same through the whole extent of the system;

The sum of all the similar terms of the equation [110″], being divided by $\delta\varpi$, becomes as in [121], which is the same as the first of the equations [122]. The other two equations are found in precisely the same manner, changing the co-ordinates as in [121′].

therefore we shall have*

$$S = S' = S'' = \&c.\,;$$

$$\left(\frac{\delta s}{\delta x}\right) = \left(\frac{\delta s'}{\delta x'}\right) = \left(\frac{\delta s''}{\delta x''}\right) = \&c.\,;$$

$$\left(\frac{\delta s}{\delta y}\right) = \left(\frac{\delta s'}{\delta y'}\right) = \left(\frac{\delta s''}{\delta y''}\right) = \&c.\,;$$

$$\left(\frac{\delta s}{\delta z}\right) = \left(\frac{\delta s'}{\delta z'}\right) = \left(\frac{\delta s''}{\delta z''}\right) = \&c.\,;$$ [123]

the three equations (n) [122], will be satisfied, whatever be the direction of s, or, in other words, whatever be the direction of gravity, by means of the three following equations :†

$$0 = \Sigma\,.\,mx\,; \qquad 0 = \Sigma\,.\,my\,; \qquad 0 = \Sigma\,.\,mz\,; \qquad (o)$$ [124]

the origin of the co-ordinates, being supposed fixed, will destroy the three forces $S.\left(\frac{\delta s}{\delta x}\right).\Sigma.m$; $S.\left(\frac{\delta s}{\delta y}\right).\Sigma.m$; $S.\left(\frac{\delta s}{\delta z}\right).\Sigma.m$; parallel to each

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* (55*a*) The action of gravity being the same upon all the bodies, makes

$$S = S' = S'' = \&c.;$$

as in the first of the equations [123]. If we refer to the figure in page 8, we shall have [13*c*], $\left(\frac{\delta s}{\delta x}\right) = \frac{x-a}{s} = \frac{AD}{Ac} = \cos.\,c\,A\,D$, $\left(\frac{\delta s}{\delta y}\right) = \frac{y-b}{s} = \frac{Aa}{Ac} = \cos.\,c\,A\,a$,

$$\left(\frac{\delta s}{\delta z}\right) = \frac{z-c}{s} = \frac{AB}{Ac} = \cos.\,c\,A\,B\,;$$

therefore $\left(\frac{\delta s}{\delta x}\right)$, $\left(\frac{\delta s}{\delta y}\right)$, $\left(\frac{\delta s}{\delta z}\right)$, represent the cosines of the angles which the line s makes [123*a*] with lines drawn parallel to the axes of x, y, z, respectively. In like manner $\left(\frac{\delta s'}{\delta x'}\right)$, $\left(\frac{\delta s'}{\delta y'}\right)$, $\left(\frac{\delta s'}{\delta z'}\right)$, represent the cosines of the angles, which the line s' makes, with lines drawn parallel to the same axis, and so on for the other lines, s'', s''', &c. Now since the lines s, s', &c. are parallel [122^{iv}], we shall have, $\left(\frac{\delta s}{\delta x}\right) = \left(\frac{\delta s'}{\delta x'}\right) = \&c.$, $\left(\frac{\delta s}{\delta y}\right) = \left(\frac{\delta s'}{\delta y'}\right) = \&c.$, $\left(\frac{\delta s}{\delta z}\right) = \left(\frac{\delta s'}{\delta z'}\right) = \&c.$, as in [123].

† (56) The forces S, S', S'', &c., being equal [123], as well as $\left(\frac{\delta s}{\delta x}\right)$, $\left(\frac{\delta s'}{\delta x'}\right)$, &c. $\left(\frac{\delta s}{\delta y}\right)$, $\left(\frac{\delta s'}{\delta y'}\right)$, &c., $\left(\frac{\delta s}{\delta z}\right)$, $\left(\frac{\delta s'}{\delta z'}\right)$; these quantities may be brought from under the sign Σ,

of the three axes;* by the composition of these three forces, they will
[124′] produce the single force $S\,.\,\Sigma\,.\,m$, which is equal to the weight of the system.

[124″] This origin of the co-ordinates, about which we have supposed the system to be in equilibrium, is a very remarkable point, because if this point is sustained, and gravity acts only upon the system, it will remain in equilibrium, in whatever situation we may place it about this point, which is called the
Centre of Gravity. *centre of gravity* of the system. The position is found by the condition, that if any plane whatever be made to pass through the centre of gravity, the sum of the products of each body, by its distance from that plane will be nothing.
[124‴] For this distance is a linear function of the co-ordinates of the body $x, y, z,$†

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in the equations [122], which will then become

$$S\,.\left(\frac{\delta s}{\delta x}\right).\,\Sigma\,.\,m\,y - S\,.\left(\frac{\delta s}{\delta y}\right).\,\Sigma\,.\,m\,x = 0\,;$$

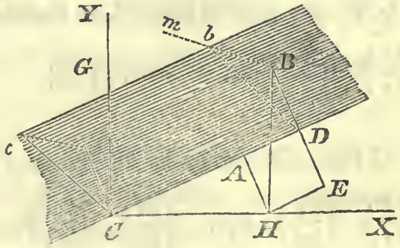
$$S\,.\left(\frac{\delta s}{\delta x}\right).\,\Sigma\,.\,m\,z - S\,.\left(\frac{\delta s}{\delta z}\right).\,\Sigma\,.\,m\,x = 0\,;$$

$$S\,.\left(\frac{\delta s}{\delta z}\right).\,\Sigma\,.\,m\,y - S\,.\left(\frac{\delta s}{\delta y}\right).\,\Sigma\,.\,m\,z = 0\,;$$

these three equations are evidently satisfied by means of the equations [124].

* (57) These forces are similar to those used in [119″ &c.], bringing the same
[124a] terms from under the sign Σ, as in the preceding note. Now $S\,.\left(\frac{\delta s}{\delta x}\right)$, $S\,.\left(\frac{\delta s}{\delta y}\right)$, $S\,.\left(\frac{\delta s}{\delta z}\right)$, [13], represent the force S, resolved in a direction parallel to x, y, z, and the composition of these three forces will again produce the single force S. Multiplying all these by $\Sigma\,.\,m$, it will follow that the three forces $S\,.\left(\frac{\delta s}{\delta x}\right).\,\Sigma\,.\,m$; $S\,.\left(\frac{\delta s}{\delta y}\right).\,\Sigma\,.\,m$; $S\,.\left(\frac{\delta s}{\delta z}\right).\,\Sigma\,.\,m$, in the directions parallel to x, y, z, will produce the single force $S\,.\,\Sigma\,.\,m$, in the direction of the origin of that force.

† (57a) Suppose the body m to be placed at m, in the annexed figure, (which is the same as that in page 13), upon the continuation of the line $B\,b$, so that its rectangular co-ordinates may be $C\,H = x$, $H\,B = y$, $B\,m = z$, and let the co-ordinate $B\,m$ be intersected in b, by a plane $C\,D\,b\,c$ passing through the centre of gravity of the system C, the ordinate $B\,b$ being denoted by the accented letter z'. Then



and by multiplying this distance by the mass of the body, the sum of these products will be nothing, in consequence of the equations (*o*) [124.]

To determine the position of the centre of gravity, let X, Y, Z, be its three co-ordinates, referred to a given point; x, y, z, the co-ordinates of m, referred to the same point; x', y', z', those of m', and in the same manner for the rest, the equations (*o*) [124], will give*

$$0=\Sigma . m . (x-X); \qquad [125]$$

but we have $\Sigma . m . X = X . \Sigma . m$, $\Sigma . m$ being the whole mass of the system; hence we shall have

$$X=\frac{\Sigma . mx}{\Sigma . m}. \qquad [126]$$

In like manner,

$$Y=\frac{\Sigma . my}{\Sigma . m}; \qquad Z=\frac{\Sigma . mz}{\Sigma . m}; \qquad [127]$$

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the general equation of the plane [19*c*] $z'=Ax+By$, gives

$$bm=(Bm-Bb=z-z')=z-Ax-By.$$

Now if from the point m we let fall, upon the plane $CDbc$, a perpendicular p, this perpendicular, or distance of the body m from the plane, will be equal to bm multiplied by the sine of the inclination of bm to that plane; and this inclination is evidently equal to the angle DbB, whose complement bDB was named φ in [19*b*‴], hence

$$p=bm . \sin. DbB=(z-Ax-By) . \cos. \varphi, \qquad [125a]$$

which is linear, or of the first degree, in x, y, z, as was observed above. Accenting the letters p, z, x, y, with one accent, for the body m', and with two accents for m'', &c., we obtain Linear Function.

$$p'=(z'-Ax'-By') . \cos. \varphi, \quad p''=(z''-Ax''-By'') . \cos. \varphi, \text{ \&c.}$$

Multiplying these respectively, by m, m', m'', &c., and adding these products together, we get

$$\Sigma . mp=\cos. \varphi . \Sigma . mz-A . \cos. \varphi . \Sigma . mx-B . \cos. \varphi . \Sigma . my.$$

Which, by substituting the values of Σmx, Σmy, Σmz, [124], becomes $\Sigma . mp=0$, as in [124‴].

* (57*b*) In the equations [124], the co-ordinates, x, y, z, are referred to the centre of gravity of this system, [124″], but if we count them from another point, which would make the co-ordinates of that centre X, Y, Z, it is evident, that the co-ordinates of the body, referred to that centre, would be $x-X$, $y-Y$, $z-Z$, which are to be substituted in [124], for x, y, z; and the first equation [124] becomes as in [125], which gives

$$\Sigma . mx=\Sigma . mX,$$

and as X is the same for all the bodies m, m', &c., we may put $\Sigma . m X=X . \Sigma . m$, hence $\Sigma . mx=X . \Sigma . m$, as above.

Therefore the co-ordinates X, Y, Z, correspond but to one point,* consequently there is but one centre of gravity of a system of bodies. The three preceding equations give

[128] $$X^2+Y^2+Z^2=\frac{(\Sigma.mx)^2+(\Sigma.my)^2+(\Sigma.mz)^2}{(\Sigma.m)^2},$$

which may be put under this form,†

[129] $$X^2+Y^2+Z^2=\frac{\Sigma.m.(x^2+y^2+z^2)}{\Sigma.m}-\frac{\Sigma.mm'.\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}{(\Sigma.m)^2},$$

the finite integral $\Sigma.mm'.\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}$ expressing the sum of all the products similar to that under the sign Σ, formed by combining all the bodies, two by two. We shall therefore have the distance of the centre of gravity, from any fixed point whatever, by means of the distances of the bodies of the system from the same fixed point, and from each other. By

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* (57c) Because the equations [126, 127], give but one value of X, one of Y, and one of Z.

† (58) Both these expressions of $X^2+Y^2+Z^2$, [128, 129], are symmetrical in x, y, z, x', y', z', &c. To prove therefore their identity, it is only necessary to show, that the coefficient of any one of these quantities, as x, is the same in the second members of both these values. This requires that the coefficient of x should be the same in both members of $\frac{(\Sigma.mx)^2}{(\Sigma.m)^2}=\frac{\Sigma.mx^2}{\Sigma.m}-\frac{\Sigma.mm'(x'-x)^2}{(\Sigma.m)^2}$, or

[129a] $$(\Sigma.mx)^2=\Sigma m.\Sigma mx^2-\Sigma mm'(x'-x)^2.$$

Substituting the values of $\Sigma\, mx$, Σm, &c., and retaining only the terms multiplied by x, we get

[129b] $$\begin{aligned}(\Sigma.mx)^2&=(mx+m'x'+m''x''+\&c.)^2=m^2x^2+2mx(m'x'+m''x''+\&c);\\ \Sigma m.\Sigma mx^2&=(m+m'+\&c.).(mx^2+m'x'^2+\&c.)=m^2x^2+mx^2(m'+m''+\&c.)\\ -\Sigma mm'(x'-x)^2&=-mm'(x'-x)^2-mm''(x''-x)^2-\&c.\\ &=-mx^2(m'+m''+\&c.)+2mx(m'x'+m''x''+\&c.;\end{aligned}$$

hence

$$\Sigma.m.\Sigma.mx^2-\Sigma.mm'(x'-x)^2=m^2x^2+2mx(m'x'+m''x''+\&c.),$$

which, being equal to the development of $(\Sigma mx)^2$ [129b], proves that the coefficients of x, in both members of [129a] are equal.

It may be observed, that the quantities which occur in the second member of [129], are the squares of the distances of the bodies m, m', &c. from the origin, represented by $x^2+y^2+z^2$, $x'^2+y'^2+z'^2$, &c. [19e], and the squares of their mutual distances f, f', &c. [118]; as is observed in [129'].

determining in this manner the distance of the centre of gravity from any [129′]
three fixed points whatever, we shall have its position in space; which
furnishes a new method of determining it.*

New method of finding the Centre of Gravity.

The name of *centre of gravity* has been extended to the point determined by the three co-ordinates X, Y, Z, of any system of bodies, whether they are acted upon by gravity or not.

Equilibrium of a solid body.

16. It is easy to apply the preceding results, to the equilibrium of a solid body of any figure whatever, by supposing it to be formed of an infinite number of particles, invariably connected together. Let dm be one of these
points or infinitely small particles of the body; x, y, z the rectangular [129″]
co-ordinates of that particle; P, Q, R the forces acting upon it, in directions parallel to the axes x, y, z; the equations (m) [119″], and (n) [122] of the preceding article will become†

$$0=\int P.dm\,; \qquad 0=\int Q.dm\,; \qquad 0=\int R.dm\,; \qquad [130]$$

$$0=\int(Py-Qx).dm\,; \qquad 0=\int(Pz-Rx).dm\,; \qquad 0=\int(Ry-Qz).dm\,; \qquad [131]$$

the sign of integration $\int$ refers to the particle dm, and must be extended to the whole mass of the solid.

If the body be so fixed, that it can only turn about the origin of the [131′]
co-ordinates, the three last equations will be sufficient for its equilibrium.

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* (59) These three fixed points may be considered as the angular points of the base of a triangular pyramid, whose vertex is the centre of gravity; and it is evident, that when the base is given, the vertex may be found, by means of the length of the three lines, drawn from those angular points to the vertex.

† (59a) Substituting for the forces $S.\left(\frac{\delta s}{\delta x}\right)$, $S.\left(\frac{\delta s}{\delta y}\right)$, $S.\left(\frac{\delta s}{\delta z}\right)$, [124$a$], their values P, Q, R, respectively [129″]; also putting dm for m, and $\int$ for Σ. This changes [119″] into [130], and [122] into [131].

CHAPTER IV.

ON THE EQUILIBRIUM OF FLUIDS.

17. To obtain the laws of the equilibrium and of the motion of each of the particles of a fluid, it would be necessary to ascertain their figure, which is impossible; but as these laws are required only for the fluids considered in a mass, the knowledge of the figure of the particles becomes useless. Whatever may be these figures, and the dispositions which result in the separate particles, all fluids, taken in a mass, must present the same phenomena, in their equilibrium, and in their motions; so that the observation of these phenomena will not enable us to discover anything respecting the configuration of the particles of the fluid. These general phenomena depend on the perfect mobility of the particles, which yield to the least pressure. This mobility is the characteristic property of fluids; it distinguishes them from solid bodies, and serves to define them. Hence it follows, that to maintain the equilibrium of a fluid mass, each particle ought to be held in equilibrium, by means of all the forces acting on it, and the pressure which it sustains from the surrounding particles. Let us now investigate the equations resulting from this property.

Mobility of Fluids.

[131″]

Equilibrium of Fluids.

For that purpose we shall consider a system of fluid particles, forming an infinitely small rectangular parallelopiped. Let x, y, z, be the three rectangular co-ordinates of that angle of the parallelopiped, which is nearest to the origin of the co-ordinates; dx, dy, dz the three dimensions of the parallelopiped; p the mean of all the pressures upon the different points of the face $dy \,.\, dz$ of the parallelopiped, nearest to the origin of the co-ordinates; and p' the same quantity relative to the opposite face. The parallelopiped will therefore be urged, by these pressures, in a direction parallel to the axis of x, by a force equal to $(p - p') \,.\, dy \,.\, dz$. $p' - p$ is the differential of p, considering x only as variable; for, although the pressure p' acts in a contrary direction to p, yet the pressure upon a particle of fluid being the same in all directions, $p'-p$ may be considered as the difference of two forces, acting in the same direction, at an infinitely small distance from each other; therefore we shall have, $p'-p=\left(\frac{dp}{dx}\right).dx$; and* $(p-p').dy.dz=-\left(\frac{dp}{dx}\right).dx.dy.dz.$ [131‴]

Let P, Q, R, be the three accelerating forces, which also act on the fluid particle, parallel to the axes of x, y, z;† if we call the density of the parallelopiped ρ, its mass will be $\rho.dx.dy.dz$, and the product of the force P by this mass, will be the whole resulting force which tends to move it; consequently the mass will be urged in a direction parallel to the axis of x, by the force $\left\{\rho P-\left(\frac{dp}{dx}\right)\right\}.dx.dy.dz.$ In like manner it will be urged [131^iv] in directions parallel to the axes of y and z, by the forces

$\left\{\rho Q-\left(\frac{dp}{dy}\right)\right\}.dx.dy.dz$; and $\left\{\rho R-\left(\frac{dp}{dz}\right)\right\}.dx.dy.dz$; [131^v]

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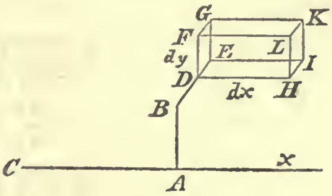
* (60) Let $DEFGHIKL$ be the infinitely small rectangular parallelopiped, the co-ordinates of its angular point D being $CA=x$, $AB=y$, $BD=z$, its sides $DH=dx$, $DF=dy$, $DE=dz$; area of the parallel faces $DEGF$, $HIKL=dy.dz$. Now the pressure upon the face of $DEGF$ is p, in the direction parallel to DH or x, and tending to increase x; p being, in general, a function of x, y, z. Therefore the parallelopiped is pressed in the direction parallel to DH or x, by the force $p.dy.dz$. Now if x were increased by dx, without varying y, z, the point for which the pressure is computed, would be changed from D to H, and we should obtain the pressure at the point H, from the preceding value of p, which would become $p'=p+\left(\frac{dp}{dx}\right)dx$, by the common principles of the differential calculus, the direction of the pressure being the same. But as fluids press in every direction, the face $HIKL$ must be pressed backwards, towards the origin of x, by the force $p'.dy.dz$; the difference of these two forces, $(p'-p).dy.dz$ or $\left(\frac{dp}{dx}\right)dx.dy.dz$ represents the whole pressure, suffered by the parallelopiped, in the direction HD, and as this tends to decrease x, the negative sign must be prefixed, and it becomes



$$-\left(\frac{dp}{dx}\right).dx.dy.dz,$$

as in [131‴].

† (60*a*) These forces are supposed to tend to *increase* the co-ordinates.

we shall therefore have, by means of the equation (*b*) § 3 [18],

[132] $$0=\left\{\rho P-\left(\frac{dp}{dx}\right)\right\}.\delta x+\left\{\rho Q-\left(\frac{dp}{dy}\right)\right\}.\delta y+\left\{\rho R-\left(\frac{dp}{dz}\right)\right\}.\delta z\,;$$

or*

[133] $$\delta p=\rho.\{P.\delta x+Q.\delta y+R.\delta z\}.$$

The second member of this equation ought to be, like the first, an exact variation,† which gives the following equations of partial differentials:

[134] $$\left(\frac{d.\rho P}{dy}\right)=\left(\frac{d.\rho Q}{dx}\right);\qquad \left(\frac{d.\rho P}{dz}\right)=\left(\frac{d.\rho R}{dx}\right);\qquad \left(\frac{d.\rho Q}{dz}\right)=\left(\frac{d.\rho R}{dy}\right);$$

whence we deduce

[135] $$0=P.\left(\frac{dQ}{dz}\right)-Q.\left(\frac{dP}{dz}\right)+R.\left(\frac{dP}{dy}\right)-P.\left(\frac{dR}{dy}\right)+Q.\left(\frac{dR}{dx}\right)-R.\left(\frac{dQ}{dx}\right).\ddagger$$

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* (60*b*) Substituting in [132], for $\left(\frac{dp}{dx}\right)\delta x+\left(\frac{dp}{dy}\right)\delta y+\left(\frac{dp}{dz}\right)\delta z$, its value, δp, [14*a*], it becomes as in [133].

† (61) The second member of [133] being an exact variation of p, gives $\left(\frac{dp}{dx}\right)=\rho P$,
[133*a*] $\left(\frac{dp}{dy}\right)=\rho Q$, $\left(\frac{dp}{dz}\right)=\rho R$. The differential of the first being taken, relative to y, and that of the second, relative to x, the first members of both expressions will be $\left(\frac{ddp}{dxdy}\right)$; hence $\left(\frac{d.\rho P}{dy}\right)=\left(\frac{d.\rho Q}{dx}\right)$. In a similar way, the other equations [134] were deduced from
[134*a*] $\left(\frac{ddp}{dxdz}\right)$, $\left(\frac{ddp}{dydz}\right)$. These are the well known equations of condition, of the integrability of a function p of three variable quantities x, y, z.

‡ (62) Developing the three equations [134], and transposing the terms to one side, we get

$$P.\left(\frac{d\rho}{dy}\right)+\rho.\left(\frac{dP}{dy}\right)-Q.\left(\frac{d\rho}{dx}\right)-\rho.\left(\frac{dQ}{dx}\right)=0;$$
$$P.\left(\frac{d\rho}{dz}\right)+\rho.\left(\frac{dP}{dz}\right)-R.\left(\frac{d\rho}{dx}\right)-\rho.\left(\frac{dR}{dx}\right)=0;$$
$$Q.\left(\frac{d\rho}{dz}\right)+\rho.\left(\frac{dQ}{dz}\right)-R.\left(\frac{d\rho}{dy}\right)-\rho.\left(\frac{dR}{dy}\right)=0.$$

Multiply the first by R, the second by $-Q$, and the third by P, and add these products together; the coefficients of the terms $\left(\frac{d\rho}{dx}\right)$, $\left(\frac{d\rho}{dy}\right)$, $\left(\frac{d\rho}{dz}\right)$, will vanish; and the rest, divided by ρ, will become as in [135].

This equation expresses the relation which ought to exist between the forces P, Q, R, to render the equilibrium possible.

If the fluid is free at its surface, or in any parts of its surface, the value of p will be nothing in those parts; in which case we shall have* $\delta p=0$, provided we take the variations δx, δy, δz, so as to appertain to this surface; therefore, by fulfilling these conditions, we shall have

$$0=P.\delta x+Q.\delta y+R.\delta z. \qquad [136]$$

Let $\delta u=0$ be the differential equation of the surface, we shall have†

$$P.\delta x+Q.\delta y+R.\delta z=\lambda.\delta u, \qquad [137]$$

λ being a function of x, y, z; hence it follows, from § 3, that the resultant of the forces P, Q, R, ought to be perpendicular to those parts of the surface where the fluid is free.‡ [137′]

Equation of Equilibrium at the surface.

Suppose that the quantity $P.\delta x+Q.\delta y+R.\delta z$ is an exact variation, which is the case by § 2, when P, Q, R, are the result of attractive forces.§ Put this variation equal to $\delta\varphi$, or

$$\delta\varphi=P.\delta x+Q.\delta y+R.\delta z, \qquad [137'']$$

---

* (63) For if the pressure p, *in the direction of the tangent of the surface*, is of any magnitude, the fluid would yield to that pressure, in those parts of the surface where it is free, and this motion would continue till the particles had assumed the state corresponding to $\delta p=0$, and then [133] would change into [136].

† (63*a*) From $\delta u=0$, and the formula [136], we obtain [137], as [19″] was found in note 15.

‡ (64) The three forces P, Q, R, acting in directions parallel to x, y, z, may be reduced to one force V, acting in the direction r; so that by the formula [16], we should have $P.\delta x+Q.\delta y+R.\delta z=V.\delta r$, and this, by means of [136], becomes $V.\delta r=0$; hence in general, V being finite, we shall have $\delta r=0$. Now δr cannot be equal to nothing, unless the line r be drawn perpendicular to the surface [19*a*]. Therefore the resultant of the forces P, Q, R, must be perpendicular to the surface, in those parts where the fluid is free. This is also evident of itself. For if the resultant of the forces, acting upon a particle of the fluid, at the surface, was not in the direction of the normal, it might be resolved into two forces, the one in the direction of the normal, the other in the direction of the tangent, and this last would, as was observed above, cause the particle to move, on the surface, and destroy the equilibrium.

§ (64*a*) As is shown in note 13*f*.

and we shall have

[137'''] $$\delta p = \rho . \delta \varphi ;$$

therefore ρ must be a function of p and φ;* and since the integration of this equation gives φ in terms of p, we shall also have p expressed in a function of ρ. Consequently the pressure p is the same for all particles of the same density; therefore dp is nothing, relative to the surfaces of the strata of the fluid mass, in which the density is constant, and as it respects these surfaces we shall have†

[138] $$0 = P . \delta x + Q . \delta y + R . \delta z.$$

---

* (65) This is evident, because the equation $\delta p = \rho \delta \varphi$, contains the variations δp and $\delta \varphi$, which could not be integrated, unless ρ was a function of p, φ, and any constant quantities.

† (65*a*) By hypothesis dp or $\delta p = 0$. If we substitute this in [133], and divide by ρ, we shall get [138], and from this last equation we find, as in note 64, that the result of the forces P, Q, R, is perpendicular to the level surface at that part.

It follows from what has been said in this chapter, that no heterogeneous mass of fluid can remain in equilibrium, unless each level stratum be homogeneous throughout its whole extent. This is the only condition required, when the fluid completely fills a vessel, which is closed on every side; but if any part remain open, it is also necessary, [137'], that the resultant of all the forces, at that part, should be perpendicular to the surface.

As an example of the use of the formula [138], we may apply it to the investigation of the form of the level strata, when the force, acting upon the particles of the fluid, is reduced to one single force S, tending towards the origin of the co-ordinates x, y, z, and varying, as any function of the distance s, of the particle from that origin. This force, resolved in

[137*a*] directions parallel to the axes x, y, z, will be $S . \frac{x}{s}$, $S . \frac{y}{s}$, $S . \frac{z}{s}$, respectively, as in [13*a*]; observing that, the force S is supposed to be situated at the origin of the co-ordinates, and we may therefore put $a = 0$, $b = 0$, $c = 0$. Substitute these for P, Q, R, in the equation of the stratum [138], and it will become $\frac{S}{s} . (x \delta x + y \delta y + z \delta z) = 0$, or

$$x \delta x + y \delta y + z \delta z = 0;$$

whose integral is $x^2 + y^2 + z^2 =$ constant. This corresponds to a spherical surface [19*e*],

[137*b*] the constant quantity being equal to the square of the radius s. Therefore the level strata will, in this hypothesis, be concentrical spherical surfaces.

The equation [136] might also be applied to the computation of the figure of the upper surface of a fluid, contained in a vertical cylinder, open at the top, and revolving uniformly about its vertical axis z, with the angular velocity n. In this case the centrifugal force,

Hence it follows, that the resultant of the forces acting upon each particle of the fluid, when in a state of equilibrium, is perpendicular to the surfaces of those strata; which, for that reason, are called *level strata*, or *level* [138′]
surfaces. This condition is always fulfilled, when the fluid is homogeneous and incompressible; since then the strata, to which this resultant is perpen- [138″]
dicular, are all of the same density.

Therefore, to support the equilibrium of a homogeneous mass of fluid, whose exterior surface is free, and which contains within it a fixed solid nucleus, of any figure whatever, it is requisite, and it is sufficient; *First*, [138‴]
that $P \cdot \delta x + Q \cdot \delta y + R \cdot \delta z$ should be an exact differential; *Second*, that the resultant of the forces acting on the exterior surface should be perpen- [138^iv]
dicular to the surface, and should be directed towards the inner part of the fluid.

---

arising from the rotation, may be considered as an actual force applied to the particles. Now if ρ be the distance of a particle from the axis of the cylinder, its rotatory velocity will be $n \rho$, and its centrifugal force [54′], being equal to the square of the velocity, divided by the radius, will be $n^2 . \rho$. This force is in the direction of the radius ρ, or AW, (Fig. 2, [138a]
page 20); it may be resolved into two forces, parallel to the ordinates $AX = x$, $XW = y$, and will be represented by $P = n^2 . x$, $Q = n^2 . y$, [11]; these forces tending to increase the co-ordinates. Moreover, the force of gravity g tends to decrease the ordinate z, so that $R = -g$. Substituting these in the differential equation of the surface [136], it becomes $n^2 . (x \, \delta x + y \, \delta y) - g \, \delta z = 0$, whose integral is $\frac{1}{2} . n^2 . (x^2 + y^2) - g . z = \text{constant}$. If we suppose x, y, z to commence together, the constant quantity will be nothing, and by putting $n^2 = 2 g a$, this equation will become $z = a (x^2 + y^2)$, but $x^2 + y^2 = \rho^2$ [27], therefore the equation of the surface will be $z = a . \rho^2$. This is the equation of a parabola [64b]. Therefore the figure of the upper surface of the fluid is that of an inverted parabolic conoid. This subject is treated of in a different manner in [323a].

It may be observed that the preceding values of P, Q, R, satisfy the equation of condition [135], since each term of that equation vanishes.

CHAPTER V.

GENERAL PRINCIPLES OF THE MOTION OF A SYSTEM OF BODIES.

18. We have, in § 7, reduced the laws of motion of a point, or particle, to those of its equilibrium, by resolving its motion, at any instant of time, into two others, one of which remains in the next instant, and the other is destroyed by the forces acting on that point; the equilibrium between these forces and the motion lost by the body, has given us the differential equations of its motion. We shall now make use of the same method, to determine
[138$^{\mathrm{v}}$] the motion of a system of bodies m, m', m'', &c. Therefore let mP, mQ, mR, be the forces which act on the body m, parallel to the axes of its rectangular co-ordinates x, y, z; $m'P'$, $m'Q'$, $m'R'$ the forces which act on m', parallel to the same axes,* and in the same manner for the others; and let t be the time. The partial forces $m.\frac{dx}{dt}$, $m.\frac{dy}{dt}$, $m.\frac{dz}{dt}$, of the body m, at any instant, will become, in the next instant,†

[139]
$$m.\frac{dx}{dt}+m.d.\frac{dx}{dt}-m.d.\frac{dx}{dt}+mP.dt;$$
$$m.\frac{dy}{dt}+m.d.\frac{dy}{dt}-m.d.\frac{dy}{dt}+mQ.dt;$$
$$m.\frac{dz}{dt}+m.d.\frac{dz}{dt}-m.d.\frac{dz}{dt}+mR.dt;$$

and as the following forces only are retained,

[140]
$$m.\frac{dx}{dt}+m.d.\frac{dx}{dt}; \qquad m.\frac{dy}{dt}+m.d.\frac{dy}{dt}; \qquad m.\frac{dz}{dt}+m.d.\frac{dz}{dt};$$

---

* (66) It may be observed that these forces are supposed to tend to increase the co-ordinates.

[139a] † (66a) The reasoning in this article is like that in page 31. The expressions [139], being like [36], [140] are like [36′], &c.

the other forces

$$-m.d.\frac{dx}{dt}+mP.dt;\qquad -m.d.\frac{dy}{dt}+mQ.dt;\qquad -m.d.\frac{dz}{dt}+mR.dt;$$ [141]

will be destroyed. By marking successively, in these expressions, the letters m, x, y, z, P, Q, R, with one accent, two accents, &c., we shall have the forces destroyed in the bodies m', m'', &c. This being premised, if we multiply these forces respectively by the variations $\delta x, \delta y, \delta z, \delta x'$, &c. of their directions, the principle of virtual velocities explained in § 14 [114'], will give, by supposing dt constant, the following equation:

$$\left.\begin{array}{l} 0=m.\delta x.\left\{\frac{ddx}{dt^2}-P\right\}+m.\delta y.\left\{\frac{ddy}{dt^2}-Q\right\}+m.\delta z.\left\{\frac{ddz}{dt^2}-R\right\} \\ \quad +m'.\delta x'.\left\{\frac{ddx'}{dt^2}-P'\right\}+m'.\delta y'.\left\{\frac{ddy'}{dt^2}-Q'\right\}+m'.\delta z'.\left\{\frac{ddz'}{dt^2}-R'\right\} \end{array}\right\};\quad (P)$$ [141']

&c., or*

$$0=\Sigma.m.\delta x.\left\{\frac{ddx}{dt^2}-P\right\}+\Sigma.m.\delta y.\left\{\frac{ddy}{dt^2}-Q\right\}+\Sigma.m.\delta z.\left\{\frac{ddz}{dt^2}-R\right\};\quad (P)$$ [142]

General Equation of the motion of a system of bodies.

we must exterminate from this equation, by means of the particular conditions of the system, as many variations as we have conditions, and then put the coefficients of the remaining variations separately equal to nothing; we shall thus have all the necessary equations, to determine the motions of the different bodies of the system.

19. The equation (P) [142] contains several general principles of motion, which we shall now proceed to develop. We shall evidently subject the variations $\delta x, \delta y, \delta z, \delta x'$, &c. to all the conditions of the connexion of the parts of the system, by supposing them equal to the differentials dx, dy, dz, dx', &c. This supposition can therefore be made, and the equation (P) [142] will give, by integration,†

$$\Sigma.m.\frac{(dx^2+dy^2+dz^2)}{dt^2}=c+2.\Sigma.\int m\,(P.dx+Q.dy+R.dz);\qquad (Q)$$ [143]

c being the constant quantity to complete the integral.

---

* (66b) The equation P [141'], is again repeated in [142], though it is not in the original. This was done because it is most commonly referred to under this last form.

† (66c) The equation [143] is found in the same manner as [39], in note 17h.

If P, Q, R, are the result of attractive forces, directed towards fixed points, and the mutual attractions of the bodies on each other; the function

[143'] $$\Sigma . \int m . (P . dx + Q . dy + R . dz)$$

will be an exact integral. For the parts of this function, depending on attractive forces, directed towards fixed points, are, by § 8,* exact integrals. The same is true with respect to the parts, depending on the mutual attractions of the bodies of the system upon each other. For, if we denote by f, the distance from m to m', $m'F$ the attraction of m' upon m; the part
[143''] of $m . (P . dx + Q . dy + R . dz)$, depending on the attraction of m' upon m,
will, by the article just named, be equal to† $-mm' . F . df$, the differential df being taken supposing only the co-ordinates x, y, z to be variable. But the reaction being equal and contrary to the action, the part of

$$m' . (P' . dx' + Q' . dy' + R' . dz'),$$

depending on the attraction of m upon m', is equal to $-mm' . F . df$, supposing only the co-ordinates x', y', z' to be variable in f; the part of the function
[143'''] $\Sigma . m . (P . dx + Q . dy + R . dz)$, depending on the reciprocal attraction of
m and of m', is therefore $-mm' . F . df$, all being supposed to vary in f. This quantity is an exact differential, when F is a function of f, or when the attraction varies as a function of the distance, which we shall suppose to be
[143^{iv}] always the case; the function $\Sigma . m . (P . dx + Q . dy + R . dz)$ is therefore an
exact differential, whenever the forces which act upon the bodies of the system, result from their mutual attraction, or from attractive forces, directed towards certain fixed points. Put therefore for this differential

[143^{v}] $$d\varphi = P . dx + Q . dy + R . dz;$$

---

* (66*d*) As was proved in note 13*f*.

† (66*e*) Suppose the force $m'F$, in the direction f, to be resolved into three forces P, Q, R, parallel to the axes of x, y, z, respectively; we should have, by formula [16], $-m'F . \delta f = P . \delta x + Q . \delta y + R . \delta z$. The negative sign being prefixed to δf, because the force $m'F$ tends to decrease f. This, multiplied by m, gives

$$m (P . \delta x + Q . \delta y + R . \delta z) = -mm' . F . \delta f;$$

consequently, the part of the general formula $m . (P . dx + Q . dy + R . dz)$, [143''], must evidently be of the form $-mm' . F . df$.

and let v be the velocity of m; v' that of m' &c.; we shall have*

$$\Sigma . m v^2 = c + 2\varphi. \qquad (R)$$ [144]

Vis viva, or living force.

This equation is analogous to the equation (g) § 8 [40]; it is the analytical expression of the principle of living forces. *The product of the mass of a body by the square of its velocity is called its living force*, and the principle just mentioned consists in this; that the sum of the living forces, or the living force of the whole system, is constant, if the system is not urged by any forces; and if the bodies are urged by any forces whatever, the sum of the increments of the whole living force is the same, whatever may be the curves described by the bodies, provided the points departed from and arrived at are the same.†

The living force is diminished by sudden changes.

This principle takes place only in those cases where the motions of the bodies change by insensible gradations. If these motions suffer sudden changes, the living force is diminished, by a quantity, which may be determined in the following manner. The analysis which led to the equation (P) [142] of the preceding article, will give, instead of it, the following:‡

$$0 = \Sigma . m . \left\{ \frac{\delta x}{dt} . \Delta . \frac{dx}{dt} + \frac{\delta y}{dt} . \Delta . \frac{dy}{dt} + \frac{\delta z}{dt} . \Delta . \frac{dz}{dt} \right\} - \Sigma . m . (P . \delta x + Q . \delta y + R . \delta z);$$ [145]

$\Delta . \frac{dx}{dt}$, $\Delta . \frac{dy}{dt}$, $\Delta . \frac{dz}{dt}$, being the differentials of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from one instant to another; which differences become finite, when the motions of the bodies receive finite alterations in an instant. We may suppose, in this equation,

$$\delta x = dx + \Delta . dx; \qquad \delta y = dy + \Delta . dy; \qquad \delta z = dz + \Delta . dz;$$ [146]

---

* (66f) The formula [144] is deduced from [143] in the same manner as [40] from [39] in note 17i.

† (67) This is demonstrated as in note 18b.

‡ (67a) The terms $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, &c., which occur in [142], are given in [139], under the form $d . \frac{dx}{dt}$, $d . \frac{dy}{dt}$, &c. If in these we change d into Δ, because the differentials of the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c. are finite, they become $\Delta . \frac{dx}{dt}$, $\Delta . \frac{dy}{dt}$, &c. and [142] changes into [145].

because the values of dx, dy, dz, become in the following instant, $dx+\Delta.dx$, $dy+\Delta.dy$, $dz+\Delta.dz$. These values of δx, δy, δz, satisfy the conditions of the connexion of the parts of the system; hence we shall have

[147] $$0=\Sigma.m.\left\{\left(\frac{dx}{dt}+\Delta.\frac{dx}{dt}\right).\Delta.\frac{dx}{dt}+\left(\frac{dy}{dt}+\Delta.\frac{dy}{dt}\right).\Delta.\frac{dy}{dt}+\left(\frac{dz}{dt}+\Delta.\frac{dz}{dt}\right).\Delta.\frac{dz}{dt}\right\}$$
$$-\Sigma.m.\{P.(dx+\Delta.dx)+Q.(dy+\Delta.dy)+R.(dz+\Delta.dz)\}.$$

This equation ought to be integrated like an equation of finite differences as it respects the time t, whose variations are infinitely small, as well as those of x, y, z, x', &c. We shall denote by $\Sigma_{,}$ the finite integrals resulting from this integration, in order to distinguish them from the preceding finite integrals relative to all the bodies of the system. The integral of [147′]

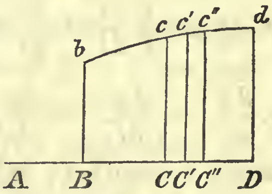
$$mP.(dx+\Delta.dx)$$

[147″] is evidently the same as $\int m.P.dx$;* therefore we shall have†

[148] $$\text{constant}=\Sigma.m.\frac{(dx^2+dy^2+dz^2)}{dt^2}+\Sigma_{,}.\Sigma.m.\left\{\left(\Delta.\frac{dx}{dt}\right)^2+\left(\Delta.\frac{dy}{dt}\right)^2+\left(\Delta.\frac{dz}{dt}\right)^2\right\}$$
$$-2.\Sigma.\int.m.(P.dx+Q.dy+R.dz);$$

---

* (68) Put for brevity $dx+\Delta.dx=\Delta'.dx$, then the integral of $m.P.(dx+\Delta.dx)$, relative to the characteristic $\Sigma_{,}$, becomes $\Sigma_{,}.m.P.\Delta'.dx$. The value of this expression may be conceived of, by supposing the curve bcd to be of such a nature that to any absciss $AC=x$, the perpendicular ordinate Cc may be $m.P$. For by taking the infinitely small quantity $CC''=\Delta'.dx$, and drawing the ordinate $C''c''$, the space $Ccc''C''$ will be



$$m.P.\Delta'.dx,$$

and the sum of all these elements relative to the characteristic $\Sigma_{,}$, will represent the whole curvilinear space $BbcC$; so that by taking the integral between the limits $x=AB$, and $x=AD$, we shall have $\Sigma_{,}.m.P.\Delta'.dx=$ space $BbdD$. In a similar manner, by taking $CC'=dx$, and drawing the ordinate $C'c'$, we shall have the space $Ccc'C'=m.P.dx$, and its integral relative to $\int$, taken between the same limits, will, by the usual rules of integration, give $\int m.P.dx=$ space $BbdD$, whence we have $\Sigma_{,}.m.P.\Delta'.dx=\int m.P.dx$, as in [147″]. [147a]

† (69) Multiplying the equation [147] by 2, and changing the arrangement of the terms, we shall find

$$0=\Sigma.m.\left\{\left(2.\frac{dx}{dt}+\Delta.\frac{dx}{dt}\right).\Delta.\frac{dx}{dt}+\left(2.\frac{dy}{dt}+\Delta.\frac{dy}{dt}\right).\Delta.\frac{dy}{dt}+\left(2.\frac{dz}{dt}+\Delta.\frac{dz}{dt}\right).\Delta.\frac{dz}{dt}\right\}$$
$$+\Sigma.m.\left\{\left(\Delta.\frac{dx}{dt}\right)^2+\left(\Delta.\frac{dy}{dt}\right)^2+\left(\Delta.\frac{dz}{dt}\right)^2\right\}$$
$$-2.\Sigma.m.\{P.(dx+\Delta.dx)+Q.(dy+\Delta.dy)+R.(dz+\Delta.dz)\}.$$

denoting therefore by v, v', v'', &c., the velocities of m, m', m'', &c., we shall have

$$\Sigma . m v^2 = \text{constant} - \Sigma_{,} . \Sigma . m . \left\{ \left(\Delta . \frac{dx}{dt}\right)^2 + \left(\Delta . \frac{dy}{dt}\right)^2 + \left(\Delta . \frac{dz}{dt}\right)^2 \right\} + 2 . \Sigma . \int . m . (P . dx + Q . dy + R . dz).$$ [149]

The quantity contained under the sign $\Sigma_{,}$ being necessarily positive,* it is evident, that the living force of the system is diminished by the mutual action of the bodies, whenever any of the variations $\Delta . \frac{dx}{dt}$, $\Delta . \frac{dy}{dt}$, &c., become finite, during the motion of the system. The preceding equation furnishes a very simple method for determining this diminution.

At each sudden change of motion in the system, we may suppose the velocity of m to be resolved into two others, one of which v remains in the following instant; the other V is destroyed by the action of the bodies; now the velocity of m being $\frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$, before this resolution [40a], and becoming afterwards† $\frac{\sqrt{(dx + \Delta . dx)^2 + (dy + \Delta . dy)^2 + (dz + \Delta . dz)^2}}{dt}$, it is [149']

Now

$$\left(2 . \frac{dx}{dt} + \Delta . \frac{dx}{dt}\right) . \Delta . \frac{dx}{dt} = \left(\frac{dx}{dt} + \Delta . \frac{dx}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 = \Delta . \left(\frac{dx}{dt}\right)^2,$$

whose integral relative to $\Sigma_{,}$ is $\left(\frac{dx}{dt}\right)^2$; and the similar terms relative to y, z, furnish the terms $\left(\frac{dy}{dt}\right)^2$, $\left(\frac{dz}{dt}\right)^2$. Again, the integral of $-2 . \Sigma . m . P . (dx + \Delta . dx)$, relative to the characteristic $\Sigma_{,}$, is by [147a], equal to $-2 . \Sigma . \int . m . P . dx$, and the similar terms in y, z, produce $-2 . \Sigma . \int . m . Q . dy$, $-2 . \Sigma . \int . m . R . dz$; with these reductions the integral of the preceding equation, relative to $\Sigma_{,}$, becomes as in [148], and by substituting for $\frac{(dx^2 + dy^2 + dz^2)}{dt^2}$ its value v^2, &c. [40a] it changes into [149].

* (70) Because each term of the expression, as $\left(\Delta . \frac{dx}{dt}\right)^2$, is a square, consequently positive.

† (71) This expression of the velocity at the second instant, is of the same form as that in [40a], changing the elements dx, dy, dz, corresponding to the first instant, into $dx + \Delta . dx$, $dy + \Delta . dy$, $dz + \Delta . dz$, corresponding to the second instant.

easy to perceive that we shall have*

[150] $$V^2 = \left(\Delta.\frac{dx}{dt}\right)^2 + \left(\Delta.\frac{dy}{dt}\right)^2 + \left(\Delta.\frac{dz}{dt}\right)^2;$$

the preceding equation can therefore be put under this form,

[151] $$\Sigma.mv^2 = \text{constant} - \Sigma_{,}.\Sigma.m\,V^2 + 2\Sigma.\textstyle\int.m(P.dx + Q.dy + R.dz.)$$

20. If in the equation (P) § 18 [142], we suppose

[152] $$\begin{array}{lll} \delta x' = \delta x + \delta x_{,}'; & \delta y' = \delta y + \delta y_{,}'; & \delta z' = \delta z + \delta z_{,}'; \\ \delta x'' = \delta x + \delta x_{,}''; & \delta y'' = \delta y + \delta y_{,}''; & \delta z'' = \delta z + \delta z_{,}''; \\ \&c. & & \end{array}$$

by substituting these variations in the expression of the variations δf, $\delta f'$, $\delta f''$, &c. of the mutual distances of the bodies of the system, whose values are given in § 15 [118]; we shall find that the variation δx, δy, δz, will disappear from these expressions. If the system is free, that is, if no one of its parts has any connexion with foreign bodies; the conditions relative to the mutual connexion of the bodies, will depend only upon their distances from each other, and the variations δx, δy, δz, will therefore be independent of these conditions;† whence it follows, that, if we substitute the preceding values of $\delta x'$, $\delta y'$, $\delta z'$, $\delta x''$, &c., in the equation P [142], we ought to put the coefficients of the variations of δx, δy, δz, separately equal to nothing; which will give these three equations,

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* (71*a*) The primitive velocity of the body m, in a direction parallel to the axis of x, is $\frac{dx}{dt}$, which after the first instant becomes $\frac{dx+\Delta.dx}{dt}$, consequently the loss of velocity, in that direction is $\frac{-\Delta.dx}{dt}$. In a similar manner the losses of velocity in the directions parallel to the axes of y and z, are respectively $\frac{-\Delta.dy}{dt}$, $\frac{-\Delta.dz}{dt}$, and the sum of the squares of these expressions is, as in [40*a*], evidently equal to the square of the whole loss of velocity, or V^2, as in [150]. The substitution of this, in [149], gives [151].

† (72) As the system is supposed to be wholly unconnected with any foreign body, we can suppose each one of the bodies to be moved through an equal space, in a parallel direction, without producing any change in their relative situations, or in their mutual actions upon each other. Therefore we may suppose each of the bodies to be moved through the arbitrary spaces δx, δy, δz, parallel to the three axes x, y, z, respectively; and these spaces may be varied at pleasure, without affecting the relative situation of the bodies; or, in other words, without affecting the values of $\delta x_{,}'$, $\delta x_{,}''$, &c. $\delta y_{,}'$, $\delta y_{,}''$, &c. $\delta z_{,}'$, $\delta z_{,}''$, &c. [152]. Therefore, if we substitute the values of [152] in the equation [142], it will not generally be satisfied, unless the coefficients of these arbitrary quantities δx, δy, δz, are put separately equal to nothing, hence we obtain the three equations [153].

$$0=\Sigma.m.\left(\frac{ddx}{dt^2}-P\right); \qquad 0=\Sigma.m.\left(\frac{ddy}{dt^2}-Q\right); \qquad 0=\Sigma.m.\left(\frac{ddz}{dt^2}-R\right). \qquad [153]$$

Let X, Y, Z be the three co-ordinates of the centre of gravity of the system, we shall have, by § 15, [126, 127], [153']

$$X=\frac{\Sigma.mx}{\Sigma.m}; \qquad Y=\frac{\Sigma.my}{\Sigma.m}; \qquad Z=\frac{\Sigma.mz}{\Sigma.m}; \qquad [154]$$

consequently*

$$0=\frac{ddX}{dt^2}-\frac{\Sigma.mP}{\Sigma.m}; \qquad 0=\frac{ddY}{dt^2}-\frac{\Sigma.mQ}{\Sigma.m}; \qquad 0=\frac{ddZ}{dt^2}-\frac{\Sigma.mR}{\Sigma.m}; \qquad [155]$$

therefore, the centre of gravity will move in the same manner, as if all the bodies m, m', m'', &c., were collected in that centre, and all the forces which act upon the different bodies of the system, were directly applied to the whole mass collected in that centre. [155']

Motion of the Centre of Gravity.

If the system is affected only by the mutual action of the bodies upon each other, and their reciprocal attractions; we shall have

$$0=\Sigma.mP; \qquad 0=\Sigma.mQ; \qquad 0=\Sigma.mR; \qquad [155'']$$

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* (73) The second differential of $X=\frac{\Sigma.mx}{\Sigma.m}$, is $ddX=\frac{\Sigma.mddx}{\Sigma.m}$, which, multiplied by $\frac{\Sigma.m}{dt^2}$, gives $\Sigma.m.\frac{ddx}{dt^2}=\frac{ddX}{dt^2}.\Sigma.m$. Substituting this in $0=\Sigma.m.\left(\frac{ddx}{dt^2}-P\right)$ [153], it becomes $0=\frac{ddX}{dt^2}.\Sigma.m-\Sigma.mP$, which, divided by $\Sigma.m$, gives $0=\frac{ddX}{dt^2}-\frac{\Sigma.mP}{\Sigma.m}$ [155]. The two other equations, in Y, Z, are found in the same manner. Now if all the bodies were collected in the centre of gravity, and all the forces applied to it, as above, the sum of all the forces in the direction parallel to the axis of x, would be $\Sigma.mP$. This divided by the sum of the masses, $\Sigma.m$, would give the accelerative force, acting upon *one* particle, equal to $\frac{\Sigma.mP}{\Sigma.m}$, which is what is called P in [38]. Therefore the first of the equations [38], for finding the motion of a particle, is similar to the first of [155]. In like manner the second and third of the equations [38] become like those of [155]; consequently the motion of the centre of gravity is found by the same equations as that of a single particle of the mass $\Sigma.m$, collected at the centre of gravity. The motion of the centre of gravity will, therefore, be exactly the same, as that of the congregated mass, supposing the forces to be applied at that centre, in the manner mentioned above. [155a]

For, if we put p to denote the reciprocal action of m upon m', whatever be its nature, and f for the mutual distance of these two bodies; we shall have, by means of this action only,*

[156]
$$mP=\frac{p.(x-x')}{f}; \qquad mQ=\frac{p.(y-y')}{f}; \qquad mR=\frac{p.(z-z')}{f};$$
$$m'P'=\frac{p.(x'-x)}{f}; \qquad m'Q'=\frac{p.(y'-y)}{f}; \qquad m'R'=\frac{p.(z'-z)}{f};$$

whence we deduce

[157]
$$0=mP+m'P'; \qquad 0=mQ+m'Q'; \qquad 0=mR+m'R';$$

and it is evident, that these equations exist, even when the bodies instantaneously exert, upon each other, a finite action; so that their reciprocal action must disappear from the integrals $\Sigma.mP$, $\Sigma.mQ$, $\Sigma.mR$, therefore these integrals will become nothing, when the system is not acted upon by extraneous forces. In this case, we shall have†

[158]
$$0=\frac{ddX}{dt^2}; \qquad 0=\frac{ddY}{dt^2}; \qquad 0=\frac{ddZ}{dt^2};$$

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* (74) Using the figure in page 8, let the body m, whose co-ordinates are x, y, z, be at A; the body m' whose co-ordinates are x', y', z', be at c, and the distance $Ac=f$. Then the force p, which we shall suppose to act upon the body m' in the directions Ac, would produce a force in a direction parallel to AD, represented by $p.\frac{AD}{Ac}=p.\frac{(x'-x)}{f}$, as is evident by the first of the formulas, [13]; this force is what is called above $m'P'$. From the same formula it follows, that the force p, acting upon the body m, in the direction cA, would produce a force in a direction parallel to DA, represented by

[156a]
$$p.\frac{AD}{Ac}=p.\frac{(x'-x)}{f},$$

or in other words, a force in the opposite direction AD represented by $p.\frac{(x-x')}{f}$, and called above mP. Adding this to the preceding value of $m'P'$, the sum becomes nothing, as in the first equation [157], and the two other equations [157], are found in the same manner for the other axes y, z.

† (74a) The equations [158] are deduced from [155], by substituting the values [155″]. The first integrals of [158], are $\frac{dX}{dt}=b$, $\frac{dY}{dt}=b'$, $\frac{dZ}{dt}=b''$. The square root of the sum of the squares of these is $\sqrt{\left(\frac{dX}{dt}\right)^2+\left(\frac{dY}{dt}\right)^2+\left(\frac{dZ}{dt}\right)^2}=\sqrt{b^2+b'^2+b''^2}$, the first

and by integration,

$$X=a+b\,t,\qquad\qquad Y=a'+b'\,t,\qquad\qquad Z=a''+b''\,t;$$ [159]

a, b, a', b', a'', b'', being arbitrary constant quantities. By exterminating the time t, we shall have an equation of the first order, between X and Y, or between X and Z; whence it follows, that the motion of the centre of gravity is rectilineal. Moreover, its velocity being equal to

$$\sqrt{\left(\frac{dX}{dt}\right)^2+\left(\frac{dY}{dt}\right)^2+\left(\frac{dZ}{dt}\right)^2},$$ [159′]

or to $\sqrt{b^2+b'^2+b''^2}$, it is constant, and the motion is uniform.

It is evident, from the preceding analysis, that this permanency in the motion of the centre of gravity of a system of bodies, whatever be their mutual action, exists even in the case, where some of the bodies lose [159″] instantaneously, by this action, a finite quantity of motion.*

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member of which represents the velocity of the centre of gravity, as is shown in [40a], therefore that velocity is equal to the constant quantity $\sqrt{b^2+b'^2+b''^2}$. Taking the integrals of $\frac{dX}{dt}=b$, &c., we obtain the expressions [159]. The value of t, deduced from the first, being substituted in the second and third, gives Y, Z, in equations of the form [159a] $Y=AX+A'$; $Z=BX+B'$, which are the equations of a *right* line [19b″], therefore the motion of the centre of gravity must be in a right line.

* (75) That the uniform motion of the centre of gravity, is not disturbed by a sudden change of the motions of some of the bodies of the system, arising from their mutual attractions, or impact, &c. may be proved, by means of the equation [145], in which this sudden change of motion is supposed to take place. For, by substituting, in this equation, the values of $\delta x'$, $\delta y'$, $\delta z'$, $\delta x''$, &c. [152], and putting, as in note 72, the coefficients of δx, δy, δz, equal to nothing, we get

$$0=\Sigma\,.\,m\,.\left(\Delta.\frac{dx}{dt}-P\right);\quad 0=\Sigma\,.\,m.\left(\Delta.\frac{dy}{dt}-Q\right);\quad 0=\Sigma\,.\,m\,.\left(\Delta.\frac{dz}{dt}-R\right).$$

The first of these equations gives $\Sigma\,.\,m\,.\,\Delta\,.\,\frac{dx}{dt}=\Sigma\,.\,mP$. Now the differential of the first of the equations [154], divided by $\frac{dt}{\Sigma.m}$ is $\Sigma\,.\,m\,.\,\frac{dx}{dt}=\frac{dX}{dt}.\Sigma\,.\,m$, and its differential relative to Δ, is $\Sigma\,.\,m\,.\,\Delta\,.\,\frac{dx}{dt}=\Delta\,.\,\frac{dX}{dt}.\Sigma\,.\,m$. Substituting this, in the former equation, it becomes $\Delta.\frac{dX}{dt}.\Sigma\,.\,m=\Sigma\,.\,mP$, or $\Delta\,.\,\frac{dX}{dt}=\frac{\Sigma.mP}{\Sigma.m}$; and in like manner, $\Delta\,.\,\frac{dY}{dt}=\frac{\Sigma.mQ}{\Sigma.m}$, and

21. If we put*

[160]
$$\delta x = \frac{y.\delta x}{y} + \delta x_{,}; \qquad \delta x' = \frac{y'.\delta x}{y} + \delta x_{,}'; \qquad \delta x'' = \frac{y''.\delta x}{y} + \delta x_{,}''; \ \&c.;$$
$$\delta y = \frac{-x.\delta x}{y} + \delta y_{,}; \qquad \delta y' = \frac{-x'.\delta x}{y} + \delta y_{,}'; \qquad \delta y'' = \frac{-x''.\delta x}{y} + \delta y_{,}'', \ \&c.;$$

the variation δx will again disappear from the expressions† of δf, $\delta f'$, $\delta f''$,

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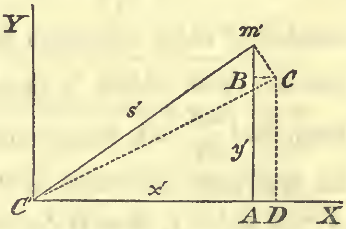
$\Delta.\frac{dZ}{dt} = \frac{\Sigma.mR}{\Sigma.m}$. But when the system is subjected only to the mutual action of the bodies upon each other, we shall have [155″], $\Sigma.mP = 0$, $\Sigma.mQ = 0$, $\Sigma.mR = 0$; hence $\Delta.\frac{dX}{dt} = 0$, $\Delta.\frac{dY}{dt} = 0$, $\Delta.\frac{dZ}{dt} = 0$. The integrals being taken, relative to the characteristic of finite differences Δ, give $\frac{dX}{dt} = b$, $\frac{dY}{dt} = b'$, $\frac{dZ}{dt} = b''$. Integrating these, relative to t, we obtain $X = a + bt$, $Y = a' + b't$, $Z = a'' + b''t$, as above.



* (76) For the sake of symmetry, the value

$$\delta x = \frac{y.\delta x}{y} + \delta x_{,},$$

which is not in the original, is here inserted, supposing $\delta x_{,} = 0$. It may be observed that in the formulas [160], the whole system is supposed to have an angular rotatory motion, equal to $\frac{\delta x}{y}$, about the axis of z; so that for any one of the bodies, as m', whose co-ordinates projected upon the plane of x, y, are $C'A = x'$, $Am' = y'$, and distance from that axis $C'm' = s'$, this rotatory motion would be represented by the arch $C\ m' = s'.\frac{\delta x}{y}$. This would *increase* the ordinate x' by the quantity

$$AD = BC = Cm' \times \frac{y'}{s'} = \frac{y'.\delta x}{y},$$

and would *decrease* the ordinate y' by the quantity $Bm' = Cm' \times \frac{x'}{s'} = \frac{x'.\delta x}{y}$. These are the first terms of $\delta x'$, $\delta y'$, [160]. Those of $\delta x''$, $\delta y''$, are found in the same manner, or by merely adding another accent to the letters x', y', &c.

† (77) The expressions of f, f', f'', &c. [118], and the assumed values of the variations δx, $\delta x'$, &c., δy, $\delta y'$, &c., [160], being of a symmetrical form, it will only be necessary to prove that δx disappears from any one of the quantities δf, $\delta f'$, $\delta f''$, &c., as $\delta f''$. Now the value of $\delta f''$, found as in [109*b*], supposing z', z'', invariable, is

$$\frac{1}{f''}(x''-x').(\delta x''-\delta x') + \frac{1}{f''}(y''-y').(\delta y''-\delta y'),$$

&c.; supposing therefore the system to be free, the conditions relative to the connexion of the parts of the system, affecting only the variations δf, $\delta f'$, &c., the variation δx is independent and arbitrary; therefore by substituting, in the equation (P) § 18 [142], the preceding values of δx, $\delta x'$, $\delta x''$, &c., [160] δy, $\delta y'$, $\delta y''$, &c., we ought to put the coefficient of δx equal to nothing; hence we get*

$$0 = \Sigma . m . \frac{x\,ddy - y\,ddx}{dt^2} + \Sigma . m . (Py - Qx); \qquad [161]$$

whence we deduce, by integrating relative to the time t,

$$c = \Sigma . m . \frac{(x\,dy - y\,dx)}{dt} + \Sigma . \int . m . (Py - Qx) . dt; \qquad [162]$$

c being an arbitrary constant quantity.

We may, in this integral, change the co-ordinates y, y', &c., into z, z', &c., provided that the forces R, R', &c., parallel to the axis of z, are substituted instead of Q, Q', &c., which are parallel to the axis of y; hence we get

$$c' = \Sigma . m . \frac{(x\,dz - z\,dx)}{dt} + \Sigma . \int . m . (Pz - Rx) . dt; \qquad [163]$$

c' being another arbitrary constant quantity. We shall have, in like manner,

$$c'' = \Sigma . m . \frac{(y\,dz - z\,dy)}{dt} + \Sigma . \int . m . (Qz - Ry) . dt; \qquad [164]$$

c'' being a third arbitrary constant quantity.†

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which, by substituting the above values of $\delta x'$, $\delta x''$, $\delta y'$, $\delta y''$, and retaining only the terms multiplied by δx, becomes $\frac{\delta x}{yf''}\left\{(x''-x') . (y''-y') + (y''-y') . (-x''+x')\right\}$, which is evidently $= 0$, because the terms between the braces mutually destroy each other.

* (77a) Substituting the values [160] in [142], retaining only the terms multiplied by δx, putting the coefficient of δx equal to 0, and multiplying by $-y$, we get [161], whose integral, relative to dt, gives [162]. The same method of reasoning, applied to the co-ordinates x, z, combined together gives [163], and applied to those of y, z, gives [164].

† (77b) If we put $c' = -c_{,,}$ and $c'' = c_{,}$, the equations [162, 163, 164] may be put [161a] under the following more symmetrical form,

$$c = \Sigma . m . \frac{(x\,dy - y\,dx)}{dt} + \Sigma . \int . m . (Py - Qx) . dt. \qquad [162a]$$

$$c_{,} = \Sigma . m . \frac{(y\,dz - z\,dy)}{dt} + \Sigma . \int . m . (Qz - Ry) . dt. \qquad [163a]$$

$$c_{,,} = \Sigma . m . \frac{(z\,dx - x\,dz)}{dt} + \Sigma . \int . m . (Rx - Pz) . dt. \qquad [164a]$$

Suppose the bodies of the system to be affected only by their mutual action upon each other, and by a force directed towards the origin of the co-ordinates. If we put, as above, p for the reciprocal action of m on m', we shall have, by means of this action alone,*

[165] $$0 = m\,.\,(P\,y - Q\,x) + m'\,.\,(P'\,y' - Q'\,x')\,;$$

consequently the mutual action of the bodies will disappear from the finite integral $\Sigma\,.\,m\,.\,(P\,y - Q\,x)$. Let S be the force which attracts the body m towards the origin of the co-ordinates; we shall have, by means of this force alone,†

[166] $$P = \frac{-S\,.\,x}{\sqrt{x^2+y^2+z^2}}\,; \qquad Q = \frac{-S\,.\,y}{\sqrt{x^2+y^2+z^2}}\,;$$

therefore the force S will disappear from the expression $P\,y - Q\,x$, and if

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[165a] In which each equation may be derived from the preceding one, by taking in these three series of letters, $c, c_{\prime}, c_{\prime\prime}$; x, y, z; P, Q, R; the next letters in order, observing to begin the series of letters again, when it is required to change the last terms $c_{\prime\prime}, z, R$, which become respectively c, x, P.

* (78) By [156*a*] it appears that this force p produces the forces $m\,P = p\,.\,\frac{(x-x')}{f}$, $m'\,P' = p\,.\,\frac{(x'-x)}{f}$, parallel to the axis of x; and in a similar manner, $m\,Q = p\,.\,\frac{(y-y')}{f}$, $m'\,Q' = p\,.\,\frac{(y'-y)}{f}$, parallel to the axis of y. These give

$$m\,.\,(P\,y - Q\,x) = \frac{p}{f}\,.\,\left\{y\,.\,(x-x') - x\,.\,(y-y')\right\} = \frac{p}{f}\,.\,(x\,y' - y\,x'),$$

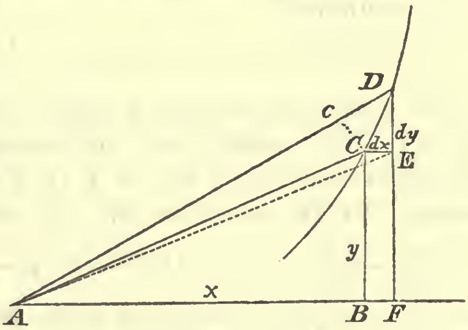
and $m'\,.\,(P'\,y' - Q'\,x') = \frac{p}{f}\,.\,\left\{y'\,.\,(x'-x) - x'\,.\,(y'-y)\right\} = \frac{p}{f}\,.\,(-x\,y' + y\,x')$, and the sum of these two is $m\,.\,(P\,y - Q\,x) + m'\,.\,(P'\,y' - Q'\,x') = 0$, since the terms of the second member mutually destroy each other.

† (78*a*) The force S, in the direction of the origin of the co-ordinates, may be resolved into the forces $S\,.\,\frac{x}{s}$, $S\,.\,\frac{y}{s}$, $S\,.\,\frac{z}{s}$ [137*a*], parallel to the co-ordinates x, y, z, respectively; and since $s = \sqrt{x^2+y^2+z^2}$, [137*b*], the two former forces become as in [166], the negative sign being prefixed because this force is supposed to decrease the co-ordinates. Substituting these values of P, Q, [166], in $P\,y - Q\,x$ it becomes 0. Noticing, therefore, only the mutual action p, of the bodies on each other, and the force S, we may neglect $P\,.\,Q\,.\,R$, in [162, 163, 164], and they will become as in [167].

we suppose the different bodies of the system to be affected only by their mutual attraction, and by forces directed towards the origin of the co-ordinates, we shall have

$$c=\Sigma.m.\frac{(xdy-ydx)}{dt}; \qquad c'=\Sigma.m.\frac{(xdz-zdx)}{dt}; \qquad c''=\Sigma.m.\frac{(ydz-zdy)}{dt}; \qquad (Z)$$ [167]

If the path of the body m be projected, on the plane of x, y, the differential $\frac{x\,dy-y\,dx}{2}$, will be the area described in the time dt, by the radius vector drawn from the origin of the co-ordinates,* to the projection [167']



* (79) In the adjoining figure let A be the origin of the co-ordinates, CD the projection of the part of the path of the body m, on the plane of xy, described in the time dt. Draw the ordinates CB, DEF; also, CE parallel to BF, and join AE. Then the triangle ACD, described in the time dt, is equal to

$$AED-AEC-DEC.$$

But

$$AED=\tfrac{1}{2}.AF.DE=\tfrac{1}{2}.(AB+BF).DE;$$

$$AEC=\tfrac{1}{2}.BC.CE=\tfrac{1}{2}.BC.BF; \qquad DEC=\tfrac{1}{2}.DE.EC=\tfrac{1}{2}.DE.BF.$$

Hence,

$$ACD=\tfrac{1}{2}.DE(AB+BF)-\tfrac{1}{2}.BC.BF-\tfrac{1}{2}.DE.BF=\tfrac{1}{2}.(DE.AB-BC.BF),$$

which, by substituting $AB=x$, $BF=dx$, $BC=y$, $DE=dy$, becomes

$$ACD=\tfrac{1}{2}.(x\,dy-y\,dx).$$ [167a]

This might have been simplified a little, by neglecting wholly the infinitely small triangle CDE, of the second order.

If the angle $BAC=\varpi$, and $AC=\rho$, we shall have $x=\rho.\cos.\varpi$, $y=\rho.\sin.\varpi$, [27], and the area dA, of the infinitely small triangle ACD, may be found by describing about the point A as a centre, with the radius AC, the circular arch $Cc=\rho.d\varpi$, to meet AD in c. This arch, multiplied by $\tfrac{1}{2}.AC=\tfrac{1}{2}.\rho$, gives the area of the triangle ACc, or $ACD=\tfrac{1}{2}.\rho^2.d\varpi$, which is to be put equal to the value above found; hence

$$dA=\tfrac{1}{2}.\rho^2.d\varpi=\tfrac{1}{2}.(x\,dy-y\,dx).$$ [167b]

In like manner, since $CD=\sqrt{CE^2+DE^2}=\sqrt{Dc^2+Cc^2}$, we shall have in symbols,

$$CD=\sqrt{dx^2+dy^2}=\sqrt{d\rho^2+\rho^2.d\varpi^2}.$$ [167c]

[167''] of m; the sum of these areas, multiplied respectively by the masses of these bodies, is therefore proportional to the element of the time dt; whence it follows, that in a finite time this sum is proportional to the time. In this consists the principle of the *preservation of areas.*

Principle of the preservation of areas.

The fixed plane of x, y, being arbitrary, this principle exists for any plane whatever; and if the force S is nothing, that is, if the bodies are subjected only to their action upon each other, and to their mutual attraction, the origin of the ordinates will be arbitrary, and we may take this fixed point at pleasure. Lastly, it is easy to perceive, by what precedes, that this principle holds good even in the case in which, by the mutual action of the bodies of the system upon each other, sudden changes are produced in their motions.* [167''']

The velocity of the body m being v, the space $C\,D$, described in the time $d\,t$, will be $v\,d\,t$. If we multiply this by the perpendicular p, let fall from A, upon the tangent at C, or on the continuation of the line $C\,D$, it will give another expression of the area of the triangle $A\,C\,D$, represented by $\frac{1}{2}.p.v\,d\,t$. Putting this equal to the former value $\frac{1}{2}.(x\,dy-y\,dx)$, we get $pv.d\,t=x\,dy-y\,d\,x$, hence $m\,v.p=m.\frac{(x\,dy-y\,dx)}{d\,t}$. Now $m\,v.p$ represents the momentum of the body m, about the axis of z, [29'], and if we put this equal to M, we shall have

[167d] $$M=m.\frac{(x\,dy-y\,dx)}{d\,t}.$$

Accenting these with one accent for the body m', two for the body m'', &c. and taking the sum of all, we shall have $\Sigma.M=\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}$; hence, from the first of the equations, [167], we get $\Sigma.M=c$. A similar result might be obtained relative to the other axes. Thus it appears, that the principle of the preservation of areas is equivalent to the supposition, that the sum of the momenta of all the bodies, about any axis, which is nothing in the case of equilibrium, is constant, in the case of motion here treated of. [167e] [167f]

* (80) Substituting the values $\delta x'$, $\delta x''$, &c., $\delta y'$, $\delta y''$, &c. [160], in the equation [145]; then putting the co-efficient of δx equal to nothing, and multiplying by $-y$, we shall get

$$0=\Sigma.m.\frac{(x.\Delta dy-y.\Delta dx)}{dt}+\Sigma.m.(Py-Qx).dt;$$

in which the terms $\Sigma.m.(Py-Qx).dt$ vanish, when the bodies only act on each other by impulse or attraction, or are impelled by forces tending to the centre of the co-ordinates [165, 166]. Hence we have $0=\Sigma.m.\left(\frac{x.\Delta dy-y.\Delta dx}{dt}\right)$. This is to be integrated as an

[167^iv] There is a plane with respect to which the quantities c', c'' are nothing, and on this account it is interesting to know it; for it is evident that this supposition of $c'=0$, $c''=0$, must tend greatly to simplify the investigation of the motions of a system of bodies. To determine this plane, it is necessary to refer the co-ordinates x, y, z, to three other axes, having the same origin as the preceding. Therefore let θ be the inclination of the required plane, which is formed by two of these new axes, to the plane of x, y; and ψ the angle which the axis of x forms with the intersection of [167^v] these two planes, so that $\frac{\pi}{2}-\theta$ may be the inclination of the third new axis to the plane of x and y, and $\frac{\pi}{2}-\psi$ may be the angle which its projection upon the same plane, makes with the axis of x, π being the semi-circumference.

Plane in which the sum of the areas described, multiplied by the masses respectively, is a maximum.

To assist the imagination, suppose the origin of the co-ordinates is at the centre of the earth, and the plane of x, y, is that of the ecliptic, the axis z being the line drawn from the centre of the earth to the north pole of the ecliptic; suppose also that the required plane is that of the equator, and that the new third axis is the axis of rotation of the earth, directed towards the north pole; θ will then be the obliquity of the ecliptic, and ψ [167^vi] the longitude of the fixed axis of x, counted from the moveable vernal equinox. The two first of these new axes will be in the plane of the equator, and by putting φ for the angular distance of the first of these axes from that equinox, φ will represent the rotation of the earth counted from [167^vii] that equinox, and $\frac{\pi}{2}+\varphi$ will be the angular distance of the second of

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equation of finite differences, upon the principles adopted in [147′]. From which it would follow, since the differentials of x, y, t, are infinitely small, that we may put $\Delta.(x\,d\,y)$ for $x.\Delta\,d\,y$; and $\Delta.(y\,d\,x)$ for $y.\Delta\,d\,x$; by which means, the preceding expression will become, $0=\Sigma.m.\Delta.\left(\frac{x\,dy-y\,dx}{dt}\right)$, whose integral, relative to the characteristic Δ, or $\Sigma_{,,}$ is $c=\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}$; as above. In like manner we may find the other two equations [167].

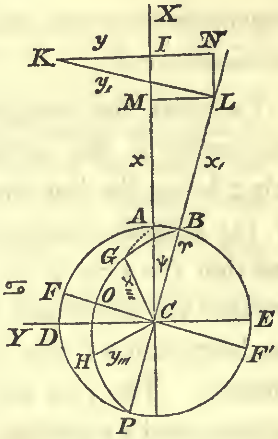
Principal Axes. these axes from the same equinox. We shall name these three new axes the *principal axes*.[*] This being premised,

[167viii] Let $x_{\prime}, y_{\prime}, z_{\prime}$ be the co-ordinates of m referred, *first*, to the line drawn from the origin of the co-ordinates, to the vernal equinox, the positive values of x being taken in the direction of that equinox; *second*, to the projection of the third principal axis, upon the plane of x, y; *third*, to the axis of z; we shall have[†]

$$x = x_{\prime} . \cos. \psi + y_{\prime} . \sin. \psi ;$$

[168] $$y = y_{\prime} . \cos. \psi - x_{\prime} . \sin. \psi ;$$

$$z = z_{\prime}.$$



[*] (81) Suppose that $BFPE$ represents the ecliptic, or plane of x,y; $CAIX$ the axis of x, CDY the axis of y, the axis of z falling *above* C, perpendicularly to the plane of the figure. Also BOP the plane of the two principal axes $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, the part BOP being supposed *below* the plane of the figure, and making with it an angle $FBO = \theta$, and the angle $ACB = \psi$. Let CG be the first principal axis $x_{\prime\prime\prime}$, CH the second, or axis of $y_{\prime\prime\prime}$, the third being drawn through C perpendicular to the plane of the equator $BGHP$, and falling above the plane of the figure, its projection on this plane being on the line CF, drawn at right angles to CB, making, with the axis of x, the angle $FCA = \frac{\pi}{2} - \psi$. Lastly, φ is the angle BCG, which the first principal axis CG, makes with CB; the angle, which the second principal axis CH makes with the same line CB, is $\frac{\pi}{2} + \varphi$.

[†] (82) Let K be the projection of the place of the body m upon the plane of the ecliptic, in the above figure, the co-ordinates of this point being either $CI = x$, $IK = y$; or, $CL = x_{\prime}$, $LK = y_{\prime}$. Through L draw LM parallel to KI, and LN parallel to CI, to meet KI produced in N. Then in the right angled triangles CML, KNL, we have the angle $MCL = NKL = \psi$. Hence $CM = CL . \cos. MCL = x_{\prime} . \cos. \psi$;

$$ML = IN = CL . \sin. MCL = x_{\prime} . \sin. \psi ;$$

$$KN = KL . \cos. NKL = y_{\prime} . \cos. \psi ; \quad LN = MI = KL . \sin. NKL = y_{\prime} . \sin. \psi .$$

Substitute these in $x = CI = CM + MI$, $y = KI = KN - IN$, and they become as in [168]. As the axis of z is not changed, we shall have $z = z_{\prime}$, [168].

Let $x_{\prime\prime}$, $y_{\prime\prime}$, $z_{\prime\prime}$, be the co-ordinates referred, *first*, to the line of the vernal equinox ; *second*, to the perpendicular to this line, in the plane of the equator ; *third*, to the third principal axis ; we shall have*

$$x_{\prime} = x_{\prime\prime} ;$$
$$y_{\prime} = y_{\prime\prime} \,.\, \cos.\, \theta + z_{\prime\prime} \,.\, \sin.\, \theta ; \qquad [169]$$
$$z_{\prime} = z_{\prime\prime} \,.\, \cos.\, \theta - y_{\prime\prime} \,.\, \sin.\, \theta.$$

Lastly, let $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, be the co-ordinates of m, referred to the first, second and third principal axes, we shall have†

$$x_{\prime\prime} = x_{\prime\prime\prime} \,.\, \cos.\, \varphi - y_{\prime\prime\prime} \,.\, \sin.\, \varphi ;$$
$$y_{\prime\prime} = y_{\prime\prime\prime} \,.\, \cos.\, \varphi + x_{\prime\prime\prime} \,.\, \sin.\, \varphi ; \qquad [170]$$
$$z_{\prime\prime} = z_{\prime\prime\prime}.$$

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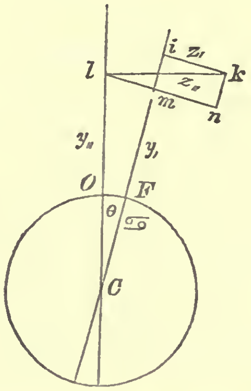
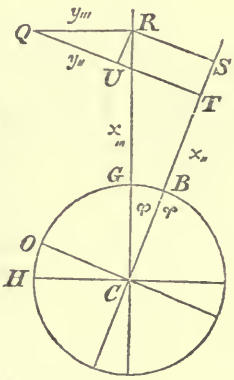
* (83) In this part of the computation the place of the body is supposed to be projected upon the plane of the solstitial colure *F C O*, (Fig. page 112), which we shall suppose to be the plane of the annexed figure. The point of projection being k, its co-ordinates may be either $C\,i = y_{\prime}$, $i\,k = z_{\prime}$; or $C\,l = y_{\prime\prime}$, $l\,k = z_{\prime\prime}$. Draw $l\,m\,n$ parallel to $i\,k$, and $k\,n$ parallel to $C\,i$. Then in the right angled triangles $C\,m\,l$, $l\,n\,k$, we have the angle $l\,C\,m = n\,l\,k = \theta$, hence



$$C\,m = C\,l \,.\, \cos.\, l\,C\,m = y_{\prime\prime} \,.\, \cos.\, \theta ;$$
$$l\,m = C\,l \,.\, \sin.\, l\,C\,m = y_{\prime\prime} \,.\, \sin.\, \theta ;$$
$$l\,n = l\,k \,.\, \cos.\, k\,l\,n = z_{\prime\prime} \,.\, \cos.\, \theta ;$$
$$k\,n = m\,i = l\,k \,.\, \sin.\, k\,l\,n = z_{\prime\prime} \,.\, \sin.\, \theta.$$

Substitute these in $y_{\prime} = C\,i = C\,m + m\,i$, and $z_{\prime} = i\,k = m\,n = l\,n - l\,m$, we get $y_{\prime}$, $z_{\prime}$, [169]. The axis of $x_{\prime}$ not being altered we get $x_{\prime} = x_{\prime\prime}$ [169].

† (84) In this case, the axes in the plane of the equator, are changed from *CB*, *CO* to *CG*, *CH*, (Fig. page 112, 113). The ordinates in the first case being $CT = x_{\prime\prime}$, $TQ = y_{\prime\prime}$, and in the second $CR = x_{\prime\prime\prime}$, $RQ = y_{\prime\prime\prime}$. Draw RS perpendicular to CT, and RU equal and parallel to ST. Then, in the right angled triangles CSR, QUR, we have the angle $RCS = RQU = \varphi$; hence,



$$C\,S = C\,R \,.\, \cos.\, R\,C\,S = x_{\prime\prime\prime} \,.\, \cos.\, \varphi ;$$
$$R\,S\,(= T\,U) = C\,R \,.\, \sin.\, R\,C\,S = x_{\prime\prime\prime} \,.\, \sin.\, \varphi ;$$
$$Q\,U = R\,Q \,.\, \cos.\, R\,Q\,U = y_{\prime\prime\prime} \,.\, \cos.\, \varphi ;$$
$$R\,U = S\,T = R\,Q \,.\, \sin.\, R\,Q\,U = y_{\prime\prime\prime} \,.\, \sin.\, \varphi.$$

Substitute these in

$$x_{\prime\prime} = C\,T = C\,S - S\,T, \qquad y_{\prime\prime} = T\,Q = Q\,U + T\,U,$$

they become as in [170]. The axis of $z_{\prime\prime}$ remaining unchanged gives $z_{\prime\prime} = z_{\prime\prime\prime}$ [170].

Formulas for the change of the rectangular axes.

Whence it is easy to conclude*

$$x = x_{,,,}.\{\cos.\theta.\sin.\psi.\sin.\varphi + \cos.\psi.\cos.\varphi\}$$
$$+ y_{,,,}.\{\cos.\theta.\sin.\psi.\cos.\varphi - \cos.\psi.\sin.\varphi\} + z_{,,,}.\sin.\theta.\sin.\psi;$$

[171]
$$y = x_{,,,}.\{\cos.\theta.\cos.\psi.\sin.\varphi - \sin.\psi.\cos.\varphi\}$$
$$+ y_{,,,}.\{\cos.\theta.\cos.\psi.\cos.\varphi + \sin.\psi.\sin.\varphi\} + z_{,,,}.\sin.\theta.\cos.\psi;$$
$$z = z_{,,,}.\cos.\theta - y_{,,,}.\sin.\theta.\cos.\varphi - x_{,,,}.\sin.\theta.\sin.\varphi.$$

Multiplying these values of x, y, z, respectively by the coefficients of $x_{,,,}$ in those values, and adding these products together, we shall obtain†

[172]
$$x_{,,,} = x.\{\cos.\theta.\sin.\psi.\sin.\varphi + \cos.\psi.\cos.\varphi\}$$
$$+ y.\{\cos.\theta.\cos.\psi.\sin.\varphi - \sin.\psi.\cos.\varphi\} - z.\sin.\theta.\sin.\varphi.$$

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* (85) Substituting in x, [168], the values $x_{,}$, $y_{,}$, [169], it becomes

$$x = x_{,,}.\cos.\psi + \sin.\psi.\{y_{,,}.\cos.\theta + z_{,,}.\sin.\theta\} = x_{,,}.\cos.\psi + y_{,,}.\cos.\theta.\sin.\psi + z_{,,}.\sin.\theta.\sin.\psi,$$

and by substituting in this last the values of $x_{,,}$, $y_{,,}$, $z_{,,}$, [170], we obtain

$$x = \cos.\psi.\{x_{,,,}.\cos.\varphi - y_{,,,}.\sin.\varphi\} + \cos.\theta.\sin.\psi.\{y_{,,,}.\cos.\varphi + x_{,,,}.\sin.\varphi\} + z_{,,,}.\sin.\theta.\sin.\psi,$$

by reduction it becomes as in [171].

In a similar way we may find y; or, more briefly, by changing, in this value of x, $\sin.\psi$ into $\cos.\psi$, and $\cos.\psi$ into $-\sin.\psi$; for these changes being made in the values of x [168], it becomes $-x_{,}.\sin.\psi + y_{,}.\cos.\psi$, which is equal to the value of y, [168], therefore the same changes being made in x, [171], it will become the same as y, [171]. Lastly, the values z, $z_{,}$, $z_{,,}$, [168, 169, 170], give successively

$$z = z_{,} = z_{,,}.\cos.\theta - y_{,,}.\sin.\theta = z_{,,,}.\cos.\theta - \sin.\theta.\{y_{,,,}.\cos.\varphi + x_{,,,}.\sin.\varphi\}$$
$$= z_{,,,}.\cos.\theta - y_{,,,}.\sin.\theta.\cos.\varphi - x_{,,,}.\sin.\theta.\sin.\varphi, \text{ [171].}$$

† (86) If we for brevity put

[171a]
$$\begin{aligned}
A_0 &= \cos.\theta.\sin.\psi.\sin.\varphi + \cos.\psi.\cos.\varphi,\\
A_1 &= \cos.\theta.\cos.\psi.\sin.\varphi - \sin.\psi.\cos.\varphi,\\
A_2 &= -\sin.\theta.\sin.\varphi;\\
B_0 &= \cos.\theta.\sin.\psi.\cos.\varphi - \cos.\psi.\sin.\varphi,\\
B_1 &= \cos.\theta.\cos.\psi.\cos.\varphi + \sin.\psi.\sin.\varphi,\\
B_2 &= -\sin.\theta.\cos.\varphi;\\
C_0 &= \sin.\theta.\sin.\psi,\\
C_1 &= \sin.\theta.\cos.\psi,\\
C_2 &= \cos.\theta.
\end{aligned}$$

In the same manner, if we multiply the values of x, y, z, respectively by the coefficients of $y_{\prime\prime\prime}$ in these values; and also by the coefficients of $z_{\prime\prime\prime}$, we

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The equations [171] will become

$$\begin{aligned} x &= A_0 . x_{\prime\prime\prime} + B_0 . y_{\prime\prime\prime} + C_0 . z_{\prime\prime\prime}, \\ y &= A_1 . x_{\prime\prime\prime} + B_1 . y_{\prime\prime\prime} + C_1 . z_{\prime\prime\prime}, \\ z &= A_2 . x_{\prime\prime\prime} + B_2 . y_{\prime\prime\prime} + C_2 . z_{\prime\prime\prime}. \end{aligned} \qquad [172a]$$

If we put r for the distance of the body m from the origin, we shall have, by [19e], $r^2 = x^2 + y^2 + z^2$; also $r^2 = x_{\prime\prime\prime}^2 + y_{\prime\prime\prime}^2 + z_{\prime\prime\prime}^2$, whence $x^2 + y^2 + z^2 = x_{\prime\prime\prime}^2 + y_{\prime\prime\prime}^2 + z_{\prime\prime\prime}^2$. Substituting the above values of x, y, z, [172a], we shall get an identical equation in $x_{\prime\prime\prime}, y_{\prime\prime\prime}, z_{\prime\prime\prime}$, in which the coefficients of $x_{\prime\prime\prime}^2, y_{\prime\prime\prime}^2, z_{\prime\prime\prime}^2$, in both members, must be 1, and the coefficients of the products of $x_{\prime\prime\prime}, y_{\prime\prime\prime}, z_{\prime\prime\prime}$, must be nothing; whence we obtain,

$$\begin{aligned} & A_0^2 + A_1^2 + A_2^2 = 1, \\ & B_0^2 + B_1^2 + B_2^2 = 1, \\ & C_0^2 + C_1^2 + C_2^2 = 1; \\ & A_0 . B_0 + A_1 . B_1 + A_2 . B_2 = 0, \\ & A_0 . C_0 + A_1 . C_1 + A_2 . C_2 = 0, \\ & B_0 . C_0 + B_1 . C_1 + B_2 . C_2 = 0. \end{aligned} \qquad [172b]$$

Multiply the equations [172a], by A_0, A_1, A_2, respectively, and add the products, we shall get by means of the equations [172b], the value of $x_{\prime\prime\prime}$, [172]. In like manner, multiply the same equations [172a], by B_0, B_1, B_2, respectively, and add the products, we shall get $y_{\prime\prime\prime}$ [173]. Lastly, multiply the same equations by C_0, C_1, C_2, respectively, and add the products, we shall get, by the same reductions, the values of $z_{\prime\prime\prime}$, [174], all of which are in the following table;

$$\begin{aligned} x_{\prime\prime\prime} &= A_0 . x + A_1 . y + A_2 . z, \\ y_{\prime\prime\prime} &= B_0 . x + B_1 . y + B_2 . z, \\ z_{\prime\prime\prime} &= C_0 . x + C_1 . y + C_2 . z. \end{aligned} \qquad [172c]$$

If we substitute these values in the above equation $x^2 + y^2 + z^2 = x_{\prime\prime\prime}^2 + y_{\prime\prime\prime}^2 + z_{\prime\prime\prime}^2$, we shall obtain an identical equation in x, y, z, in which the coefficients of x^2, y^2, z^2, are 1, and the coefficients of the products of x, y, z, are nothing, whence we get the following equations, similar to [172b],

$$\begin{aligned} & A_0^2 + B_0^2 + C_0^2 = 1, \\ & A_1^2 + B_1^2 + C_1^2 = 1, \\ & A_2^2 + B_2^2 + C_2^2 = 1; \\ & A_0 . A_1 + B_0 . B_1 + C_0 . C_1 = 0, \\ & A_0 . A_2 + B_0 . B_2 + C_0 . C_2 = 0, \\ & A_1 . A_2 + B_1 . B_2 + C_1 . C_2 = 0. \end{aligned} \qquad [172d]$$

The quantities A_0, A_1, &c. represent the cosines of the angles, formed by the axes shall successively find

[173] $$y_{\prime\prime\prime} = x\,.\{\cos.\,\theta\,.\,\sin.\,\psi\,.\,\cos.\,\varphi - \cos.\,\psi\,.\,\sin.\,\varphi\} + y\,.\{\cos.\,\theta\,.\,\cos.\,\psi\,.\,\cos.\,\varphi + \sin.\,\psi\,.\,\sin.\,\varphi\} - z\,.\,\sin.\,\theta\,.\,\cos.\,\varphi\,;$$

[174] $$z_{\prime\prime\prime} = x\,.\,\sin.\,\theta\,.\,\sin.\,\psi + y\,.\,\sin.\,\theta\,.\,\cos.\,\psi + z\,.\,\cos.\,\theta.$$

These different transformations of the co-ordinates will hereafter be very

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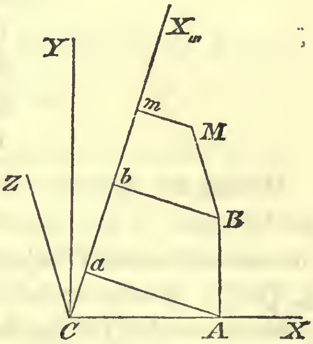
[172e] of x, y, z, with those of $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$; so that if we represent, by $(x, y_{\prime\prime\prime})$, the angle formed by the axes x, $y_{\prime\prime\prime}$, and in like manner for the angles formed by the other axes, we shall have,

[172f] $$\begin{array}{lll} A_0 = \cos.\,(x, x_{\prime\prime\prime})\,; & A_1 = \cos.\,(y, x_{\prime\prime\prime})\,; & A_2 = \cos.\,(z, x_{\prime\prime\prime})\,; \\ B_0 = \cos.\,(x, y_{\prime\prime\prime})\,; & B_1 = \cos.\,(y, y_{\prime\prime\prime})\,; & B_2 = \cos.\,(z, y_{\prime\prime\prime})\,; \\ C_0 = \cos.\,(x, z_{\prime\prime\prime})\,; & C_1 = \cos.\,(y, z_{\prime\prime\prime})\,; & C_2 = \cos.\,(z, z_{\prime\prime\prime})\,; \end{array}$$



To prove this, we shall refer to the adjoined figure, in which CX, CY, CZ, are the rectangular axes of x, y, z, and $CX_{\prime\prime\prime}$ the axis of $x_{\prime\prime\prime}$. Let M be the place of the body m, whose co-ordinates are $CA = x$, $AB = y$, parallel to Cy, and $BM = z$, parallel to CZ. From the points A, B, M, let fall, upon $CX_{\prime\prime\prime}$, the perpendiculars Aa, Bb, Mm; then Cm will be the value of $x_{\prime\prime\prime}$, corresponding to the body at M. Hence $x_{\prime\prime\prime} = Cm = Ca + ab + bm$. Now by the principles of the orthographic projection, we have $Ca = CA\,.\,\cos.\,(x, x_{\prime\prime\prime}) = x\,.\,\cos.\,(x, x_{\prime\prime\prime})\,;$ $ab = AB\,.\,\cos.\,(y, x_{\prime\prime\prime}) = y\,.\,\cos.\,(y, x_{\prime\prime\prime})$, and $bm = BM\,.\,\cos.\,(z, x_{\prime\prime\prime}) = z\,.\,\cos.\,(z, x_{\prime\prime\prime})$. Substituting these in the preceding value of $x_{\prime\prime\prime}$, we get

[172g] $$x_{\prime\prime\prime} = x\,.\,\cos.\,(x, x_{\prime\prime\prime}) + y\,.\,\cos.\,(y, x_{\prime\prime\prime}) + z\,.\,\cos.\,(z, x_{\prime\prime\prime}).$$

Comparing this value of $x_{\prime\prime\prime}$, with the first of the equations [172c], which is identical with it, we get the values of A_0, A_1, A_2 [172f]. In like manner, using the axis of $y_{\prime\prime\prime}$, instead of $CX_{\prime\prime\prime}$, we get the values of B_0, B_1, B_2; and the axis of $z_{\prime\prime\prime}$ gives the values of C_0, C_1, C_2.

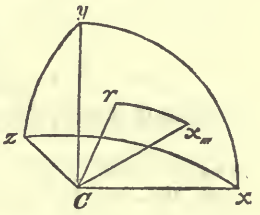
If we divide the value $x_{\prime\prime\prime}$, [172g], by $r = \sqrt{x^2 + y^2 + z^2}$, observing, that by the principle of orthographic projection, $\frac{x_{\prime\prime\prime}}{r}$ represents the cosine of the angle, which the line r makes with the axis of $x_{\prime\prime\prime}$, or $\cos.\,(r, x_{\prime\prime\prime})$; in like manner $\frac{x}{r} = \cos.\,(x, r)$, $\frac{y}{r} = \cos.\,(y, r)$, $\frac{z}{r} = \cos.\,(z, r)$, we shall get,

[172h] $$\cos.\,(r, x_{\prime\prime\prime}) = \cos.\,(x, r)\,.\,\cos.\,(x, x_{\prime\prime\prime}) + \cos.\,(y, r)\,.\,\cos.\,(y, x_{\prime\prime\prime}) + \cos.\,(z, r)\,.\,\cos.\,(z, x_{\prime\prime\prime}).$$

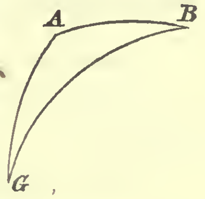
useful. By marking, at the top of the letters, x, y, z, $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, one accent, two accents, &c., we shall have the co-ordinates corresponding to the bodies m', m'', &c.

Hence it is easy to deduce, by substituting c, c', c'', instead of

$$\Sigma . m . \frac{(x\,dy - y\,dx)}{dt}; \qquad \Sigma . m . \frac{(x\,dz - z\,dx)}{dt}; \qquad \Sigma . m . \frac{(y\,dz - z\,dy)}{dt}, \qquad [174']$$



This is a well known formula of spherics, which will be used hereafter. For the sake of illustration, we may refer this formula to the arches of a spherical surface $x\,y\,z\,r\,x_{\prime\prime\prime}$, described about the centre C, with the radius unity; since the angles (x, r), (y, r), (z, r), $(r, x_{\prime\prime\prime})$, $(x, x_{\prime\prime\prime})$, &c. will correspond respectively to the arches $x\,r$, $y\,r$, $z\,r$, &c.



In Fig. page 112, the arch $A\,G$, drawn upon the spherical surface $A\,B\,G$, whose radius is unity, is the same as the angle $(x, x_{\prime\prime\prime})$, so that we shall have, as in [172f], $\cos. A\,G = \cos. (x, x_{\prime\prime\prime}) = A_0$. If we make use of the value of A_0 [171a], and resubstitute $\psi = A\,B$, $\varphi = B\,G$, $\theta =$ angle $A\,B\,G$, we shall obtain the following fundamental theorem of spherics,

$$\cos. A\,G = \cos. A\,B\,G \,.\, \sin. A\,B \,.\, \sin. B\,G + \cos. A\,B \,.\, \cos. B\,G, \qquad [172i]$$

from which all other formulas of spherics may be deduced.

It would have been easy to derive from this formula, and other well known formulas of spherics, the whole of the values [171a]. Thus since by [172f],

$$B_0 = \cos. (x, y_{\prime\prime\prime}) = \cos.\ \text{arch}\ A\,H,$$

in Fig. page 112, it is evident that in the triangle $A\,B\,H$, we shall have the same data, as in the preceding example, except that for $B\,G = \varphi$, we must use φ increased by a right angle, and this evidently changes A_0 into B_0 [171a].

Before concluding this note we may observe, that the values $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, [172—174], might be derived from those of x, y, z, [171], by changing in those formulas x, y, z, into $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, respectively, and making the changes of φ into ψ, ψ into φ, and θ into $-\theta$, which would follow from this supposition, as will easily appear, from the situations of those axes, as marked down in Fig. page 112. Observing also that if the inclination θ is taken positive, as it regards the plane $B\,F\,P$, it must be considered as negative as it regards the plane $B\,O\,P$.

[175] $$\Sigma . m . \frac{\{x_{\prime\prime\prime} . dy_{\prime\prime\prime} - y_{\prime\prime\prime} . dx_{\prime\prime\prime}\}}{dt} = c . \cos. \theta - c' . \sin. \theta . \cos. \psi + c'' . \sin. \theta . \sin. \psi ;^{*}$$

[176] $$\Sigma . m . \frac{\{x_{\prime\prime\prime} . dz_{\prime\prime\prime} - z_{\prime\prime\prime} . dx_{\prime\prime\prime}\}}{dt} = c . \sin. \theta . \cos. \varphi + c' . \{\sin. \psi . \sin. \varphi + \cos. \theta . \cos. \psi . \cos. \varphi\} + c'' . \{\cos. \psi . \sin. \varphi - \cos. \theta . \sin. \psi . \cos. \varphi\} ;$$

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* (87) The differentials of the values $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, [172*c*], being taken, considering x, y, z, $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$ as variable, give

$$dx_{\prime\prime\prime} = A_0 . dx + A_1 . dy + A_2 . dz ;$$
$$dy_{\prime\prime\prime} = B_0 . dx + B_1 . dy + B_2 . dz ;$$

substituting these values in $x_{\prime\prime\prime}\, dy_{\prime\prime\prime} - y_{\prime\prime\prime}\, dx_{\prime\prime\prime}$, it becomes

[175*a*] $$\begin{aligned} & x_{\prime\prime\prime}\, dy_{\prime\prime\prime} - y_{\prime\prime\prime}\, dx_{\prime\prime\prime} \\ & = (A_0 . B_1 - A_1 . B_0) . (x\,dy - y\,dx) + (A_2 . B_0 - A_0 . B_2) . (z\,dx - x\,dz) + (A_1 . B_2 - A_2 . B_1) . (y\,dz - z\,dy) \end{aligned}$$

Now the values [171*a*], give

[175*b*] $$\begin{aligned} A_1 . B_2 &= - \sin. \theta . \cos. \theta . \cos. \psi . \sin. \varphi . \cos. \varphi + \sin. \theta . \sin. \psi . \cos.^2 \varphi , \\ - A_2 . B_1 &= \sin. \theta . \cos. \theta . \cos. \psi . \sin. \varphi . \cos. \varphi + \sin. \theta . \sin. \psi . \sin.^2 \varphi , \end{aligned}$$

whose sum is

$$A_1 . B_2 - A_2 . B_1 = \sin. \theta . \sin. \psi . (\cos. \varphi^2 + \sin.^2 \varphi) = \sin. \theta . \sin. \psi = C_0 , \text{ [171}a\text{]}.$$

Therefore we shall have in general $C_0 = A_1 . B_2 - A_2 . B_1$, and from the perfect symmetry of the formulas [171*a*, 172*a*], this formula would exist if we changed the axes of x, y into those of y, z, respectively, which would have the effect to increase the index of the letters by 1, rejecting 3, when the index exceeds 3, so that from derivations of this kind, we shall obtain the following system of equations.

[175*c*] $$\begin{array}{lll} A_0 = B_1 . C_2 - B_2 . C_1 ; & B_0 = C_1 . A_2 - C_2 . A_1 ; & C_0 = A_1 . B_2 - A_2 . B_1 ; \\ A_1 = B_2 . C_0 - B_0 . C_2 ; & B_1 = C_2 . A_0 - C_0 . A_2 ; & C_1 = A_2 . B_0 - A_0 . B_2 ; \\ A_2 = B_0 . C_1 - B_1 . C_0 ; & B_2 = C_0 . A_1 - C_1 . A_0 ; & C_2 = A_0 . B_1 - A_1 . B_0 ; \end{array}$$

which may also be easily proved, by substituting the values [171*a*]. Hence the above expression [175*a*] becomes

[175*d*] $$x_{\prime\prime\prime}\, dy_{\prime\prime\prime} - y_{\prime\prime\prime}\, dx_{\prime\prime\prime} = C_2 . (x\,dy - y\,dx) + C_1 . (z\,dx - x\,dz) + C_0 . (y\,dz - z\,dy).$$

The inspection of the formulas [172*c*], shows that we may change $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, into $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, $x_{\prime\prime\prime}$, respectively, without changing the values of x, y, z, or altering the indexes of the letters A, B, C, provided we change A, B, C, into B, C, A, respectively; that is, we must put the letters one term forward, beginning the series again when we come to the last term;

$$\Sigma . m . \frac{\{y_{,,,} . dz_{,,,} - z_{,,,} . dy_{,,,}\}}{dt} = -c . \sin.\theta . \sin.\varphi + c' . \{\sin.\psi . \cos.\varphi - \cos.\theta . \cos.\psi . \sin.\varphi\}$$ [177]

$$+ c'' . \{\cos.\psi . \cos.\varphi + \cos.\theta . \sin.\psi . \sin.\varphi\} .$$

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since the same three equations [172*c*], will exist after these changes. We may therefore make the same changes in [175*d*], which will give

$$y_{,,,} d z_{,,,} - z_{,,,} d y_{,,,} = A_2 . (x\,dy - y\,dx) + A_1 . (z\,dx - x\,dz) + A_0 . (y\,dz - z\,dy),$$ [175*e*]

and by a similar process,

$$z_{,,,} d x_{,,,} - x_{,,,} d z_{,,,} = B_2 . (x\,dy - y\,dx) + B_1 . (z\,dx - x\,dz) + B_0 . (y\,dz - z\,dy)$$ [175*f*]

If we now multiply these expressions [175*d*, *e*, *f*], by $\frac{m}{dt}$, then mark, at the top of the letters, x, y, z, $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, one accent for the body m', two accents for the body m'', and take the sum of all these products, using as before the characteristic Σ, and putting also as in [167],

$$c = \Sigma . m . \frac{(x\,dy - y\,dx)}{dt}; \qquad c_{,} = \Sigma . m . \frac{(y\,dz - z\,dy)}{dt}; \qquad c_{,,} = \Sigma . m . \frac{(z\,dx - x\,dz)}{dt};$$ [175*g*]

observing that $c_{,} = c''$, $c_{,,} = -c'$, [161*a*] we shall get

$$\Sigma . m . \frac{(x_{,,,} dy_{,,,} - y_{,,,} dx_{,,,})}{dt} = c . C_2 + c_{,,} . C_1 + c_{,} . C_0;$$ [175*h*]

$$\Sigma . m . \frac{(y_{,,,} dz_{,,,} - z_{,,,} dy_{,,,})}{dt} = c . A_2 + c_{,,} . A_1 + c_{,} . A_0;$$ [175*i*]

$$\Sigma . m . \frac{(z_{,,,} dx_{,,,} - x_{,,,} dz_{,,,})}{dt} = c . B_2 + c_{,,} . B_1 + c_{,} . B_0 .$$ [175*k*]

Substituting the values [171*a*], and putting $c_{,,} = -c'$, $c_{,} = c''$, it becomes as in [175—177]. [175*l*]

It may be observed that the formulas [175*d*, *e*, *f*] may be very easily found from geometrical considerations. For if the body m move from a point whose co-ordinates are x, y, z, and whose radius vector, drawn from the origin of the co-ordinates, is

$$r = \sqrt{x^2 + y^2 + z^2},$$

[19*e*], and at the end of the time dt arrive at a point whose co-ordinates are $x + dx$, $y + dy$, $z + dz$, and radius vector $r + dr$, the angle included between the radii r and $r + dr$, being $d\varpi$, the described area will be $\frac{1}{2} r^2 d\varpi$, [167*b*]. Now it follows, from the principles of orthographic projection, that if this area be projected on any other plane, the projected area will be equal to the described area $\frac{1}{2} r^2 d\omega$, multiplied by the cosine of the inclination of the two planes. But the inclination of two planes, passing through the origin of the co-ordinates, is evidently equal to that of the two lines drawn through the origin perpendicular to these planes. Suppose now that a line R is drawn through the origin, perpendicular to the plane of the described area, included by the radii r, $r + dr$; the angle formed by the line R and the axis of z, will be represented by (z, R), [172*e*], and the area

If we determine ψ and θ, so that we may have*

[178] $$\sin.\theta\,.\,\sin.\psi=\frac{c''}{\sqrt{c^2+c'^2+c''^2}}\,; \qquad \sin.\theta\,.\,\cos.\psi=\frac{-c'}{\sqrt{c^2+c'^2+c''^2}}\,;$$

which gives

[179] $$\cos.\theta=\frac{c}{\sqrt{c^2+c'^2+c''^2}}\,;$$

we shall have

[180] $$\begin{aligned} \Sigma\,.\,m\,.\,\frac{\{x_{\prime\prime\prime}\,.\,dy_{\prime\prime\prime}-y_{\prime\prime\prime}\,.\,dx_{\prime\prime\prime}\}}{dt}&=\sqrt{c^2+c'^2+c''^2}\,;\\ \Sigma\,.\,m\,.\,\frac{\{x_{\prime\prime\prime}\,.\,dz_{\prime\prime\prime}-z_{\prime\prime\prime}\,.\,dx_{\prime\prime\prime}\}}{dt}&=0\,;\\ \Sigma\,.\,m\,.\,\frac{\{y_{\prime\prime\prime}\,.\,dz_{\prime\prime\prime}-z_{\prime\prime\prime}\,.\,dy_{\prime\prime\prime}\}}{dt}&=0\,; \end{aligned}$$

$\frac{1}{2}r^2\,.\,d\varpi$, projected upon the plane of x, y, which is perpendicular to the axis of z, will be represented by $\frac{1}{2}r^2\,.\,d\varpi\,.\,\cos.\,(z,R)$; and this, by [167$a$], is $=\frac{1}{2}(x\,dy-y\,dx)$, therefore we shall have $r^2\,.\,d\varpi\,.\,\cos.\,(z,R)=(x\,dy-y\,dx)$. Changing in succession x, y, z into y, z, x, respectively, we get

$$r^2\,.\,d\varpi\,.\cos.(x,R)=(y\,dz-z\,dy)\,; \qquad r^2\,.\,d\varpi\,.\,\cos.(y,R)=(z\,dx-x\,dy),$$

also $r^2\,.\,d\varpi\,.\,\cos.\,(R,x_{\prime\prime\prime})=y_{\prime\prime\prime}\,dz_{\prime\prime\prime}-z_{\prime\prime\prime}\,dy_{\prime\prime\prime})$. Now if in the formula [172h], we change r into R, and multiply the whole by $r^2\,.\,d\varpi$, we shall get,

$$r^2\,.\,d\varpi\,.\,\cos.\,(R,x_{\prime\prime\prime})$$
$$=r^2\,.\,d\varpi\,.\,\{\cos.\,(x,R)\,.\cos.\,(x,x_{\prime\prime\prime})+\cos.\,(y,R)\,.\,\cos.\,(y,x_{\prime\prime\prime})+\cos.\,(z,R)\,.\,\cos.\,(z,x_{\prime\prime\prime})\},$$

substituting the values just found, it becomes

$$y_{\prime\prime\prime}dz_{\prime\prime\prime}-z_{\prime\prime\prime}dy_{\prime\prime\prime}=(ydz-zdy)\cos.(x,x_{\prime\prime\prime})+(zdx-xdy)\,.\cos.(y,x_{\prime\prime\prime})+(xdy-ydx)\,.\cos.(z,x_{\prime\prime\prime}),$$

and by using the values $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ [172f], it becomes

$$y_{\prime\prime\prime}\,dz_{\prime\prime\prime}-z_{\prime\prime\prime}\,dy_{\prime\prime\prime}=\mathcal{A}_0\,(y\,dz-z\,dy)+\mathcal{A}_1\,(z\,dx-x\,dy)+\mathcal{A}_2\,(x\,dy-y\,dx),$$

being the same as in [175e], and from this the others [175d, f], may be derived, as above.

* (88) If we put, as in [161a], $c'=-c_{\prime\prime}$, $c''=c_{\prime}$; also for brevity,

$$m=\sqrt{c^2+c'^2+c''^2}=\sqrt{c^2+c_{\prime}^2+c_{\prime\prime}^2}\,.$$

The equations [178], will become $\sin.\theta\,.\,\sin.\psi=\frac{c_{\prime}}{m}$, $\sin.\theta\,.\,\cos.\psi=\frac{c_{\prime\prime}}{m}$, the sum of whose squares is $\sin.^2\theta=\frac{c_{\prime}^2+c_{\prime\prime}^2}{m^2}$, whence $\cos.\theta=\sqrt{1-\sin.^2\theta}=\frac{c}{m}$. Multiply these three equations by m, and use the values, C_0, C_1, C_2, [171a], we get $c_{\prime}=m\,C_0$, $c_{\prime\prime}=m\,C_1$, $c=m\,C_2$. Substitute these in the second members of the equations [175i, k], and they will become respectively, $m(\mathcal{A}_2\,C_2+\mathcal{A}_1\,C_1+\mathcal{A}_0\,C_0)$, $m\,(B_2\,C_2+B_1\,C_1+B_0\,C_0,)$ which, The values of c' and of c'' are therefore nothing with respect to the plane of $x_{\prime\prime\prime}$ and $y_{\prime\prime\prime}$, determined in this manner. There is but one plane which [180'] possesses this property; for, by supposing it to be that of x and y, we shall have*

$$\Sigma . m . \frac{\{x_{\prime\prime\prime} . dz_{\prime\prime\prime} - z_{\prime\prime\prime} . dx_{\prime\prime\prime}\}}{dt} = c . \sin. \theta \cos. \varphi ;$$
$$\Sigma . m . \frac{\{y_{\prime\prime\prime} . dz_{\prime\prime\prime} - z_{\prime\prime\prime} . dy_{\prime\prime\prime}\}}{dt} = - c . \sin. \theta . \sin. \varphi. \qquad [181]$$

By putting these two functions equal to nothing, we shall have, $\sin. \theta = 0$; whence it follows, that the plane of $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, coincides, in that case, with the plane of x, y.

The value of $\Sigma . m . \frac{\{x_{\prime\prime\prime} . dy_{\prime\prime\prime} - y_{\prime\prime\prime} . dx_{\prime\prime\prime}\}}{dt}$, being equal to $\sqrt{c^2 + c'^2 + c''^2}$, [181'] whatever be the plane of x and y; it follows that the quantity $c^2 + c'^2 + c''^2$, will be the same whatever be that plane, and that the plane of $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, found in the preceding manner, is that which renders the function

$$\Sigma . m . \frac{\{x_{\prime\prime\prime} . dy_{\prime\prime\prime} - y_{\prime\prime\prime} . dx_{\prime\prime\prime}\}}{dt}$$

a maximum. Hence this plane has the following remarkable properties, namely, *First*, that the sum of the areas, traced by the projections of the radius vector of each of the bodies, multiplied by its mass, is a maximum;† [181'']

Properties of the plane of maximum areas.

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by [172b], are nothing, as in [180]. The same values of c, $c_{\prime}$, $c_{\prime\prime}$, being substituted in the second member of [175h], it becomes $m . (C_2 C_2 + C_1 C_1 + C_0 C_0)$, which, by [172b], is simply m, or $\sqrt{c^2 + c'^2 + c''^2}$, as in [180].

* (89) These equations are obtained by putting $c' = 0$, $c'' = 0$, in [176, 177]. If we now find the value of θ, which will render the second members of [181] nothing, that is, $0 = c . \sin. \theta . \cos. \varphi$; $0 = - c . \sin. \theta . \sin. \varphi$, it will give $\theta = 0$. For the sum of the squares of these equations becomes, by putting $\cos.^2 \varphi + \sin.^2 \varphi = 1$, $0 = c^2 . \sin.^2 \theta$, whence $\sin. \theta = 0$, and $\theta = 0$, c being finite and φ indeterminate.

† (89a) Upon any plane taken as that of x, y, we have $\Sigma . m . \frac{(x dy - y dx)}{dt} = c$, and c must be less than the quantity $\sqrt{c^2 + c'^2 + c''^2}$, except $c' = 0$, $c'' = 0$, and as this quantity $\sqrt{c^2 + c'^2 + c''^2}$, is constant for every system of planes, it is evident that the maximum value of c will be obtained, by putting $c' = 0$, $c'' = 0$.

[181‴] *Second*, that the same sum, relative to any plane perpendicular to the preceding, is nothing, since the angle φ remains indeterminate. We may, by means of these properties, find this plane at any time, whatever variations may have taken place in their relative situations, by the mutual action of the bodies, in the same manner as we can easily find, at all times, the position of the centre of gravity of the system; for this reason, it is as natural to take this plane for that of x, y, as to take the centre of gravity for the origin of the co-ordinates.

Principles of living forces and areas take place when the origin has a rectilineal and uniform motion.

22. The principles of the preservation of the living forces and areas, take place also when the origin of the co-ordinates is supposed to have a rectilineal uniform motion in space. To demonstrate this, put X, Y, Z, for the co-ordinates of this moveable origin, referred to a fixed point, and suppose that

[182]
$$x = X + x_{,}\,; \qquad y = Y + y_{,}\,; \qquad z = Z + z_{,}\,;$$
$$x' = X + x'_{,}\,; \qquad y' = Y + y'_{,}\,; \qquad z' = Z + z'_{,}\,;$$
&c.;

$x_{,}$, $y_{,}$, $z_{,}$, $x'_{,}$, &c., will be the co-ordinates of m, m', &c., referred to the moveable origin. We shall have, by this hypothesis,*

[183]
$$d\,d\,X = 0\,; \qquad d\,d\,Y = 0\,; \qquad d\,d\,Z = 0\,;$$

but, by the nature of the centre of gravity, we have, when the system is free,†

[184]
$$0 = \Sigma\,.\,m\,.\{d\,d\,X + d\,d\,x_{,}\} - \Sigma\,.\,m\,.\,P\,.\,d\,t^2\,;$$
$$0 = \Sigma\,.\,m\,.\{d\,d\,Y + d\,d\,y_{,}\} - \Sigma\,.\,m\,.\,Q\,.\,d\,t^2\,;$$
$$0 = \Sigma\,.\,m\,.\{d\,d\,Z + d\,d\,z_{,}\} - \Sigma\,.\,m\,.\,R\,.\,d\,t^2\,;$$

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* (90) The velocity of this origin being rectilineal and uniform makes $\frac{dX}{d\,t} = \text{constant}$; [182a] $\frac{d\,Y}{d\,t} = \text{constant}$; $\frac{d\,Z}{d\,t} = \text{constant}$; the differentials of these being taken, supposing $d\,t$ constant, gives $d\,d\,X = 0$, $d\,d\,Y = 0$, $d\,d\,Z = 0$.

† (91) X, Y, Z, which have been usually taken for the co-ordinates of the centre of gravity, are supposed above to be the co-ordinates of the moveable origin, we shall therefore, in this note, put X', Y', Z', for the co-ordinates of the centre of gravity referred to the

the equation (*P*) § 18 [142], will therefore become, by substituting $\delta X+\delta x_{,}$, $\delta Y+\delta y_{,}$, &c., for δx, δy, &c.,*

$$0=\Sigma.m.\delta x_{,}.\left\{\frac{d\,d\,x_{,}}{d\,t^2}-P\right\}+\Sigma.m.\delta y_{,}.\left\{\frac{d\,d\,y_{,}}{d\,t^2}-Q\right\}+\Sigma.m.\delta z_{,}.\left\{\frac{d\,d\,z_{,}}{d\,t^2}-R\right\}; \qquad [185]$$

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fixed point of origin, we shall then have by [126, 127], $X'=\frac{\Sigma.mx}{\Sigma.m}$, $Y'=\frac{\Sigma.my}{\Sigma.m}$, $Z'=\frac{\Sigma.mz}{\Sigma.m}$; substituting the values [182], we get

$$X'=\frac{\Sigma.m.(X+x_{,})}{\Sigma.m}, \qquad Y'=\frac{\Sigma.m.(Y+y_{,})}{\Sigma.m}, \qquad Z'=\frac{\Sigma.m.(Z+z_{,})}{\Sigma.m}. \qquad [183a]$$

Multiplying by $\Sigma.m$, and taking the second differentials, we obtain,

$$\begin{aligned} d\,d\,X'.\Sigma.m&=\Sigma.m.(d\,d\,X+d\,d\,x_{,}); \\ d\,d\,Y'.\Sigma.m&=\Sigma.m.(d\,d\,Y+d\,d\,y_{,}); \qquad [183b] \\ d\,d\,Z'.\Sigma.m&=\Sigma.m.(d\,d\,Z+d\,d\,z_{,}); \end{aligned}$$

but by accenting *X*, *Y*, *Z*, in [155], we have

$$\begin{aligned} d\,d\,X'.\Sigma.m&=\Sigma.m.P.d\,t^2; \\ d\,d\,Y'.\Sigma.m&=\Sigma.m.Q.d\,t^2; \\ d\,d\,Z'.\Sigma.m&=\Sigma.m.R.d\,t^2. \end{aligned}$$

Substitute these and we shall obtain, by transposition, the formulas [184].

* (92) Since $x=X+x_{,}$, [182], and $d\,d\,X=0$, [183], we have

$$d\,d\,x=d\,d\,X+d\,d\,x_{,}=d\,d\,x_{,},$$

and in a similar way $d\,d\,y=d\,d\,y_{,}$, $d\,d\,z=d\,d\,z_{,}$, &c. These, together with

$$\delta x=\delta X+\delta x_{,}, \qquad \delta y=\delta Y+\delta y_{,}, \text{ \&c.}$$

being substituted in the equation [142], it becomes

$$0=\Sigma.m.\left(\delta X+\delta x_{,}\right).\left(\frac{d\,d\,x_{,}}{d\,t^2}-P\right)$$
$$+\Sigma.m.\left(\delta Y+\delta y_{,}\right).\left(\frac{d\,d\,y_{,}}{d\,t^2}-Q\right)+\Sigma.m.\left(\delta Z+\delta z_{,}\right).\left(\frac{d\,d\,z_{,}}{d\,t^2}-R\right).$$

and as δX, δY, δZ, are common to all the bodies, they may be brought from under the sign Σ, which gives

$$0=\delta X.\Sigma.m.\left(\frac{d\,d\,x_{,}}{d\,t^2}-P\right)+\delta Y.\Sigma.m.\left(\frac{d\,d\,y_{,}}{d\,t^2}-Q\right)+\delta Z.\Sigma.m.\left(\frac{d\,d\,z_{,}}{d\,t^2}-R\right)$$
$$+\Sigma.m.\delta x_{,}.\left(\frac{d\,d\,x_{,}}{d\,t^2}-P\right)+\Sigma.m.\delta y_{,}.\left(\frac{d\,d\,y_{,}}{d\,t^2}-Q\right)+\Sigma.m.\delta z_{,}.\left(\frac{d\,d\,z_{,}}{d\,t^2}-R\right).$$

But $\Sigma.m.(d\,d\,X+d\,d\,x_{,})-\Sigma.m.P.d\,t^2=0$, and $d\,d\,X=0$, [184, 183], hence we get $\Sigma.m.\left(\frac{d\,d\,x_{,}}{d\,t^2}-P\right)=0$, and in a similar manner $\Sigma.m.\left(\frac{d\,d\,y_{,}}{d\,t^2}-Q\right)=0$, $\Sigma.m.\left(\frac{d\,d\,z_{,}}{d\,t^2}-R\right)=0$; these quantities are the co-efficients of δX, δY, δZ, in the which is exactly of the same form as the equation (P) [142], if the forces P, Q, R, depend only on the co-ordinates $x_{,}$, $y_{,}$, $z_{,}$, $x'_{,}$, &c. By applying the preceding analysis, we can deduce, from this equation, the principles of the preservation of the living forces and areas, with respect to the moveable origin of the co-ordinates.

If the system is not acted upon by any extraneous forces, its centre of gravity will have a rectilineal uniform motion, as has been shown in § 20 [159'] ; by fixing, therefore, at this centre, the origin of the co-ordinates x, y, z, these principles will always subsist. X, Y, Z, being then the co-ordinates of the centre of gravity, we shall have, by the nature of the point,* [185']

[186] $$0=\Sigma.m.x_{,}\,; \qquad 0=\Sigma.m.y_{,}\,; \qquad 0=\Sigma.m.z_{,}\,;$$

hence we get†

[187] $$\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}=\frac{(X\,dY-Y\,dX)}{dt}.\Sigma.m+\Sigma.m.\frac{(x_{,}\,dy_{,}-y_{,}\,dx_{,})}{dt}\,;$$

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preceeding equation, therefore those terms must vanish, and the resulting equation will be

$$0=\Sigma.m.\delta x_{,}.\left(\frac{ddx_{,}}{dt^2}-P\right)+\Sigma.m.\delta y_{,}.\left(\frac{ddy_{,}}{dt^2}-Q\right)+\Sigma.m.\delta z_{,}.\left(\frac{ddz_{,}}{dt^2}-R\right),$$

as in [185].

* (93) By putting $X'=X$, $Y'=Y$, $Z'=Z$, in [183a], we obtain

$$X=\frac{\Sigma.m.(X+x_{,})}{\Sigma.m}\,; \qquad Y=\frac{\Sigma.m.(Y+y_{,})}{\Sigma.m}\,; \qquad Z=\frac{\Sigma.m.(Z+z_{,})}{\Sigma.m},$$

and as X, Y, Z, are common to all the terms, we may bring them from under the sign Σ, making $\frac{\Sigma.mX}{\Sigma.m}=\frac{X.\Sigma.m}{\Sigma.m}=X$, &c., consequently $X=X+\frac{\Sigma.mx_{,}}{\Sigma.m}$, or $0=\frac{\Sigma.mx_{,}}{\Sigma.m}$, or $0=\Sigma.mx_{,}$, and in a similar way, from the other two equations, we get $0=\Sigma.my_{,}$, $0=\Sigma.mz_{,}$. The differentials of these equations divided by dt, are

[186a] $$0=\Sigma.m.\frac{dx_{,}}{dt}, \qquad 0=\Sigma.m.\frac{dy_{,}}{dt}, \qquad 0=\Sigma.m.\frac{dz_{,}}{dt},$$

which will be used hereafter.

† (94) Substituting the values of x, y, z, [182], in $m.\frac{(x\,dy-y\,dx)}{dt}$, it becomes

$$m.\left\{\frac{(X\,dY-Y\,dX)}{dt}+x_{,}\frac{dY}{dt}-y_{,}\frac{dX}{dt}+X\frac{dy_{,}}{dt}-Y\frac{dx_{,}}{dt}+\frac{(x_{,}dy_{,}-y_{,}dx_{,})}{dt}\right\}.$$

Marking these letters with one, two, &c., accents, we obtain the corresponding equations for m', m'', &c., their sum gives

$$\Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} = \frac{(dX^2+dY^2+dZ^2)}{dt^2} . \Sigma . m + \Sigma . m . \frac{(dx_{,}^2+dy_{,}^2+dz_{,}^2)}{dt^2} ;^{*} \qquad [188]$$

hence the quantities resulting from the preceding principles, are composed, [188'] *First*, of quantities which would exist if all the bodies of the system were united in their common centre of gravity;† *Second*, of quantities relative to the

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$$\Sigma . m . \frac{(x\,dy-y\,dx)}{dt} = \frac{(X\,dY-Y\,dX)}{dt} . \Sigma . m . + \frac{dY}{dt} . \Sigma . m\,x_{,} - \frac{dX}{dt} . \Sigma . m\,y_{,} \qquad [187a]$$

$$+ X . \Sigma . m . \frac{dy_{,}}{dt} - Y . \Sigma . m . \frac{dx_{,}}{dt} + \Sigma . m . \left(\frac{x_{,}dy_{,}-y_{,}dx_{,}}{dt}\right),$$

the factors X, Y, $\frac{dX}{dt}$, $\frac{dY}{dt}$, being the same for all the terms, are brought from under the sign Σ, as in the preceding notes. Substituting the values [186, 186*a*], it becomes as in [187].

[*] (95) Substituting the values of x, y, z, &c. [182], in $\Sigma . m . (dx^2+dy^2+dz^2)$, we obtain

$$\Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} = \Sigma . m . \left\{ \frac{(dX+dx)^2+(dY+dy_{,})^2+(dZ+dy_{,})^2}{dt^2} \right\} .$$

Developing the second member, and bringing X, Y, dX, dY, from under the sign Σ, we get

$$\Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} = \frac{(dX^2+dY^2+dZ^2)}{dt^2} . \Sigma . m + \Sigma . m . \frac{(dx_{,}^2+dy_{,}^2+dz_{,}^2)}{dt^2}$$

$$+ 2 . dX . \Sigma . m . \frac{dx_{,}}{dt} + 2 . dY . \Sigma . m . \frac{dy_{,}}{dt} + 2 . dZ . \Sigma . m . \frac{dz_{,}}{dt}, \qquad [188a]$$

which, by means of the equations [186*a*], becomes as in [188].

[†] (95*a*) If all the bodies were situated in the centre of gravity of the system, we should have $X = x = x' = x'' = \&c.$; $Y = y = y' = y'' = \&c.$; $Z = z = z' = z'' = \&c.$; and the quantities $x_{,}$, $x_{,}'$, &c.; $y_{,}$, $y_{,}'$, &c.; $z_{,}$, $z_{,}'$, &c., [182], would vanish. Therefore the first members of the equations [187, 188], would become, respectively,

$$\Sigma . m . \frac{(X\,dY-Y\,dX)}{dt}, \qquad \Sigma . m . \frac{(dX^2+dY^2+dZ^2)}{dt^2};$$

and by bringing the terms X, Y, Z, from under the sign Σ, they become

$$\frac{(X\,dY-Y\,dX)}{dt} . \Sigma . m, \qquad \text{and} \qquad \frac{(dX^2+dY^2+dZ^2)}{dt^2} . \Sigma . m;$$

which are like the first terms of the second members of [187, 188], as is observed above. Again, if the centre of gravity is at rest, we shall have $\frac{dX}{dt}=0$; $\frac{dY}{dt}=0$; $\frac{dZ}{dt}=0$, [182*a*], and the first members of [187*a*, 188*a*], will become like the last terms of the second members of the equations [187*a*, 188*a*] or [187, 188].

Centre of Gravity and Plane of the greatest Areas. centre of gravity supposed at rest; and as the former quantities are constant, we perceive the reason why these principles exist with respect to the centre of gravity. By fixing, therefore, at this point, the origin of the co-ordinates x, y, z, x', &c. of the equations (Z) [167] of the preceding article, they will always subsist; whence it follows, that the plane passing always through that centre, and relative to which the function $\Sigma . m . \frac{(x\,dy - y\,dx)}{dt}$ is a *maximum*, remains always parallel to itself, while the system continues in motion, and that the same function, relative to any other plane perpendicular to the preceding, is nothing.

The principles of the preservation of the areas and the living forces may be reduced to certain relations between the co-ordinates of the mutual distances of the bodies of the system. For the origin of x, y, z, being always supposed to be at the centre of gravity, the equations (Z) [167] of the preceding article, may be put under the form*

[189]
$$c . \Sigma . m = \Sigma . mm' . \left\{ \frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt} \right\};$$
$$c' . \Sigma . m = \Sigma . mm' . \left\{ \frac{(x'-x).(dz'-dz)-(z'-z).(dx'-dx)}{dt} \right\};$$
$$c'' . \Sigma . m = \Sigma . mm' . \left\{ \frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt} \right\}.$$

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* (96) As some doubt of the accuracy of these equations has been expressed by a writer in an eminent scientific European journal, from a misconception of their meaning, we shall enter into some detail for illustration; and for brevity, shall put,

$$\frac{(x\,dy-y\,dx)}{dt} = C, \qquad \frac{(x'\,dy'-y'\,dx')}{dt} = C', \qquad \frac{(x''\,dy''-y''\,dx'')}{dt} = C'', \text{ \&c.}$$

[188b]
$$\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt} = [x, x'];$$
$$\frac{(x''-x).(dy''-dy)-(y''-y).(dx''-dx)}{dt} = [x, x''];$$
$$\frac{(x''-x').(dy''-dy')-(y''-y').(dx''-dx')}{dt} = [x', x''], \text{ \&c.}$$

Then the first of the above equations, by substituting for c its value [167], becomes

$$\Sigma . m . \frac{(x\,dy-y\,dx)}{dt} . \Sigma . m = \Sigma . mm' . \left\{ \frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt} \right\},$$

which, if there be only two bodies m, m', becomes $(m\,C + m'\,C').(m+m') = mm'.[x, x']$.

We may observe that the second members of these equations, multiplied by dt, express the sums of the projections of the elementary areas, described by each right line connecting any two bodies of the system, of which the [189′]

If there be three bodies m, m', m'', it is

$$(m\,C + m'\,C' + m''\,C'') \,.\, (m + m' + m'') = m\,m' \,.\, [x,x'] + m\,m'' \,.\, [x,x''] + m'\,m'' \,.\, [x',x''].$$

If there be four bodies m, m', m'', m''', it becomes,

$$(m\,C + m'\,C' + m''\,C'' + m'''\,C''') \,.\, (m + m' + m'' + m''') = m\,m' \,.\, [x,x'] + m\,m'' \,.\, [x,x''] + m\,m''' \,.\, [x,x'''] + m'\,m'' \,.\, [x',x''] + m'\,m''' \,.\, [x',x'''] + m''\,m''' \,.\, [x'',x'''],$$

and thus for any greater number of bodies. Observing that each body, in the second member of this equation, is supposed to be combined with all the others but once, and that the whole number of bodies being n, the number of terms in that second member is $\frac{n(n-1)}{2}$, as is evident by the usual rules of combination. Similar remarks may be made on the second and third equations, [189]. Having thus explained the import of these equations, we shall now proceed to the demonstration. On account of the symmetry of these equations, we might limit ourselves to the consideration of two bodies only, as m, m', but for the reason above named, we shall notice the other bodies m'', m''', &c.

We shall first prove that we have identically,

$$\Sigma \,.\, m \,.\, \frac{(x\,dy - y\,dx)}{dt} \,.\, \Sigma \,.\, m = \Sigma \,.\, m\,m' \,.\, \left\{ \frac{(x'-x)\,.\,(dy'-dy) - (y'-y)\,.\,(dx'-dx)}{dt} \right\} + \Sigma \,.\, m\,x \,.\, \Sigma \,.\, \frac{m\,dy}{dt} - \Sigma \,.\, m\,y \,.\, \Sigma \,.\, \frac{m\,dx}{dt},$$ [189a]

whatever be the origin of the co-ordinates; that is, whether the origin be at the centre of gravity or not. Now as both members of this equation are symmetrical in m, m', m'', &c. it is only necessary to prove that the terms multiplied by any one of the quantities m, m', m'', &c. as m, is the same in both.

The second member of [189a], being developed, becomes

$$m\,m'\,[x,x'] + m\,m''\,[x,x''] + \&c.$$
$$+ (m\,x + m'\,x' + \&c.) \,.\, \left(\frac{m\,dy + m'\,dy' + \&c.}{dt} \right) - (m\,y + m'\,y' + \&c.) \,.\, \left(\frac{m\,dx + m'\,dx' + \&c.}{dt} \right),$$

of which the part having the factor m is

$$m^2 \,.\, \frac{(x\,dy - y\,dx)}{dt} + m \,.\, \left\{ m'\,[x,x'] + m'' \,.\, [x,x''] + \&c. \right\}$$
$$+ m \,.\, \left\{ x \,.\, \frac{(m'\,dy' + m''\,dy'' + \&c.)}{dt} - y \,.\, \frac{(m'\,dx' + m''\,dx'' + \&c.)}{dt} \right\}$$
$$+ m \,.\, \left\{ \frac{dy}{dt} \,.\, (m'\,x' + m''\,x'' + \&c.) - \frac{dx}{dt} \,.\, (m'\,y' + m''\,y'' + \&c.) \right\};$$

one is supposed to move about the other considered as at rest, each area being multiplied by the product of the two masses which are connected by the right line.

If we apply the analysis of § 21 to the preceding equations, we shall find

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in which the terms multiplied by $m\,m'$ are $m\,m' . \left\{ [x, x'] + \frac{(x\,dy' + x'\,dy - y\,dx' - y'\,dx)}{dt} \right\}$, and by substituting the value of $[x, x']$, [188*b*], it becomes

$$mm' . \left\{ \frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)+x\,dy'+x'\,dy-y\,dx'-y'\,dx}{dt} \right\};$$

or by reduction $m\,m' . \left\{ \frac{(x\,dy - y\,dx) + (x'\,dy' - y'\,dx')}{dt} \right\}$ equal to $m\,m' . (C + C')$. In like manner, by adding one, two, &c. accents, to the letters m', C', we obtain the parts depending on m'', m''', &c., which will be $m\,m'' . (C + C'')$, $m\,m''' . (C + C''')$, &c.; consequently, the part of the second member of [189*a*], having the factor m, is

$$\begin{aligned} & m^2\,C + m\,m'\,(C + C') + m\,m''\,(C + C'') + \&c. \\ & = m\,C\,(m + m' + m'' + \&c.) + m\,(m'\,C' + m''\,C'' + \&c.). \end{aligned}$$

Again the first member of [189*a*], $\Sigma . m . \frac{(x\,dy - y\,dx)}{dt} . \Sigma . m$, being developed, is

$$(m\,C + m'\,C' + m''\,C'' + \&c.) . (m + m' + m'' + \&c.),$$

the part of it, having the factor m, is

$$m\,C . (m + m' + m'' + \&c.) + m . (m'\,C' + m''\,C'' + \&c.);$$

and as this is equal to the expression of the second member of the same equation [189*a*], just found, it will follow, that the equation [189*a*], takes place, for any origin of the co-ordinates; and by fixing the origin at the centre of gravity, we shall have, as in [186*a*], $0 = \Sigma . m . \frac{dx}{dt}$; $0 = \Sigma . m . \frac{dy}{dt}$. Substitute these in [189*a*], it becomes like the first of the equations [189]. The second is easily derived from the first, by changing y, y', &c. into z, z', &c.; and the third is obtained from the second, by changing x, x', &c. into y, y', &c. If, as in [161*a*], we put $c' = -c_{\prime\prime}$, $c'' = c_{\prime}$, the three equations [189], may be placed in the following, more symmetrical form.

[189*b*] $$c . \Sigma . m = \Sigma . m\,m' . \left\{ \frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt} \right\};$$

[189*c*] $$c_{\prime} . \Sigma . m = \Sigma . m\,m' . \left\{ \frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt} \right\};$$

[189*d*] $$c_{\prime\prime} . \Sigma . m = \Sigma . m\,m' . \left\{ \frac{(z'-z).(dx'-dx)-(x'-x).(dz'-dz)}{dt} \right\}.$$

that the plane passing constantly through any one of the bodies of the system, and relatively to which the function

$$\Sigma . mm' . \left\{ \frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt} \right\}$$ [189″]

is a *maximum*,* preserves its parallelism, during the motion of the system,

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*(97) Using the values [172*c*] for the body m, and adding one accent to x, y, z, &c. for the body m', and two accents for m'', &c., we shall get

$$x_{,,,}'-x_{,,,}=\mathcal{A}_0.(x'-x)+\mathcal{A}_1.(y'-y)+\mathcal{A}_2.(z'-z);$$
$$y_{,,,}'-y_{,,,}=B_0.(x'-x)+B_1.(y'-y)+B_2.(z'-z);$$
$$z_{,,,}'-z_{,,,}=C_0.(x'-x)+C_1.(y'-y)+C_2.(z'-z);$$

and by [175*d*], we have

$$\frac{(x_{,,,}dy_{,,,}-y_{,,,}dx_{,,,})}{dt}=C_2.\frac{(xdy-ydx)}{dt}+C_1.\frac{(zdx-xdz)}{dt}+C_0.\frac{(ydz-zdy)}{dt}.$$ [189*e*]

Now by comparing the values of $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, [172*c*] with these values of $x_{,,,}'-x_{,,,}$, $y_{,,,}'-y_{,,,}$, $z_{,,,}'-z_{,,,}$, it is evident that we may substitute in this equation, $x_{,,,}'-x_{,,,}$ for $x_{,,,}$, $y_{,,,}'-y_{,,,}$ for $y_{,,,}$, $z_{,,,}'-z_{,,,}$ for $z_{,,,}$, $x'-x$ for x, $y'-y$ for y, &c. because the quantities $\mathcal{A}_0$, $\mathcal{A}_1$, &c. are not affected by these changes. Hence, by making these substitutions in [189*e*], and multiplying by mm' we shall get,

$$\begin{aligned} & mm'.\left\{\frac{(x_{,,,}'-x_{,,,}).(dy_{,,,}'-dy_{,,,})-(y_{,,,}'-y_{,,,}).(dx_{,,,}'-dx_{,,,})}{dt}\right\} \\ & = C_2.mm'.\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\} \\ & + C_1.mm'.\left\{\frac{(z'-z).(dx'-dx)-(x'-x).(dz'-dz)}{dt}\right\} \\ & + C_0.mm'.\left\{\frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt}\right\}. \end{aligned}$$

If we change the co-ordinates of m, m', into those relative to any two other bodies of the system, we shall obtain similar expressions for them. Taking the sum of these equations, and substituting in the second member $c.\Sigma.m$, $c_,.\Sigma.m$, $c_{,,}.\Sigma.m$, for

$$\Sigma.mm'.\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\}, \text{ \&c. } [189b, c, d]$$

we shall get,

$$\begin{aligned} & \Sigma.mm'.\left\{\frac{(x_{,,,}'-x_{,,,}).(dy_{,,,}'-dy_{,,,})-(y_{,,,}'-y_{,,,}).(dx_{,,,}'-dx_{,,,})}{dt}\right\} \\ & = C_2.c.\Sigma.m+C_1.c_{,,}.\Sigma.m+C_0.c_,.\Sigma.m, \end{aligned}$$ [189*f*]

and by changing the letters $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, C_2, C_1, C_0, we shall get the following equations, in the same manner as [175*e*, *f*] were obtained from [175*d*],

$$\begin{aligned} & \Sigma.mm'.\left\{\frac{(y_{,,,}'-y_{,,,}).(dz_{,,,}'-dz_{,,,})-(z_{,,,}'-z_{,,,}).(dy_{,,,}'-dy_{,,,})}{dt}\right\} \\ & = \mathcal{A}_2.c.\Sigma.m+\mathcal{A}_1.c_{,,}.\Sigma.m+\mathcal{A}_0.c_,.\Sigma.m; \end{aligned}$$ [189*g*]

and that this plane is parallel to the plane passing through the centre of gravity, and relatively to which the function $\Sigma . m . \frac{(x\,dy - y\,dx)}{dt}$ is a maximum,* we shall also find, that the second members of the preceding equations are nothing, for all planes passing through the same body perpendicular to the plane just mentioned. [189‴]

The equation (Q) § 19 [143], may be put under this form,†

[190] $$\Sigma . mm' . \left\{ \frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2} \right\} = \text{constant} - 2 . \Sigma . m . \Sigma . \textstyle\int . mm' . F . df;$$

[189h] $$\Sigma . mm' . \left\{ \frac{(z_{\prime\prime\prime}'-z_{\prime\prime\prime}) . (dx_{\prime\prime\prime}'-dx_{\prime\prime\prime}) - (x_{\prime\prime\prime}'-x_{\prime\prime\prime}) . (dz_{\prime\prime\prime}'-dz_{\prime\prime\prime})}{dt} \right\}$$
$$= B_2 . c . \Sigma . m + B_1 . c_{\prime\prime} . \Sigma . m + B_0 . c_{\prime} . \Sigma . m,$$

which are similar to the equations [175*h*, *i*, *k*], and agree with them, by writing $x_{\prime\prime\prime}'-x_{\prime\prime\prime}$, $y_{\prime\prime\prime}'-y_{\prime\prime\prime}$, $z_{\prime\prime\prime}'-z_{\prime\prime\prime}$, $c . \Sigma . m$, $c_{\prime} . \Sigma . m$, $c_{\prime\prime} . \Sigma . m$, for $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, c, $c_{\prime}$, $c_{\prime\prime}$, respectively; consequently the results obtained from the equations [175*h*, *i*, *k*] or from their equivalent expressions [175—177], are equally applicable to the equations just found. Therefore, the second members of the two last of these equations are rendered equal to nothing, by assuming for θ and ψ, precisely the same values, as are required to make the second members of the equations [176, 177], equal to nothing. For, if we change c, c', c'', into $c . \Sigma . m$, $c' . \Sigma . m$, $c'' . \Sigma . m$, respectively, the equations [178], by which θ, ψ are determined, will remain unchanged, rejecting the factor $\Sigma . m$, common to the numerator and denominator.

* (98) This evidently follows from the calculation in the preceding note, where it is shown that the same values of θ, ψ, render each of the expressions $\Sigma . m . \frac{(x\,dy - y\,dx)}{dt}$;

$$\Sigma . mm' . \left\{ \frac{(x'-x) . (dy'-dy) - (y'-y) . (dx'-dx)}{dt} \right\},$$

a maximum.

† (99) Multiply the equation [143], by $\Sigma . m$, and substitute for

$$\Sigma . \textstyle\int . m . (P . dx + Q . dy + R . dz),$$

its value $-\Sigma . \int . mm' . F . df$, deduced from [143″], the system not being acted upon by any extraneous forces, [185′], it becomes,

[189i] $$\Sigma . m . \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} = c . \Sigma . m - 2 . \Sigma . m . \Sigma . \textstyle\int . mm' . F . df,$$

of which the second member is the same as in [190], supposing the constant quantity to be $c . \Sigma . m$, and the first members of the same equations, [190, 189*i*] will be found by developement, to be equal. For we have identically

[189k] $$\Sigma . m . \Sigma . m . dx^2 = \Sigma . mm' . (dx'-dx)^2 + (\Sigma . m . dx)^2, \quad \text{or}$$

[189l] $$(m + m' + m'' + \&c.) . (m . dx^2 + m' . dx'^2 + m'' . dx''^2 + \&c.)$$
$$= mm' . (dx'-dx)^2 + mm'' . (dx''-dx)^2 + \&c. + (m . dx + m' . dx' + \&c.)^2;$$

which contains only the co-ordinates of the mutual distances of the bodies from each other, and in which the first member expresses the sum of the squares of the relative velocities of the bodies of the system about each other, combining them in pairs, and supposing one of the two to be at rest, [190′]
each square being multiplied by the product of the two masses corresponding to it.

23. Let us now resume the equation (*R*) § 19 [144]. By taking its differential relative to the characteristic δ, we shall have*

$$\Sigma . m . v \delta v = \Sigma . m . (P . \delta x + Q . \delta y + R . \delta z) ; \qquad [191]$$

the equation (*P*) § 18 [142], thus becomes

$$0 = \Sigma . m . \left\{ \delta x . d . \frac{dx}{dt} + \delta y . d . \frac{dy}{dt} + \delta z . d . \frac{dz}{dt} \right\} - \Sigma . m dt . v \delta v ; \qquad [192]$$

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since the terms multiplied by m, in the second member of this last expression, are $m . \{ m' (dx' - dx)^2 + m'' . (dx'' - dx)^2 + \&c. + m . dx^2 + 2 . dx . (m' . dx' + m'' . dx'' + \&c.) \}$, which by reduction becomes $m . \{ dx^2 . (m' + m'' + \&c.) + (m . dx^2 + m' . dx'^2 + \&c.) \}$, and this is evidently equal to the terms multiplied by m, in the first member of [189*l*], therefore, the equation [189*k*], is identical. In like manner, by changing x', x, into y', y, and z', z, we obtain,

$$\Sigma . m . \Sigma . m . dy^2 = \Sigma . m m' . (dy' - dy)^2 + (\Sigma . m . dy)^2 ;$$

$$\Sigma . m . \Sigma . m . dz^2 = \Sigma . m m' . (dz' - dz)^2 + (\Sigma . m . dz)^2 .$$

The sum of these, divided by dt^2, gives identically,

$$\Sigma . m . \Sigma . m . \frac{(dx^2 + dy^2 + dz^2)}{dt^2} = \Sigma . m m' . \left\{ \frac{(dx' - dx)^2 + (dy' - dy)^2 + (dz' - dz)^2}{dt^2} \right\}$$

$$+ \left(\Sigma . m . \frac{dx}{dt} \right)^2 + \left(\Sigma . m . \frac{dy}{dt} \right)^2 + \left(\Sigma . m . \frac{dz}{dt} \right)^2 . \qquad [190a]$$

Now the origin of the co-ordinates being supposed at the centre of gravity, we shall have, by [186*a*], $\Sigma . m . \frac{dx}{dt} = 0$, $\Sigma . m . \frac{dy}{dt} = 0$, $\Sigma . m . \frac{dz}{dt} = 0$; all which, substituted in [189*i*], gives the equation [190]. Lastly, it is evident, by the reasoning in [40*a*], that the
relative velocity of m' about m, resolved in directions parallel to x, y, z, is $\frac{dx' - dx}{dt}$, [190*b*]
$\frac{dy' - dy}{dt}$, $\frac{dz' - dz}{dt}$; the sum of whose squares gives, as in [40*a*], the square of the whole of that relative velocity, as in [190′].

* (100) By substituting, in the differential of [144], divided by two, for $\delta \varphi$, its value [143ᵛ],

$$\delta \varphi = \Sigma . m . (P . \delta x + Q . \delta y + R . \delta z) . \qquad [190c]$$

let ds be the element of the curve described by m; ds' that described by m', &c.; we shall have [44]

[193] $$v\,dt=ds; \qquad v'\,dt=ds'; \qquad \&c.;$$

[193'] $$ds=\sqrt{dx^2+dy^2+dz^2};$$

whence we deduce, by following the analysis of § 8,*

[194] $$\Sigma.m.\delta.(v\,ds)=\Sigma.m.d.\frac{(dx.\delta x+dy.\delta y+dz.\delta z)}{dt}.$$

By integrating this equation with respect to the characteristic d, and taking the limits of the integrals to correspond to the extreme points of the curves described by the bodies m, m', &c., we shall have

[195] $$\Sigma.\delta.\textstyle\int.m\,v\,ds=\text{constant}+\Sigma.m.\dfrac{(dx.\delta x+dy.\delta y+dz.\delta z)}{dt};$$

the variations δx, δy, δz, &c., and the constant term of the second equation, correspond to the extreme points of the curves described by m, m', &c.

Hence it follows, that if these points are supposed invariable, we shall have†

[196] $$0=\Sigma.\delta.\textstyle\int.m\,v\,ds;$$

which shows that the function $\Sigma.\int.mvds$, is a minimum.‡ In this consists

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* [101] By substituting $v\,dt=ds$, $v'\,dt=ds'$, &c. in the equation [192], it becomes

$$0=\Sigma.m.\left\{\delta x.d.\frac{dx}{dt}+\delta y.d.\frac{dy}{dt}+\delta z.d.\frac{dz}{dt}\right\}-\Sigma.m.ds.\delta v,$$

which corresponds to the equation [45]. Again, the formula [47], which was easily deduced from the variation of ds^2, being multiplied by m, and added to the similar equations in m', m'', &c. gives,

$$\Sigma.mv.\delta ds=\Sigma.m.\left\{d.\frac{\{dx.\delta x+dy.\delta y+dz.\delta z\}}{dt}-\delta x.d.\frac{dx}{dt}-\delta y.d.\frac{dy}{dt}-\delta z.d.\frac{dz}{dt}\right\}.$$

Adding this equation to the preceding, transposing $\Sigma.m.ds.\delta v$, and putting $\delta.(v\,ds)$ for $v.d\delta s+ds.\delta v$, it becomes as in [194], whose integral is [195].

† (102) This will appear by the reasoning at the end of note 19.

[196a] ‡ (103) It has been remarked that the maximum, or minimum, of the expression of the living force of a system of bodies $\Sigma.m.v^2$, corresponds to that state of the system, in which it would remain in equilibrium, by means of the accelerative forces acting upon it, supposing

the principle of the least action in the motion of a system of bodies; which principle, as we have proved, is a result, deduced by mathematical principles, [196]
from the primitive laws of the equilibrium and motion of matter. We

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the system to be placed directly in that situation without any velocity. This would follow, from the expression of the living force [144], $c+2\varphi$, which, by the usual rules of the differential calculus, would become a maximum, or minimum, when

$$d\varphi = \Sigma . m . (P . dx + Q . dy + R . dz) = 0, \qquad [143^{v}], \qquad [196b]$$

Now from the principle of virtual velocities, we shall have, in the case of equilibrium, [41, 110″], $\Sigma . m . (P . \delta x + Q . \delta y + R . \delta z) = 0$; δx, δy, δz, being arbitrary variations, satisfying the conditions of the system; and since all these conditions are satisfied by putting $\delta x = dx$, $\delta y = dy$, $\delta z = dz$, we may substitute these values in the preceding equation [196c]
of equilibrium, and it will become $\Sigma . m . (P . dx + Q . dy + R . dz) = 0$, which corresponds to the maximum or minimum of the living force, [196b]. A well known example of this principle occurs, in the case of a heavy homogeneous cylinder, rolling upon a horizontal plane; the sum of the living forces of all its particles will be a maximum, or minimum, when either extremity of the conjugate, or transverse axis of the ellipsis, touches the horizontal plane; because the cylinder would remain in equilibrium, in either of those states, if it had no velocity. The equilibrium would be stable when the conjugate axis touches the plane; [196d]
but unstable when the transverse axis touches the plane. In the former case the centre of gravity would be at its lowest point, in the latter case, at its highest point. If a system of bodies is held at rest in a situation very near to its state of equilibrium, as would be the case with the cylinder just mentioned, supposing the end of the conjugate axis nearly to touch the horizontal plane, the system would, generally, upon being left free from restraint, endeavor to attain this point of equilibrium, and in doing this, the particles of the system, from being at rest, would, by the mere action of the accelerative forces, acquire a very small velocity, which would increase the living force, in approaching the point where the equilibrium would take place. Therefore the *maximum* of the living force generally corresponds to the [196e]
point of *stable* equilibrium. The contrary takes place when the system is placed at rest, in a situation infinitely near to the state of unstable equilibrium; since the system, if left to itself, would, on account of its being in an unstable state of equilibrium, generally endeavor to recede from the point of equilibrium, and in doing this, the velocity of the particles, and the whole living force, from being nothing, would become finite, while removing from the situation of *unstable* equilibrium; therefore this state must generally correspond to the *minimum* of the [196f]
living force.

If the system be acted upon only by gravity, g, which we shall suppose to act upon the bodies, in the direction of the lines z, z', &c., drawn to the centre of the earth, which may be considered as parallel; we may put $P=0$, $Q=0$, $R=g$, and we shall have, also find, that this principle, combined with that of the living forces, gives the equation (P) § 18 [142],* which contains all that is necessary

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$d\varphi = \Sigma . m . g\, dz = g . \Sigma . m\, dz$, whose integral is $\varphi = c + g . \Sigma . m z$. If we put Z for the distance of the centre of gravity of the system from the centre of the earth, we shall have $\Sigma . m z = Z . \Sigma . m$, [127]. Substitute this in the preceding value of φ, it becomes $\varphi = c + Z . g . \Sigma . m$, and as $g . \Sigma . m$ is a constant quantity, it is evident that the maximum, or minimum of φ, must correspond to the maximum or minimum of Z. Hence it follows, that the system will be in equilibrium, when the centre of gravity is at its lowest or highest point. The former case occurs in the catenarian curve, the latter in an arch or bridge composed of small globules resting upon each other, in the form of an inverted catenarian curve.

* (103a) Having, as in [44], $ds^2 = dx^2 + dy^2 + dz^2$, if we take its variation relative to δ, then divide by $2\, dt$, and substitute $\frac{ds}{dt} = v$, it becomes,

$$v . \delta\, ds = \frac{(dx . d\delta x + dy . d\delta y + dz . d\delta z)}{dt},$$

which multiplied by m, gives

$$m . v . \delta\, ds = m . \frac{(dx . d\delta x + dy . d\delta y + dz . d\delta z)}{dt},$$

adding to these one accent for the body m', two accents for the body m'', &c., and taking the sum of all these equations, we get

$$\Sigma . m . v . \delta\, ds = \Sigma . m . \left\{ \frac{dx}{dt} . \delta\, dx + \frac{dy}{dt} . \delta\, dy + \frac{dz}{dt} . \delta\, dz \right\},$$

which being integrated by parts relative to the sign d, putting for $\int \frac{dx}{dt} . \delta\, dx$, its value $\frac{dx}{dt} . \delta x - \int . \delta x . d . \frac{dx}{dt}$, &c. which is easily proved by differentiation, it becomes,

[196g] $$\Sigma . \int . m . v . \delta ds = \Sigma . m . \left\{ \frac{(dx . \delta x + dy . \delta y + dz . \delta z)}{dt} - \int \left(\delta x . d . \frac{dx}{dt} + \delta y . d . \frac{dy}{dt} + \delta z . d . \frac{dz}{dt} \right) \right\}.$$

The principle of the least action $\Sigma . \delta . \int . m v . ds = 0$, [196], by expanding the differential relative to δ, is $0 = \Sigma . \int . (m v . d\delta s + m \delta v . ds)$, whence

$$\Sigma . \int . m v . d\delta s = - \Sigma . m \delta v . ds,$$

or since $ds = v dt$, $\Sigma . \int . m v . d\delta s = - \Sigma . \int . m . v \delta v . dt$; and the principle of living forces gives, as in [191],

$$- \Sigma . \int . m . v \delta v . dt = - \Sigma . \int . m . dt . (P . \delta x + Q . \delta y + R . \delta z).$$

Substituting this in the preceding equation, we get

$$- \Sigma . \int . m . dt . (P . \delta x + Q . \delta y + R . \delta z)$$
$$= \Sigma . m . \left\{ \frac{(dx . \delta x + dy . \delta y + dz . \delta z)}{dt} - \int \left(\delta x . d . \frac{dx}{dt} + \delta y . d . \frac{dy}{dt} + \delta z . d . \frac{dz}{dt} \right) \right\},$$

to determine the motions of the system. Lastly,* it is evident that this principle also exists, when the origin of the co-ordinates is in motion,

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and by reduction this becomes

$$\Sigma . m . \left\{ \frac{(dx . \delta x + dy . \delta y + dz . \delta z)}{dt} \right\}$$

$$= \Sigma . m . \int \left\{ \left(d . \frac{dx}{dt} - P . dt \right) . \delta x + \left(d . \frac{dy}{dt} - Q . dt \right) . \delta y + \left(d . \frac{dz}{dt} - R . dt \right) . \delta z \right\}.$$

The terms of the first member correspond to the extreme points of the curves described by the bodies, that member is therefore constant, as is evident by the reasoning in note 19, and its differential relative to *d* being taken, it will become nothing, and we shall finally obtain,

$$0 = \Sigma . m . \left\{ \left(d . \frac{dx}{dt} - P . dt \right) . \delta x + \left(d . \frac{dy}{dt} - Q . dt \right) . \delta y + \left(d . \frac{dz}{dt} - R . dt \right) . \delta z \right\},$$

which is the same as the equation [142].

* (103*b*) The meaning of this proposition is, that if $v_{,}$, $v'_{,}$, $v''_{,}$, &c. are the velocities of the bodies m, m', m'', &c., referred to the origin of the co-ordinates, supposing that origin to be in motion, and $v_{,} dt = ds_{,}$, $v'_{,} dt = ds'_{,}$, $v''_{,} dt = ds''_{,}$, &c., we shall have, $0 = \Sigma . \delta . \int . m v_{,} ds_{,}$, similar to [196]. For by using the values of x, y, z, x', y', z', &c. [182], and supposing the motion of the point of origin to be uniform and rectilineal, we shall have, as in [185],

$$0 = \Sigma . m . \delta x_{,} . \left\{ \frac{ddx_{,}}{dt^2} - P \right\} + \Sigma . m . \delta y_{,} . \left\{ \frac{ddy_{,}}{dt^2} - Q \right\} + \Sigma . m . \delta z_{,} . \left\{ \frac{ddz_{,}}{dt^2} - R \right\}. \qquad [196h]$$

From this equation, which is exactly similar to [142], we may easily deduce one similar to [143], by putting $\delta x_{,} = dx_{,}$, $\delta y_{,} = dy_{,}$, &c., and integrating, which gives,

$$\Sigma . m . \frac{(dx_{,}^2 + dy_{,}^2 + dz_{,}^2)}{dt^2} = c + 2 . \Sigma . \int . m . (P . dx_{,} + Q . dy_{,} + R . dz_{,}),$$

or, by putting as in [144], the first member equal to $\Sigma . m . v_{,}^2$, and supposing

$$\Sigma . m . (P . dx_{,} + Q . dy_{,} + R . dz_{,}) = d\varphi,$$

an exact differential; also taking the variation relative to δ, we shall have, as in [191],

$$\Sigma . m v_{,} \delta v_{,} = \Sigma . m . (P . \delta x_{,} + Q . \delta y_{,} + R . \delta z_{,}).$$

This, substituted in the equation [196*h*], multiplied by dt, gives

$$0 = \Sigma . m . \left\{ \delta x_{,} . d . \frac{dx_{,}}{dt} + \delta y_{,} . d . \frac{dy_{,}}{dt} + \delta z_{,} . d . \frac{dz_{,}}{dt} \right\} - \Sigma . m dt . v_{,} \delta v_{,},$$

or, by putting $v_{,} dt = ds_{,}$,

$$0 = \Sigma . m . \left\{ \delta x_{,} . d . \frac{dx_{,}}{dt} + \delta y_{,} . d . \frac{dy_{,}}{dt} + \delta z_{,} . d . \frac{dz_{,}}{dt} \right\} - \Sigma . m ds_{,} . \delta v_{,}. \qquad [196i]$$

Again from the variation of $ds_{,}^2 = dx_{,}^2 + dy_{,}^2 + dz_{,}^2$, we obtain as in [47], or rather as in [196*g*],

$$\Sigma . m . v_{,} . \delta ds_{,} = \Sigma . m . \left\{ \frac{d . (dx_{,} . \delta x_{,} + dy_{,} . \delta y_{,} + dz_{,} . \delta z_{,})}{dt} - \delta x_{,} . d . \frac{dx_{,}}{dt} - \delta y_{,} . d . \frac{dy_{,}}{dt} - \delta z_{,} . d . \frac{dz_{,}}{dt} \right\}. \qquad [196k]$$

[196″] provided that its motion is rectilineal and uniform, and that the system is free.

Add together the equations [196*i*, *k*], transpose the term $-\Sigma . m\, d s_{,} . \delta v_{,}$, and put $\delta\, (v_{,}\, d s_{,})$ for $v_{,} . d\, \delta s_{,} + d s_{,} . \delta v_{,}$, and we shall obtain,

$$\Sigma . m . \delta . (v_{,}\, d s_{,}) = \Sigma . m . \frac{d . (d x_{,} . \delta x_{,} + d y_{,} . \delta y_{,} + d z_{,} . \delta z_{,})}{d t},$$

whose integral is $\Sigma . \delta . \int . m\, v_{,}\, d s_{,} = \text{constant} + \Sigma . m . \frac{(d x_{,} . \delta x_{,} + d y_{,} . \delta y_{,} + d z_{,} . \delta z_{,})}{d t}$, the variations in the second member correspond to the extreme points of the curves, and when these are invariable, as it respects the moveable origin of the co-ordinates, we shall have at these points, $\delta x_{,} = 0$, $\delta y_{,} = 0$, $\delta z_{,} = 0$, &c. hence $\Sigma . \delta . \int . m v_{,}\, d s_{,} = 0$, which is similar to [196].

CHAPTER VI.

OF THE LAWS OF THE MOTION OF A SYSTEM OF BODIES, IN ALL THE RELATIONS MATHEMATICALLY POSSIBLE BETWEEN THE FORCE AND VELOCITY.

24. We have observed, in § 5, that there are an infinite number of methods of expressing the force by the velocity, which imply no mathematical contradiction. The most simple is that of the force being proportional to the velocity, and we have seen that this is the law of nature. It is according to this law, we have explained, in the preceding chapter, the differential equations of the motions of a system of bodies; and it is easy to extend the analysis we have used, to all the laws mathematically possible between the force and velocity, and to present thus, in a new point of view, the general principles of motion. For this purpose, suppose that F being the force and v the velocity, we may have

$$F=\varphi(v);$$ [196‴]

$\varphi(v)$ being any function whatever of v; put $\varphi'(v)$ for the differential of $\varphi(v)$ divided by dv. The denominations of the preceding articles being used, the body m will be urged parallel to the axis of x, by the force $\varphi(v)\cdot\frac{dx}{ds}$.* [196iv]

In the following instant, this force will become $\varphi(v)\cdot\frac{dx}{ds}+d.\left(\varphi(v)\cdot\frac{dx}{ds}\right)$, or [196v]
$\varphi(v)\cdot\frac{dx}{ds}+d.\left(\frac{\varphi(v)}{v}\cdot\frac{dx}{dt}\right)$, because $\frac{ds}{dt}=v$, [40a]. Now P, Q, R, being the forces which act on the body m, parallel to the axes of the co-ordinates; the system will be, by § 18 [141], in equilibrium, by means of these forces, and

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* [103b] Found as in note 34a.

of the differentials $d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)$, $d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)$, $d.\left(\frac{dz}{dt}.\frac{\varphi(v)}{v}\right)$, taken with a contrary sign; we shall have, therefore, instead of the equation (*P*) [142] of the same article, the following:

[197] $$0=\Sigma.m.\left\{\delta x.\left\{d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)-Pdt\right\}+\delta y.\left\{d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)-Qdt\right\}+\delta z.\left\{d.\left(\frac{dz}{dt}.\frac{\varphi(v)}{v}\right)-Rdt\right\}\right\}; \quad (S)$$

which differs from it only in this respect, that $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are multiplied by the function $\frac{\varphi(v)}{v}$, which, when the force is proportional to the velocity, may be put equal to unity. But this difference renders the solution of the problems of mechanics very difficult. However we may deduce, from the equation (*S*) [197], principles analogous to those of the preservation of the living forces and of the areas, and of the centre of gravity. [197']

If we change δx into dx, δy into dy, δz into dz, &c., we shall have*

[198] $$\Sigma.m.\left\{dx.d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)+dy.d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)+dz.d.\left(\frac{dz}{dt}.\frac{\varphi(v)}{v}\right)\right\}=\Sigma.m.vdv.dt.\varphi'(v);$$

---

* (104) Put $\delta x=dx$, $\delta y=dy$, $\delta z=dz$, in the equation [197], and develop the terms like $d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)$ into the form $\frac{ddx}{dt}.\frac{\varphi(v)}{v}+\frac{dx}{dt}.d.\left(\frac{\varphi(v)}{v}\right)$ &c. we shall have

$$0=\Sigma.m.\left\{\begin{array}{l}\frac{(dx.ddx+dy.ddy+dz.ddz)}{dt}.\frac{\varphi(v)}{v}+\frac{(dx^2+dy^2+dz^2)}{dt}.d.\left(\frac{\varphi(v)}{v}\right)\\ -P.dx.dt-Q.dy.dt-R.dz.dt\end{array}\right\};$$

but $dx^2+dy^2+dz^2=v^2.dt^2$, [40*a*], and its differential gives,

$$dx.ddx+dy.ddy+dz.ddz=vdv.dt^2,$$

hence by substitution,

$$0=\Sigma.m.\left\{\frac{vdv.dt^2}{dt}.\frac{\varphi(v)}{v}+\frac{v^2.dt^2}{dt}.d.\left(\frac{\varphi(v)}{v}\right)-P.dx.dt-Q.dy.dt-R.dz.dt\right\},$$

and by substituting for $d.\left(\frac{\varphi(v)}{v}\right)$ its development $\frac{dv.\varphi'(v)}{v}-\frac{dv.\varphi(v)}{v^2}$, it becomes

$$0=\Sigma.m.\left\{\begin{array}{l}\frac{vdv.dt^2}{dt}.\frac{\varphi(v)}{v}+\frac{v^2.dt^2}{dt}.\frac{dv.\varphi'(v)}{v}-\frac{v^2.dt^2}{dt}.\frac{dv.\varphi(v)}{v^2}\\ -P.dx.dt-Q.dy.dt-R.dz.dt\end{array}\right\}.$$

in which the first and third terms destroy each other, and the expression becomes, by dividing by dt, $0=\Sigma.m.vdv.\varphi'(v)-\Sigma.m.(P.dx+Q.dy+R.dz)$, which being integrated, gives, as in [199],

$$\Sigma.\int.m.vdv.\varphi'(v)=\Sigma.\int.m.(P.dx+Q.dy+R.dz)+\text{constant}.$$

consequently,

$$\Sigma . \int . m . v dv . \varphi'(v) = \text{constant} + \Sigma . \int . m . (P . dx + Q . dy + R . dz). \qquad [199]$$

Supposing $\Sigma . m . (P . dx + Q . dy + R . dz)$ to be an exact differential, and [199'] equal to $d\lambda$, we shall have,

$$\Sigma . \int . m . v dv . \varphi'(v) = \text{constant} + \lambda ; \qquad (T) \qquad [200]$$

which is similar to the equation (R) § 19 [144], and becomes identical, in the case of nature, where $\varphi'(v) = 1$. The principle of the preservation of the living forces would therefore take place in all possible laws which might exist between the force and velocity, provided that we define the *living force* of a body by the product of its mass, by the double of the integral of its velocity multiplied by the differential of the function of the velocity which expresses the force. [200']

Living Force.

If in the equation (S) [197] we suppose $\delta x' = \delta x + \delta x'_{,}$; $\delta y' = \delta y + \delta y'_{,}$; [200''] $\delta z' = \delta z + \delta z'_{,}$; $\delta x'' = \delta x + \delta x''_{,}$; &c. [152], we shall have, by putting the coefficients of δx, δy, δz, separately equal to nothing,*

$$0 = \Sigma . m . \left\{ d . \left(\frac{dx}{dt} . \frac{\varphi(v)}{v} \right) - P . dt \right\} ; \qquad 0 = \Sigma . m . \left\{ d . \left(\frac{dy}{dt} . \frac{\varphi(v)}{v} \right) - Q . dt \right\} ;$$
$$0 = \Sigma . m . \left\{ d . \left(\frac{dz}{dt} . \frac{\varphi(v)}{v} \right) - R . dt \right\} . \qquad [201]$$

These three equations are similar to those of § 20 [153], from which we have deduced the preservation of the motion of the centre of gravity, in the case of nature, where the system is subjected to no other forces than the action and mutual attraction of the bodies of the system. In this case, $\Sigma . mP$, $\Sigma . mQ$, $\Sigma . mR$, are nothing,* and we shall have

Motion of the Centre of Gravity.

$$\text{constant} = \Sigma . m . \frac{dx}{dt} . \frac{\varphi(v)}{v} ; \qquad \text{constant} = \Sigma . m . \frac{dy}{dt} . \frac{\varphi(v)}{v} ;$$
$$\text{constant} = \Sigma . m . \frac{dz}{dt} . \frac{\varphi(v)}{v} . \qquad [202]$$

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* (105) For the reasons mentioned in note 72.

† (106) As is proved in [155'']. Substituting these in [201], and integrating, we get [202].

[202′] $m.\frac{dx}{dt}.\frac{\varphi(v)}{v}$ is equal to $m.\varphi(v).\frac{dx}{ds}$,* and this last quantity is the finite force of the body, resolved in a direction parallel to the axis of x; the force of a body being the product of its mass by the function of the velocity which expresses the force. Therefore the sum of the finite forces of the system, resolved in a direction, parallel to any axis whatever, is then constant, whatever be the relation of the force to the velocity; and what distinguishes the state of motion from that of rest, is that in the last case this sum is [202″] nothing.† These results are common to all the laws mathematically possible between the force and velocity; but it is only in the law of nature, that the centre of gravity moves with a uniform rectilineal motion.‡

Again, let us suppose in the equation (S) [197],

[203]
$$\delta x=\frac{y.\delta x}{y}+\delta x_{,}; \qquad \delta x'=\frac{y'.\delta x}{y}+\delta x_{,}'; \qquad \delta x''=\frac{y''.\delta x}{y}+\delta x_{,}''; \qquad \&c.$$
$$\delta y=\frac{-x.\delta x}{y}+\delta y_{,}; \qquad \delta y'=\frac{-x'.\delta x}{y}+\delta y_{,}'; \qquad \&c.;$$

the variation δx will disappear from the variations of the mutual distances, f, f', &c., of the bodies of the system, and from the forces which depend on these quantities.* If the system is free from external obstacles, we shall have, by putting the coefficient of δx equal to nothing,†

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* (107) This is found by putting, as in [44], ds for $v\,dt$, in $m.\frac{dx.\varphi(v)}{dt.v}$ which makes it become $m.\varphi.(v).\frac{dx}{ds}$ which, as in [196$^{\text{iv}}$], represents the force of the body resolved in a direction parallel to the axis of x.

† (108) The sum of these forces, resolved in a direction parallel to any axis, must evidently be nothing, in the case of equilibrium; since, if this was not the case, the system would have a motion, in consequence of these forces.

‡ (108a) When the centre of gravity has a uniform rectilineal motion, we shall have, as in [158, 159], $\frac{ddX}{dt^2}=0$, which, substituted in the second differential of the value of X, [154], gives $\Sigma.m.d.\frac{dx}{dt}=0$. Now the first equation [201], by putting, as above, $\Sigma.m.P=0$, becomes $\Sigma.m.d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)=0$, which cannot in general become identical with the preceding, except we have $\frac{\varphi(v)}{v}$ constant, or $\varphi(v)$ proportional to v, which is the law of nature.

$$0=\Sigma.m.\left\{x.d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)-y.d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)\right\}+\Sigma.m.\{Py-Qx\}.dt; \qquad [204]$$

whence by integration‡

$$c=\Sigma.m.\left(\frac{x\,dy-y\,dx}{dt}\right).\frac{\varphi(v)}{v}+\Sigma.\int m.(Py-Qx).dt; \qquad [205]$$

we shall likewise have

$$\begin{aligned} c' &=\Sigma.m.\left(\frac{x\,dz-z\,dx}{dt}\right).\frac{\varphi(v)}{v}+\Sigma.\int m.(Pz-Rx).dt; \\ c'' &=\Sigma.m.\left(\frac{y\,dz-z\,dy}{dt}\right).\frac{\varphi(v)}{v}+\Sigma.\int m.(Qz-Ry).dt; \end{aligned} \qquad [206]$$

c, c', c'', being arbitrary constant quantities.

If the system is subjected only to the mutual attraction of its parts, we shall have, by § 21, [165], $\Sigma.m.(Py-Qx)=0$, $\Sigma.m.(Pz-Rx)=0$; $\Sigma.m.(Qz-Ry)=0$. Again, $m.\left(x.\frac{dy}{dt}-y.\frac{dx}{dt}\right).\frac{\varphi(v)}{v}$, is the momentum of the finite force with which the body m is urged, resolved in a direction parallel to the plane of x, y, to make the system turn about the axis of z;§

Preservation of Areas.

[206′]

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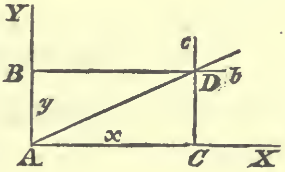
* (109) This is shown in note 77.

† (110) The reasons for putting this coefficient equal to nothing, are fully explained in [160′].

‡ (111) Integrating by parts, the expresssion $x.d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)$, it may be put under the form $x.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)-\int dx.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)$, as is easily proved by taking its differential; also the integral of $-y.d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)$ is $-y.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)+\int dy.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)$. The sum of these two integrals, is $x.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right)-y.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right)$ equal to $\frac{(x\,dy-y\,dx)}{dt}.\frac{\varphi(v)}{v}$, because the terms, under the sign of integration, destroy each other.

§ (112) Suppose a plane, $ACDB$, to be drawn through the place D, of the body m, parallel to the plane of x, y, to meet the axis of z in A. Through A draw AC, AB, parallel to the axes of x, y, respectively, and complete the parallelogram $ACBD$. Then it is shown [202′], that the body m is urged



the finite integral $\Sigma . m . \left(\frac{xdy - ydx}{dt}\right) . \frac{\varphi(v)}{v}$ is therefore the sum of the momenta of all the finite forces of the bodies of the system, exerted to make
[206″] it turn about the same axis; this sum is therefore constant. It is nothing in the state of equilibrium;* there is therefore the same difference between these two states, as there is in the sums of the forces parallel to any axis. In the law of nature, this property indicates, that the sum of the areas described about a fixed point, by the projections of the radii vectores of the
[206‴] bodies is equal in equal times, but this constancy in the described areas does not exist in other laws.

If we take the differential of $\Sigma . \int m . \varphi(v) . ds$, relative to the characteristic δ, we shall have

[207]
$$\delta . \Sigma . \int m . \varphi(v) . ds = \Sigma . \int m . \varphi(v) . \delta ds + \Sigma . \int m . \delta v . \varphi'(v) . ds ;$$

but we have†

[208]
$$\delta ds = \frac{dx . \delta dx . + dy . \delta dy + \delta z . \delta dz}{ds} = \frac{1}{v} . \left\{ \frac{dx}{dt} . d . \delta x + \frac{dy}{dt} . d . \delta y + \frac{dz}{dt} . d . \delta z \right\} ;$$

we shall therefore find, if we integrate by parts,

[209]
$$\begin{aligned} \delta . \Sigma . \int m . \varphi(v) . ds = {} & \Sigma . \frac{m . \varphi(v)}{v} . \left\{ \frac{dx}{dt} . \delta x + \frac{dy}{dt} . \delta y + \frac{dz}{dt} . \delta z \right\} \\ & - \Sigma . \int m . \left\{ \delta x . d . \left(\frac{dx}{dt} . \frac{\varphi(v)}{v}\right) + \delta y . d . \left(\frac{dy}{dt} . \frac{\varphi(v)}{v}\right) + \delta z . d . \left(\frac{dz}{dt} . \frac{\varphi(v)}{v}\right) \right\} \\ & + \Sigma . \int m . \delta v . \varphi'(v) . ds . \end{aligned}$$

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in the direction $B\,D\,b$, by the force $m . \varphi(v) . \frac{dx}{ds}$, and in the direction $C\,D\,c$, by the force $m . \varphi v . \frac{dy}{ds}$. Multiplying these forces by the corresponding perpendiculars $AB = y$, and $AC = x$, let fall upon the directions of the forces, they will give, by [29′], the momenta $m . \varphi(v) . \frac{ydx}{ds}$, and $m . \varphi(v) . \frac{xdy}{ds}$, or, by substituting for ds its value $v\,dt$, $m . \frac{\varphi(v)}{v} . \frac{ydx}{dt}$, $m . \frac{\varphi(v)}{v} . \frac{xdy}{dt}$, and as these forces tend to give to m a different motion about the axis of z, we must take their difference, or $m . \frac{(xdy - ydx)}{dt} . \frac{\varphi(v)}{v}$, for the resulting momentum, as above.

* (112*a*) In like manner as in [122″].

† (113) The variation of $ds^2 = dx^2 + dy^2 + dz^2$, [44], relative to the characteristic δ, being divided by $2\,ds$, gives the first value of δds. The second is deduced from it, by

The extreme points of the curves described by the bodies of the system, being supposed fixed, the term without the sign $\int$ disappears in this equation; we shall therefore have, by means of the equation (S) [197],*

$$\delta.\Sigma.\int m.\varphi(v).ds = \Sigma.\int m.\delta v.\varphi'(v).ds - \Sigma.\int m dt.(P.\delta x + Q.\delta y + R.\delta z); \qquad [210]$$

but the differential of the equation (T) [200], being taken relative to δ, gives†

$$\Sigma.\int m.\delta v.\varphi'(v).ds = \Sigma.\int m dt.(P.\delta x + Q.\delta y + R.\delta z); \qquad [211]$$

we therefore have

$$0 = \delta.\Sigma.\int m.\varphi(v).ds. \qquad [212]$$

Least Action.

This equation corresponds to the principle of least action in the law of
nature. $m.\varphi(v)$ is the whole force of the body m; therefore the principle
amounts to this, that the sum of the integrals of the finite forces of the
bodies of the system, multiplied respectively by the elements of their [212']
directions, is a *minimum*: in this form, it corresponds to all laws mathematically
possible between the force and velocity. In the state of equilibrium,
the sum of the forces multiplied by the elements of their directions is
nothing, by means of the principle of virtual velocities, [114']; what
distinguishes, therefore, in this respect, the state of equilibrium from that of
motion, is, that the same differential function, which is nothing in the state [212'']
of equilibrium, gives, by integration, a *minimum* in the state of motion.

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putting $ds = v dt$, [44]. Substituting this in the first term of the second member of [207], and then integrating by parts, it produces the two top lines of the second member of [209].

* (114) The integral of the equation, [197], gives,

$$-\Sigma.\int m.\left\{\delta x.d.\left(\frac{dx}{dt}.\frac{\varphi(v)}{v}\right) + \delta y.d.\left(\frac{dy}{dt}.\frac{\varphi(v)}{v}\right) + \delta z.d.\left(\frac{dz}{dt}.\frac{\varphi(v)}{v}\right)\right\}$$
$$= -\Sigma.\int m.dt.(P.\delta x + Q.\delta y + R.\delta z),$$

which being substituted in [209], gives the equation [210], rejecting, as above the term without the sign $\int$.

† (115) If we put $\int v dv.\varphi'(v) = \psi(v)$, the equation [200] would become,

$$\Sigma.m.\psi(v) = \text{constant} + \lambda,$$

and its differential relative to the characteristic δ, would give $\Sigma.m.\delta v.\left(\frac{d.\psi(v)}{dv}\right) = \delta\lambda$. But the assumed value of $\psi(v)$, gives $\frac{d.\psi(v)}{dv} = v.\varphi'(v)$, hence $\Sigma.m.\delta v.v.\varphi'(v) = \delta\lambda$. Multiplying this by dt, and in the first member, putting $v dt = ds$, [44], and in the second for $\delta\lambda$, its assumed value [199'], $\delta\lambda = \Sigma.m.(P.\delta x + Q.\delta y + R.\delta z)$, we get [211], which being substituted in [210], gives [212].

CHAPTER VII.

OF THE MOTIONS OF A SOLID BODY OF ANY FIGURE WHATEVER.

25. The differential equations of the progressive and rotatory motions of a solid body, may be easily deduced from those we have given in the fifth chapter; but their importance in the theory of the system of the world, induces us to develop them to a greater extent.

Suppose a solid body, whose particles are urged by any forces whatever. [212‴] Let x, y, z be the rectangular co-ordinates of its centre of gravity; $x+x'$, $y+y'$, $z+z'$ the co-ordinates of any one of its particles denoted by dm, so that x', y', z' may be the co-ordinates of this particle, referred to the centre [212ⁱᵛ] of gravity of the body. Moreover, let P, Q, R, be the forces which act on the particle, parallel to the axes of x, y, z.* The forces destroyed in the particle dm, at each instant, in directions parallel to those axes, will be, by § 18 [141], supposing the element of the time dt to be constant,†

[213]
$$-\left(\frac{ddx+ddx'}{dt}\right).dm+P.dt.dm;$$
$$-\left(\frac{ddy+ddy'}{dt}\right).dm+Q.dt.dm;$$
$$-\left(\frac{ddz+ddz'}{dt}\right).dm+R.dt.dm.$$

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* (116) These forces are supposed, in this computation, to tend to increase the co-ordinates.

† (117) These expressions are easily deduced from the similar quantities, [141]; changing m into dm, and x, y, z, into $x+x'$, $y+y'$, $z+z'$, respectively.

Therefore all the particles, urged by similar forces, ought mutually to be in equilibrium. We have shown in § 15 [119‴] that, for this to be the case, it is necessary that the sum of the forces parallel to the same axis should be nothing ;* which gives the three following equations :

$$S.\left(\frac{ddx+ddx'}{dt^2}\right).dm=S.Pdm;$$
$$S.\left(\frac{ddy+ddy'}{dt^2}\right).dm=S.Qdm;$$ [214]
$$S.\left(\frac{ddz+ddz'}{dt^2}\right).dm=S.Rdm;$$

the letter S being a sign of integration relative to the particle dm, which integration ought to be extended to the whole mass of the body. The variable quantities x, y, z, are the same for all the particles ; so that we may place them without the sign S; therefore by putting the mass of the body equal to m, we shall have,† [214′] Symbol S [214″]

$$S.\frac{ddx}{dt^2}.dm=m.\frac{ddx}{dt^2};\qquad S.\frac{ddy}{dt^2}.dm=m.\frac{ddy}{dt^2};\qquad S.\frac{ddz}{dt^2}.dm=m.\frac{ddz}{dt^2}.$$ [215]

We have also, by the nature of the centre of gravity,‡

$$S.x'.dm=0;\qquad S.y'.dm=0;\qquad S.z'.dm=0;$$ [216]

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* (118) This follows from what is proved in [119″]. Prefixing therefore, the sign S to each of the three forces [213], parallel to the axes of x, y, z, respectively, and putting them separately equal to nothing, then transposing and dividing by dt, we obtain the formulas [214].

† (118a) Since x is independent of the sign S, we may bring it from under the sign, and by this means $S.\frac{ddx}{dt^2}.dm$ will become $\frac{ddx}{dt^2}.S.dm$, and as $S.dm=m$, we get $S.\frac{ddx}{dt^2}.dm=\frac{ddx}{dt^2}.m$, [215], and in like manner we obtain the similar expressions in y and z, [215].

‡ (119) These are deduced from the equations [124], by writing dm for m, S for Σ, and changing x, y, z, into x', y', z', respectively. The second differential of any one of these equations, as $S.x'.dm=0$, divided by dt^2, is $S.\frac{ddx'}{dt^2}.dm=0$. For if we denote by

hence

[217] $$S.\frac{d\,d\,x'}{d\,t^2}.d\,m=0\,; \qquad S.\frac{d\,d\,y'}{d\,t^2}.d\,m=0\,; \qquad S.\frac{d\,d\,z'}{d\,t^2}.d\,m=0\,;$$

therefore we shall have

[218] $$\left.\begin{array}{l} m.\dfrac{d\,d\,x}{d\,t^2}=S.P\,d\,m \\ m.\dfrac{d\,d\,y}{d\,t^2}=S.Q\,d\,m \\ m.\dfrac{d\,d\,z}{d\,t^2}=S.R\,d\,m \end{array}\right\}\,; \qquad (A)$$

Equations for the motion of the Centre of Gravity.

these three equations determine the motion of the centre of gravity of the body; they correspond to the equations § 20 [155], relative to the motion of the centre of gravity of a system of bodies.*

We have shown in § 15 [122] that to maintain the equilibrium of a solid body, the sum of the forces parallel to the axis of x, multiplied respectively by their distances from the axis of z, less the sum of the forces parallel to the axis of y, multiplied by their distances from the axis of z, is equal to nothing; we shall therefore have,†

[219] $$\begin{array}{l} S.\left\{(x+x').\left(\dfrac{d\,d\,y+d\,d\,y'}{d\,t^2}\right)-(y+y').\left(\dfrac{d\,d\,x+d\,d\,x'}{d\,t^2}\right)\right\}.d\,m \\ =S.\{(x+x').Q-(y+y').P\}.d\,m\,; \end{array} \qquad (1)$$

$d\,m$, $d\,m_{,}$, $d\,m_{,,}$, &c., the particles of the body; x', $x'_{,}$, $x'_{,,}$, &c. the corresponding co-ordinates, in the direction of the axis x, we shall have

$$S.x'.d\,m=x'.d\,m+x'_{,}.d\,m_{,}+x'_{,,}.d\,m_{,,}+\&c.$$

The second differential of the second member of this equation being divided by $d\,t^2$ is $\frac{d\,d\,x'}{d\,t^2}.d\,m+\frac{d\,d\,x'_{,}}{d\,t^2}.d\,m_{,}+\&c.$, which is evidently of the form $S.\frac{d\,d\,x'}{d\,t^2}.d\,m$, [217]. Substituting [215, 217] in [214], we get the formulas [218].

* (119a) It follows from these equations that the motion of the centre of gravity is the same as if all the forces were applied to it which act upon the whole body, in like manner as has been proved in [155′].

† (120) This equation is easily deduced from the first of the equations [122], by writing S for Σ, $d\,m$ for m, $x+x'$, $y+y'$, $z+z'$, for x, y, z, respectively; also for $S.\left(\frac{\delta s}{\delta x}\right)$,

now we have*

$$S.(x.ddy-y.ddx).dm=m.(x.ddy-y.ddx)\,; \qquad [220]$$

likewise

$$S.(Qx-Py).dm=x.S.Qdm-y.S.Pdm\,; \qquad [221]$$

lastly we have

$$S.(x'ddy+xddy'-y'ddx-yddx').dm=ddy.S.x'dm-ddx.S.y'dm \\ +x.S.ddy'.dm-y.S.ddx'.dm\,; \qquad [222]$$

and by the nature of the centre of gravity, each of the terms of the second
member of this equation is nothing ;† the equation (1) [219] will therefore [222′]
become, by means of the equations (A) [218],‡

$$S.\left(\frac{x'ddy'-y'ddx'}{dt^2}\right).dm=S.(Qx'-Py').dm\,; \qquad [223]$$

$S.\left(\frac{\delta s}{\delta y}\right)$, $S.\left(\frac{\delta s}{\delta z}\right)$, which, by formulas [13], represent the forces acting on a particle parallel to the axes of x, y, z, their equivalent values, for this case, as they are given in [213], connected with the factor $dt.dm$.

* (121) The second members of this, and of the two following equations, are easily deduced from the first members, by bringing x, y, from under the sign S; and putting $S.dm=m$, as in note 118a.

† (122) Because $S.x'.dm=0$, $S.y'.dm=0$, [216], and $S.ddy'.m=0$, $S.ddx'.m=0$, [217].

‡ (123) Performing the multiplications, indicated in the first member of the equation [219], it becomes,

$$S.\left(\frac{xddy-yddx}{dt^2}\right).dm+S.\left(\frac{x'ddy'-y'ddx'}{dt^2}\right).dm+S.\left(\frac{x'ddy+xddy'-y'ddx-y'ddx'}{dt^2}\right).dm.$$

Of the three parts, into which this is divided by the sign S, the last is nothing, [222′], the first is equal to $m.\left(\frac{xddy-yddx}{dt^2}\right)$, [220], and if in this we substitute the values [218], we shall find $m.\left(\frac{xddy-yddx}{dt^2}\right)=x.S.Qdm-Y.S.Pdm$; hence that first member becomes $x.S.Qdm-y.S.Pdm+S.\left(\frac{x'ddy'-y'ddx'}{dt^2}\right)$. In like manner the second member of [219], becomes $x.S.Qdm-y.S.Pdm+S.(Qx'-Py').dm$. Reject $x.S.Qdm-y.S.Pdm$, common to both members, and we shall obtain [223].

taking the integral of this relative to the time t, we shall have

[224] $$S.\left(\frac{x'\,dy'-y'\,dx'}{dt}\right).dm=S.\int(Qx'-Py').dt.dm;$$

Symbol $\int$ the sign of integration $\int$ refers to the time t.

Whence it is easy to conclude that if we put

[225] $$\begin{aligned}S.\int(Qx'-Py').dt.dm&=N;\\ S.\int(Rx'-Pz').dt.dm&=N';\\ S.\int(Ry'-Qz').dt.dm&=N'';\end{aligned}$$

we shall have the three following equations:

[226] $$\left.\begin{aligned}S.\left(\frac{x'\,dy'-y'\,dx'}{dt}\right).dm&=N;\\ S.\left(\frac{x'\,dz'-z'\,dx'}{dt}\right).dm&=N';\\ S.\left(\frac{y'\,dz'-z'\,dy'}{dt}\right).dm&=N'';\end{aligned}\right\}; \qquad (B)$$

These three equations contain the principle of the preservation of areas, and [226'] are sufficient to determine the rotatory motion of the body about its centre of gravity. When combined with the equations (A) [218], they determine completely the progressive and rotatory motions of the body.*

If the body is forced to move about a fixed point, it follows from § 15 [226''] [122'''], that the equations (B) [226, &c.], are sufficient for this purpose; but then we must place the origin of the co-ordinates x', y', z', at that point.

26. Let us now consider particularly these equations, supposing this fixed origin to be at any point whatever, whether it be the centre of gravity

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Its integral relative to dt, evidently gives [224]. Substituting, in this assumed value of N, [225], we get the first of the equations [226]. The others are found in like manner, or by changing y' into z', and afterwards x' into y', &c.

* (123a) The equations [218], serve to determine the co-ordinates of the centre of gravity, x, y, z, upon which the progressive motion depends; and [226] will give the values of the co-ordinates x', y', z', referred to that centre, from which may be found the rotatory motion of the body about that point. Moreover, the remarks made in [155', 159', 167''], relative to the centre of gravity of a system of bodies, may also be applied to the case of a solid body.

or not. We shall refer the position of each particle to three axes, perpendicular to each other, and fixed in the body, but moveable in space. Let θ be the inclination of the plane formed by the two first axes upon the plane of x, y; φ the angle formed by the line of intersection of these two planes, and by the first axis; lastly, let ψ be the complement of the angle which the projection of the third axis, upon the plane of x, y, makes with the axis of x.* We shall call these three new axes, *principal axes*, and we shall [$226'''$]

Principal Axes of Rotation.

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* (124) This change of co-ordinates is precisely the same as that in [$167^{\rm v}$ &c.], writing x', y', z', for x, y, z, and x'', y'', z'', for $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, respectively. In this case, by referring to the figure page 112, CX will be the axis of x', CY that of y', the axis of z' being above C, perpendicular to the plane of the figure; CG is the axis of x'', CH the axis of y'', the axis of z'' being drawn above C, perpendicular to the plane BOP, so that its projection upon the plane of the figure shall fall on CF, the part of the plane BOP, falling below the plane of the figure. The angles $FBO = \theta$, $ACB = \psi$, $GCB = \varphi$. [227a]

It is of importance to notice particularly, the different kind of axes mentioned in this chapter, which are of frequent use throughout the rest of the work. If the origin of the co-ordinates is supposed to be fixed in space, the rectangular axes x, y, z, will also be fixed, but if their origin is supposed to be in any way connected with the body, and to move with it, these axes will continue to pass through the moveable origin always retaining situations parallel to their original directions. The co-ordinates of the centre of gravity of the body [$212'''$], being x, y, z; and those of any particle dm being $x + x'$, $y + y'$, $z + z'$, it will follow, as in [$212^{\rm iv}$], that x', y', z', are the co-ordinates of the particle dm, referred to three axes, drawn through the centre of gravity, parallel to the axes x, y, z. On the contrary, the directions of the axes x'', y'', z'', [$226^{\rm iv}$], vary with the motion, being *fixed* in the body and moveable with it; the situation of these axes, relative to the axes x', y', z', being determined by means of the *variable* angles θ, ψ, φ. So that the place of any particle dm, may be determined two ways; *first*, by means of the variable co-ordinates x', y', z', corresponding to that particle; or, *second*, by the constant co-ordinates x'', y'', z'', corresponding to the same particle, taken in connexion with the variable angles θ, ψ, φ, which determine the positions of the axes of x'', y'', z''. We must therefore, in finding the differentials of x', y', z', [227], suppose x'', y'', z'', to be constant and θ, ψ, φ, to be variable, as is done in [230*b*, &c.] [227b]

Remarks on the axes x', y', z', x'', y'', z''.

Before closing this note, it may not be amiss to remark, that in the calculations [175—181], the angles θ, ψ, φ, were supposed to be constant, the object being merely to change the system of co-ordinates, x, y, z, into another system, $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, entirely similar, and in which the directions of the axes of $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, should be invariable.

[226$^{\rm iv}$] denote by x'', y'', z'', the three co-ordinates of the particle dm, referred to these axes; we shall have, by § 21,*

[227]
$$\begin{aligned}
x' &= x''.\{\cos.\theta.\sin.\psi.\sin.\varphi+\cos.\psi.\cos.\varphi\} \\
&\qquad +y''.\{\cos.\theta.\sin.\psi.\cos.\varphi-\cos.\psi.\sin.\varphi\}+z''.\sin.\theta.\sin.\psi; \\
y' &= x''.\{\cos.\theta.\cos.\psi.\sin.\varphi-\sin.\psi.\cos.\varphi\} \\
&\qquad +y''.\{\cos.\theta.\cos.\psi.\cos.\varphi+\sin.\psi.\sin.\varphi\}+z''.\sin.\theta.\cos.\psi; \\
z' &= z''.\cos.\theta-y''.\sin.\theta.\cos.\varphi-x''.\sin.\theta.\sin.\varphi.
\end{aligned}$$

By means of these equations we can develop the first members of the equations (B) [226], in functions of θ, ψ, and φ, and their differentials. But we may simplify the calculation considerably, by observing, that the position of the three principal axes depends on three constant quantities, which may always be determined so as to satisfy the three equations

[228]
$$S.x''y''.dm=0; \qquad S.x''z''.dm=0; \qquad S.y''z''.dm=0.$$

Put†

[229]
$$\begin{aligned}
S.(y''^2+z''^2).dm&=A; \\
S.(x''^2+z''^2).dm&=B; \\
S.(x''^2+y''^2).dm&=C;
\end{aligned}$$

and for brevity‡

[230]
$$\begin{aligned}
d\varphi-d\psi.\cos.\theta&=p\,dt; \\
d\psi.\sin.\theta.\sin.\varphi-d\theta.\cos.\varphi&=q\,dt; \\
d\psi.\sin.\theta.\cos.\varphi+d\theta.\sin.\varphi&=r\,dt;
\end{aligned}$$

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* (125) These values of x', y', z', are deduced from those of x, y, z, [171], by writing x', y', z', for x, y, z, and x'', y'', z'', for $x_{,,,}$, $y_{,,,}$, $z_{,,,}$, &c., as in [227a].

† (125a) If we put ρ for the distance of a particle dm, from the axis of z'', we shall have as in [27], $\rho^2=x''^2+y''^2$, and the expression of C, [229], becomes $C=S.\rho^2.dm$; so that C represents the sum of the products of each particle dm, by the square of its distance from the axis of z'', and this is what is called in [245'''], the momentum of inertia of the solid body about the axis of z''. In like manner, B represents the momentum of inertia
[227c] about the axis of y'', and A the momentum about x''. This is analogous to the definition [29']; for the velocity of a particle dm, revolving about the axis z'', at the distance ρ, will be proportional to ρ, and the corresponding force acting at the end of the lever ρ, in a perpendicular direction, its momentum will be $\rho^2.dm$, [29'], and the sum, for the whole body, $S.\rho^2.dm$, as above.

‡ (126) The importance of this substitution, for abridging and simplifying the calculations, will be seen in § 28 [259], where it will be proved that p, q, r, are proportional to the

the equations (*B*) [226] will become, by reduction, as follows :*

Equations of the Rotatory Motion.

$$\left.\begin{aligned} &A.q.\sin.\theta.\sin.\varphi+Br.\sin.\theta.\cos.\varphi-Cp.\cos.\theta=-N; \\ &\cos.\psi.\{Aq.\cos.\theta.\sin.\varphi+Br.\cos.\theta.\cos.\varphi+Cp.\sin.\theta\} \\ &\quad+\sin.\psi.\{Br.\sin.\varphi-Aq.\cos.\varphi\}=-N'; \\ &\cos.\psi.\{Br.\sin.\varphi-Aq.\cos.\varphi\} \\ &\quad-\sin.\psi.\{Aq.\cos.\theta.\sin.\varphi+Br.\cos.\theta.\cos.\varphi+Cp.\sin.\theta\}=-N'' \end{aligned}\right\}; \quad (C) \qquad [231]$$

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cosines of the angles formed by the momentary axis of rotation, and the three principal axes. These quantities p, q, r, might have been found, *a priori*, being the co-efficients of x'', y'', z'', in the equations $0=px''-qz''$, $0=py''-rz''$, $0=qy''-rx''$, [256, 257, 258], computed by putting $dx'=0$, $dy'=0$, $dz'=0$, [256*a*], but we shall, in the notes on this part, follow precisely the method of the author, and, on account of the importance of the subject, shall give the calculations at full length, using however the abridged expressions [171*a*].

* (127) For greater symmetry we shall put $\mathcal{N}'=-\mathcal{N}_2$, $\mathcal{N}''=\mathcal{N}_1$, and the equations [226], by altering the order of the two last equations, will become, [228*a*]

$$\begin{aligned} -S.\left(\frac{x'dy'-y'dx'}{dt}\right).dm&=-\mathcal{N}, \\ -S.\left(\frac{y'dz'-z'dy'}{dt}\right).dm&=-\mathcal{N}_1, \\ -S.\left(\frac{z'dx'-x'dz'}{dt}\right).dm&=-\mathcal{N}_2. \end{aligned} \qquad [229a]$$

Under this form any one of the equations can be derived from the preceding, as in [165*a*], by taking the next letters in order in the two series x', y', z', ; $\mathcal{N}$, $\mathcal{N}_1$, $\mathcal{N}_2$. If we use the values [171*a*], the co-ordinates x', y', z', [227], will become,

$$\begin{aligned} x'&=\mathcal{A}_0.x''+B_0.y''+C_0.z'', \\ y'&=\mathcal{A}_1.x''+B_1.y''+C_1.z'', \\ z'&=\mathcal{A}_2.x''+B_2.y''+C_2.z''. \end{aligned} \qquad [230a]$$

The differentials of these equations, supposing x'', y'', z'', constant, and θ, ψ, φ, variable, [227*b*], will be

$$\begin{aligned} dx'&=d\mathcal{A}_0.x''+dB_0.y''+dC_0.z'', \\ dy'&=d\mathcal{A}_1.x''+dB_1.y''+dC_1.z'', \\ dz'&=d\mathcal{A}_2.x''+dB_2.y''+dC_2.z'', \end{aligned} \qquad [230b]$$

Substitute these in [229*a*], neglecting the products $x''y''$, $x''z''$, $y''z''$, which produce nothing

By taking the differentials of these three equations and supposing $\psi = 0$,

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in the result, in consequence of the equations [228], and we shall obtain.

[230c]
$$S.dm.\left\{\left(\frac{\mathcal{A}_1\,d\mathcal{A}_0 - \mathcal{A}_0\,d\mathcal{A}_1}{dt}\right).x''^2 + \left(\frac{B_1\,dB_0 - B_0\,dB_1}{dt}\right).y''^2 + \left(\frac{C_1\,dC_0 - C_0\,dC_1}{dt}\right).z''^2\right\} = -\mathcal{N},$$
$$S.dm.\left\{\left(\frac{\mathcal{A}_2\,d\mathcal{A}_1 - \mathcal{A}_1\,d\mathcal{A}_2}{dt}\right).x''^2 + \left(\frac{B_2\,dB_1 - B_1\,dB_2}{dt}\right).y''^2 + \left(\frac{C_2\,dC_1 - C_1\,dC_2}{dt}\right).z''^2\right\} = -\mathcal{N}_1,$$
$$S.dm.\left\{\left(\frac{\mathcal{A}_0\,d\mathcal{A}_2 - \mathcal{A}_2\,d\mathcal{A}_0}{dt}\right).x''^2 + \left(\frac{B_0\,dB_2 - B_2\,dB_0}{dt}\right).y''^2 + \left(\frac{C_0\,dC_2 - C_2\,dC_0}{dt}\right).z''^2\right\} = -\mathcal{N}_2,$$

If we take the differentials of $\mathcal{A}_0$, B_0, C_0, &c. [171*a*], supposing θ, ψ, φ, variable, and substitute the values p, q, r, [230], we may obtain the following system of equations,

[230d]
$$\begin{array}{lll} d\mathcal{A}_0 = (B_0 p - C_0 r)\,.\,dt, & dB_0 = (C_0 q - \mathcal{A}_0 p)\,.\,dt, & dC_0 = (\mathcal{A}_0 r - B_0 q)\,.\,dt, \\ d\mathcal{A}_1 = (B_1 p - C_1 r)\,.\,dt, & dB_1 = (C_1 q - \mathcal{A}_1 p)\,.\,dt, & dC_1 = (\mathcal{A}_1 r - B_1 q)\,.\,dt, \\ d\mathcal{A}_2 = (B_2 p - C_2 r)\,.\,dt, & dB_2 = (C_2 q - \mathcal{A}_2 p)\,.\,dt, & dC_2 = (\mathcal{A}_2 r - B_2 q)\,.\,dt, \end{array}$$

For, if we take the differential of $\mathcal{A}_0$, [171*a*], and afterwards substitute the coefficients C_0, $\mathcal{A}_1$, B_0, [171*a*], it will become,

[230e]
$$d\mathcal{A}_0 = -d\theta\,.\,\sin.\varphi\,.\,C_0 + d\psi\,.\,\mathcal{A}_1 + d\varphi\,.\,B_0.$$

Now if we multiply the first and third of the equations [230], by B_0 and $-C_0$, respectively, and add the products we shall find,

[230f]
$$(B_0 p - C_0 r)\,.\,dt = -d\theta\,.\,\sin.\varphi\,.\,C_0 + d\psi\,.\,(-C_0\,.\,\sin.\theta\,.\,\cos.\varphi - B_0\,.\,\cos.\theta) + B_0\,.\,d\varphi,$$

and if we substitute, in the coefficient of $d\psi$, the values [171*a*], $-\sin.\theta\,.\,\cos.\varphi = B_2$, $\cos.\theta = C_2$, it will become $B_2\,C_0 - B_0\,C_2$, which is equal to $\mathcal{A}_1$, [175*c*]; hence the second member of [230*f*], will become like that of [230*e*], and we shall get

[230g]
$$d\mathcal{A}_0 = (B_0 p - C_0 r)\,.\,dt,$$

This is the first of the equations [230*d*], and the others may be found in a similar manner. But the labor may be much abridged, by observing that if we increase ψ, by a right angle, it would change the values of $\mathcal{A}_0$, B_0, C_0, [171*a*], into $\mathcal{A}_1$, B_1, C_1, respectively, without altering the values of p, q, r, [230], and it is evident, from the manner in which [230*e*, *f*, *g*] were found, that we may make the same changes in [230*g*], by which means it would
[230h] become $d\mathcal{A}_1 = (B_1 p - C_1 r)\,.\,dt$, which is the second of the equations [230*d*]. In like manner, if we put $\psi = 0$, and then increase θ by a right angle, it will change the quantities $\mathcal{A}_1$, B_1, C_1, [171*a*], into $\mathcal{A}_2$, B_2, C_2, respectively, without altering the values of p, q, r, provided the terms depending on $d\psi$, are neglected, which can be done in making this derivation, because these terms vanish from both members of the expressions, $d\mathcal{A}_2$, dB_2, dC_2, [230*d*]. For if we notice only the terms depending upon $d\psi$, in [230], using also the values, [171*a*], we shall have $p\,dt = -C_2.d\psi$; $q\,dt = -\mathcal{A}_2.d\psi$; $r\,dt = -B_2.d\psi$. Substituting these in $d\mathcal{A}_2$, dB_2, dC_2, [230*d*], they mutually destroy each other. Making therefore, the change of $\mathcal{A}_1$, B_1, C_1, into $\mathcal{A}_2$, B_2, C_2, the expression $d\mathcal{A}_1$, changes into
[230i] $d\mathcal{A}_2 = (B_2\,.\,p - C_2\,.\,r)\,.\,dt$, which is the third of the equations [230*d*].

after taking the differentials, which is the same thing as to take the axis of [231']

Again, if we increase φ by a right angle, it will change the quantities A_0, B_0, C_0, p, q, r, [171*a*, 230], into $B_0, -A_0, C_0, p, r, -q$, respectively, and the equation [230*g*], will become $dB_0=(-A_0p+C_0q).dt$, which is the fourth of the equations [230*d*]. From this we may derive dB_1, dB_2, by increasing the index of the letters, A_0, B_0, C_0, as was done in [230*h*, *i*], since the method of derivation, there used, can be applied here without alteration.

The differential of $C_0=\sin.\theta.\sin.\psi$, [171*a*], is

$$dC_0=d\theta.\cos.\theta.\sin.\psi+d\psi.\sin.\theta.\cos.\psi, \qquad [230k]$$

and if we multiply the two lower equations [230], by $-B_0, A_0$, respectively, the sum of these products will become,

$$(A_0r-B_0q).dt=d\theta.(A_0.\sin.\varphi+B_0.\cos.\varphi)+d\psi.\sin.\theta.(A_0\cos.\varphi-B_0.\sin.\varphi), \qquad [230l]$$

but from [171*a*], we get, by reduction,

$$A_0.\sin.\varphi+B_0.\cos.\varphi=\cos.\theta.\sin.\psi, \qquad A_0.\cos.\varphi-B_0.\sin.\varphi=\cos.\psi, \qquad [230m]$$

hence, the second member of [230*l*], will become like that of [230*k*], and we shall find $dC_0=(A_0r-B_0q).dt$. This is the seventh of the equations [230*d*], and from it we may derive dC_1, dC_2, by increasing the index of the letters A, B, C, as in [230*h*, *i*]. It may be observed that the system of equations [230*d*] is symmetrical, either by increasing the indexes of A, B, C, without changing the letters, or by changing in the two series of letters, p, q, r, A, B, C, any letter into the following one of the series, without altering the indexes of A, B, C.

The values [230*d*], being substituted in the factors of x''^2, y''^2, z''^2, [230*c*], they will become as in the first of the following forms, and these may be reduced to the second form by using the equations [175*c*],

$$\begin{aligned}
A_1dA_0-A_0dA_1&=\{(A_1B_0-A_0B_1).p+(A_0C_1-A_1C_0).r\}.dt=-(C_2p+B_2r).dt,\\
B_1dB_0-B_0dB_1&=\{(B_1C_0-B_0C_1).q+(A_1B_0-A_0B_1).p\}.dt=-(A_2q+C_2p).dt, \qquad [230n]\\
C_1dC_0-C_0dC_1&=\{(A_0C_1-A_1C_0).r+(B_1C_0-B_0C_1).q\}.dt=-(B_2r+A_2q).dt,
\end{aligned}$$

and the others may be found in the same manner, or more simply, by the method of derivation above used, adding one or two to the index of the letters, rejecting three when the index is equal to that number. The quantities thus obtained are to be substituted in [230*c*], and we shall get,

$$\begin{aligned}
S.dm.\{-(C_2p+B_2r).x''^2-(A_2q+C_2p).y''^2-(B_2r+A_2q).z''^2\}&=-N,\\
S.dm.\{-(C_0p+B_0r).x''^2-(A_0q+C_0p).y''^2-(B_0r+A_0q).z''^2\}&=-N_1, \qquad [230o]\\
S.dm.\{-(C_1p+B_1r).x''^2-(A_1q+C_1p).y''^2-(B_1r+A_1q).z''^2\}&=-N_2,
\end{aligned}$$

x' infinitely near to the line of intersection of the plane of x', y', with that

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Connecting the terms depending on $\mathcal{A}_0$, B_0, &c., and bringing the quantities p, q, r, from under the sign S, because they are the same for all parts of the body, we get,

[230p]
$$\begin{aligned}
&-\mathcal{A}_2\, q.S.dm.(y''^2+z''^2)-B_2\, r.S.dm.(x''^2+z''^2)-C_2\, p.S.dm.(x''^2+y''^2)=-\mathcal{N},\\
&-\mathcal{A}_0\, q.S.dm.(y''^2+z''^2)-B_0\, r.S.dm.(x''^2+z''^2)-C_0\, p.S.dm.(x''^2+y''^2)=-\mathcal{N}_1,\\
&-\mathcal{A}_1\, q.S.dm.(y''^2+z''^2)-B_1\, r.S.dm.(x''^2+z''^2)-C_1\, p.S.dm.(x''^2+y''^2)=-\mathcal{N}_2,
\end{aligned}$$

Substituting the values [229], we shall find,

$$\begin{aligned}
&-\mathcal{A}_2\,.\,q\,.\,A-B_2\,.\,r\,.\,B-C_2\,.\,p\,.\,C=-\mathcal{N},\\
&-\mathcal{A}_0\,.\,q\,.\,A-B_0\,.\,r\,.\,B-C_0\,.\,p\,.\,C=-\mathcal{N}_1=-\mathcal{N}'',\\
&-\mathcal{A}_1\,.\,q\,.\,A-B_1\,.\,r\,.\,B-C_1\,.\,p\,.\,C=-\mathcal{N}_2=\mathcal{N}',
\end{aligned}$$

and by using the values [171a], connecting the terms multiplied by sin. ψ, cos. ψ, they will become as in [231], the order of the second and third equations being changed.

If we multiply $d\mathcal{A}_0$, $d\mathcal{A}_1$, $d\mathcal{A}_2$, [230d], by B_0, B_1, B_2, respectively, and add the products together, the coefficient of $p\,dt$, in the sum will be 1, and that of $r\,dt$ nothing, in consequence of the second and sixth of the equations [172b], and in like manner we may obtain the rest of the following system of equations, which are easily proved by the substitution of the values of $d\mathcal{A}_0$, $d\mathcal{A}_1$, &c., [230d], and reducing by means of [172b],

[230q]
$$\begin{aligned}
p\,dt&=B_0\,.\,d\mathcal{A}_0+B_1\,.\,d\mathcal{A}_1+B_2\,.\,d\mathcal{A}_2=-\mathcal{A}_0\,.\,dB_0-\mathcal{A}_1\,.\,dB_1-\mathcal{A}_2\,.\,dB_2,\\
q\,dt&=C_0\,.\,dB_0+C_1\,.\,dB_1+C_2\,.\,dB_2=-B_0\,.\,dC_0-B_1\,.\,dC_1-B_2\,.\,dC_2,\\
r\,dt&=\mathcal{A}_0\,.\,dC_0+\mathcal{A}_1\,.\,dC_1+\mathcal{A}_2\,.\,dC_2=-C_0\,.\,d\mathcal{A}_0-C_1\,.\,d\mathcal{A}_1-C_2\,.\,d\mathcal{A}_2.
\end{aligned}$$

(127a) The angles θ, ψ, φ, used by the author in computing the rotatory motions of a solid body, which is at liberty to move in any direction, are peculiarly well adapted to astronomimal uses, but for other purposes, the following notation has been generally used. It consists in putting, as usual, x, y, z, for the rectangular co-ordinates of any particle dm of the body, and then changing them successively into polar co-ordinates, as in [27]. If we put $\rho=\sqrt{x^2+y^2}$, and change ϖ into φ, the expressions of x, y, [27], will become,

$$x=\rho\,.\,\cos.\,\varphi, \qquad y=\rho\,.\,\sin.\,\varphi,$$

[230r] Taking the differentials of these, supposing x, y, φ, to be variable, we get the values of dx, dy, corresponding to a rotatory motion $d\varphi$ about the axis of z.

$$dx=-d\varphi\,.\,\rho\,.\,\sin.\,\varphi, \qquad dy=d\varphi\,.\,\rho\,.\,\cos.\,\varphi,$$

and by using the values of x, y, we shall find

[230s]
$$dx=-y\,.\,d\varphi, \qquad dy=x\,.\,d\varphi.$$

In like manner the differentials of y, z, depending on a rotatory motion $d\psi$, about the axis of x, will be obtained, by changing, as in note 87, x, y, φ, into the letters immediately following y, z, ψ, by which means we shall find,

[230t]
$$dy=-z\,d\psi, \qquad dz=y\,d\psi.$$

of x'', y'', we obtain

Lastly, the differentials of z, x, depending on a rotatory motion $d\omega$, about the axis of y, will be found, by changing, in like manner, y, z, ψ, into z, x, ω, respectively, hence we shall have,

$$dz = -x\,d\omega, \qquad dx = z\,d\omega.$$ [230u]

Connecting together these partial differentials [230s, t, u], we shall obtain the complete differentials of x, y, z, corresponding to the element of the time dt, namely,

$$dx = z\,d\omega - y\,d\varphi, \qquad dy = x\,d\varphi - z\,d\psi, \qquad dz = y\,d\psi - x\,d\omega.$$ [230v]

To find the points of the body in which these variations are nothing, we must put $dx = 0$, $dy = 0$, $dz = 0$, hence

$$0 = z\,d\omega - y\,d\varphi, \qquad 0 = x\,d\varphi - z\,d\psi, \qquad 0 = y\,d\psi - x\,d\omega.$$ [230w]

Which equations may be satisfied for various values of x, y, z, corresponding to the points of the body which remain at rest, during the rotatory motions, $d\varphi$, $d\psi$, $d\omega$, about the axes z, x, y. If we put $\frac{d\varphi}{d\omega} = C$, $\frac{d\varphi}{d\psi} = B$, $\frac{d\omega}{d\psi} = A$, these equations may be put under the forms $z = Cy$, $z = Bx$, $y = Ax$, which are the equations of a right line passing through the origin of the co-ordinates, [19b']. In all parts of this right line, we shall have $dx = 0$, $dy = 0$, $dz = 0$, and this line will therefore be the momentary axis of revolution, [230x] corresponding to these three angular motions $d\varphi$, $d\psi$, $d\omega$.

The two first of the equations [230w], give $y = z\,.\frac{d\omega}{d\varphi}$, $x = z\,.\frac{d\psi}{d\varphi}$. Substituting [230y] these in $r = \sqrt{x^2+y^2+z^2}$, [19e], putting also $d\theta = \sqrt{d\varphi^2 + d\psi^2 + d\omega^2}$, we shall get $r = z\,.\sqrt{\frac{d\psi^2}{d\varphi^2} + \frac{d\omega^2}{d\varphi^2} + 1} = z\,.\frac{d\theta}{d\varphi}$, hence $\frac{d\varphi}{d\theta} = \frac{z}{r}$; but $\frac{z}{r}$, represents, as in page 116, the cosine of the angle formed by the lines r, z, represented by cos. (r, z), therefore $\frac{d\varphi}{d\theta} = \cos.\,(r, z)$. Changing successively z, φ into x, ψ, and y, ω, we get,

$$\frac{d\varphi}{d\theta} = \cos.\,(r, z), \qquad \frac{d\psi}{d\theta} = \cos.\,(r, x), \qquad \frac{d\omega}{d\theta} = \cos.\,(r, y).$$ [230z]

Hence it follows that the rotatory motions $d\varphi$, $d\psi$, $d\omega$, about the axes z, x, y, respectively, are equivalent to a single rotatory motion about the momentary axis r, the situation of this axis with respect to the axes z, x, y, being determined by means of the angles (r, z), (r, x), (r, y), which depend on the equations [230z). The actual angular velocity, about this momentary axis, may be found as in [259'', 260], by considering the motion of a particle situated in the axis of z, at the distance 1 from the origin, so that $x = 0$, $y = 0$, $z = 1$. In this case the equations [230v], give $dx = d\omega$, $dy = -d\psi$, $dz = 0$, hence,

$$\sqrt{dx^2 + dy^2 + dz^2} = \sqrt{d\omega^2 + d\psi^2} = \sqrt{d\theta^2 - d\varphi^2}$$
$$= d\theta\,.\sqrt{1 - \frac{d\varphi^2}{d\theta^2}} = d\theta\,.\sqrt{1 - \cos.^2\,(r, z)} = d\theta\,.\sin.\,(r, z).$$ [231a]

$$d\theta \,.\, \cos.\theta \,.\, (Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi) + \sin.\theta \,.\, d \,.\, (Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi)$$
$$- d \,.\, (Cp \,.\, \cos.\theta) = - dN;$$

[232]
$$d\psi \,.\, (Br \,.\, \sin.\varphi - Aq \,.\, \cos.\varphi) - d\theta \,.\, \sin.\theta \,.\, (Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi)$$
$$+ \cos.\theta \,.\, d \,.\, (Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi) + d \,.\, (Cp \,.\, \sin.\theta) = - dN';$$

$$d \,.\, (Br \,.\, \sin.\varphi - Aq \,.\, \cos.\varphi) - d\psi \,.\, \cos.\theta \,.\, (Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi)$$
$$- Cp \,.\, d\psi \,.\, \sin.\theta = - dN''.\text{*}$$

This represents the motion of the proposed particle, [40*a*], in the time dt, and its distance from the axis of rotation being evidently equal to sin. (r, z), [260″], its angular motion will be obtained by dividing $d\theta \,.\, \sin.(r, z)$ by $\sin.(r, z)$; therefore this angular motion will be $d\theta$.

Hence it appears, that the same rules which prevail in the composition and resolution of rectilineal motions, are analogous to those in the composition and resolution of angular rotations. For the angular motion $d\theta = \sqrt{d\varphi^2 + d\psi^2 + d\omega^2}$, about the momentary axis [231*b*] r, may be resolved into three angular motions $d\varphi$, $d\psi$, $d\omega$, about the rectangular axes z, x, y, the cosines of the angles which the axis r makes with z, x, y, being represented by $\frac{d\varphi}{d\theta}$, $\frac{d\psi}{d\theta}$, $\frac{d\omega}{d\theta}$, as in [230*z*]. In like manner, the three angular motions $d\varphi$, $d\psi$, $d\omega$, [231*c*] about the axes z, x, y, may be composed into one, represented by $d\theta = \sqrt{d\varphi^2 + d\psi^2 + d\omega^2}$, and the situation of this axis with respect to the axes z, x, y, may be determined by means of the equations [230*z*], which give the cosines of the angles which the momentary axis of rotation makes with the axes z, x, y.

Composition and resolution of rotatory motions.

Hence the motions of any solid body, which is at liberty to move in any direction, may be resolved into a progressive motion of the centre of gravity [218], and a rotatory motion about [231*d*] a momentary axis passing through that centre. The motion of the centre of gravity may be resolved into three progressive motions, parallel to the three rectangular axes x, y, z, [218], and the rotatory motion may be resolved into three rotatory motions about these axes. The converse of this is also true, that the three motions of the centre of gravity, in the directions parallel to the axes x, y, z, may be composed into one single progressive motion of that [231*e*] centre; and the three rotatory motions about the axes x, y, z, may be resolved into one rotatory motion about the momentary axis.

* (128) Put

$$L = Br \,.\, \sin.\varphi - Aq \,.\, \cos.\varphi,$$
[232*a*]
$$L' = Br \,.\, \cos.\varphi + Aq \,.\, \sin.\varphi,$$
$$M = Aq \,.\, \cos.\theta \,.\, \sin.\varphi + Br \,.\, \cos.\theta \,.\, \cos.\varphi + Cp \,.\, \sin.\theta = L' \cos.\theta + Cp \,.\, \sin.\theta.$$

If we put

$$C p = p' ; \quad A q = q' ; \quad B r = r' ; \qquad [233]$$

these three differential equations will give*

$$\left.\begin{aligned} & d p' + \frac{(B-A)}{AB} . q' r' . d t = d N . \cos.\theta - d N' . \sin.\theta ; \\ & d q' + \frac{(C-B)}{CB} . r' p' . dt = -(dN . \sin.\theta + dN' . \cos.\theta) . \sin.\varphi + dN'' . \cos.\varphi ; \\ & d r' + \frac{(A-C)}{AC} . p' q' . dt = -(dN . \sin.\theta + dN' . \cos.\theta) . \cos.\varphi - dN'' . \sin.\varphi . \end{aligned}\right\} ; \quad (D) \qquad [234]$$

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and the equations [231] will become,

$$\begin{aligned} & L' . \sin.\theta - C p . \cos.\theta = - N, \\ & M . \cos \psi + L . \sin.\psi = - N', \\ & L . \cos.\psi - M . \sin.\psi = - N'', \end{aligned} \qquad [232b]$$

whose differentials are,

$$\begin{aligned} & d\theta . \cos.\theta . L' + d L' . \sin.\theta - d . (C p . \cos.\theta) = - d N, \\ & d M . \cos.\psi + d L . \sin.\psi - d\psi . M . \sin.\psi + d\psi . L . \cos.\psi = - d N', \\ & d L . \cos.\psi - d M . \sin.\psi - d\psi . L . \sin.\psi - d\psi . M . \cos.\psi = - d N'', \end{aligned} \qquad [232c]$$

the first of these equations is the same as in [232], and if we put $\psi = 0$, $\sin.\psi = 0$, $\cos.\psi = 1$, [231′], in the two last, they become,

$$d\psi . L + d M = - d N' ; \qquad d L - d\psi . M = - d N''. \qquad [232d]$$

Substitute M [232*a*], and its differential

$$d M = - d\theta . \sin.\theta . L' + \cos.\theta . d L' + d . (C p . \sin.\theta). \qquad [232e]$$

and we shall find,

$$\begin{aligned} & d\psi . L - d\theta . \sin.\theta . L' + d L' . \cos.\theta + d . (C p . \sin.\theta) = - d N', \\ & d L - d\psi . \cos.\theta . L' - C p . d\psi . \sin.\theta = - d N'', \end{aligned} \qquad [232f]$$

which are the same as the second and third equations [232].

* (129) Multiplying the first of the equations [232*c*] by $-\cos.\theta$, the first of [232*f*], by $\sin.\theta$, and adding the products, we shall get by reduction,

$$-d\theta . L' + d\psi . \sin.\theta . L + \cos.\theta . d . (C p . \cos.\theta) + \sin.\theta . d . (C p . \sin.\theta) = d N . \cos.\theta - d N' \sin.\theta, \qquad [234a]$$

but

$$\begin{aligned} & \cos.\theta . d . (C p . \cos.\theta) = C . \cos.\theta . (d p . \cos.\theta - p d\theta . \sin.\theta) = C d p . \cos.^2\theta - C p . d\theta . \sin.\theta . \cos.\theta, \\ & \sin.\theta . d . (C p . \sin.\theta) = C . \sin.\theta . (d p . \sin.\theta + p d\theta . \cos.\theta) = C d p . \sin.^2\theta + C p . d\theta . \sin.\theta . \cos.\theta, \end{aligned} \qquad [234b]$$

whose sum is,

$$\cos.\theta . d . (C p . \cos.\theta) + \sin.\theta . d . (C p . \sin.\theta) = C d p . (\cos.^2\theta + \sin.^2\theta) = C d p. \qquad [234c]$$

These equations are very convenient for finding the rotatory motion of a body when it turns very nearly about one of its principal axes, which is the case with the heavenly bodies.

Substitute this in [234*a*], it becomes,

[234*d*] $$-d\theta\,.\,L'+d\psi\,.\,\sin.\theta\,.\,L+C\,d\,p=d\,N\,.\,\cos.\theta-d\,N'\,.\,\sin\theta.$$

Resubstitute the values of L, L', [232*a*], it becomes,

[234*e*] $$B\,r.(d\psi.\sin.\theta.\sin.\varphi-d\theta.\cos.\varphi)-A\,q.(d\psi.\sin.\theta.\cos.\varphi+d\theta.\sin.\varphi)+C\,dp=dN.\cos.\theta-dN'.\sin.\theta).$$

The first member of this equation, by the substitution of $q\,d\,t$, $r\,d\,t$, [230], becomes, $B\,r\,q\,.\,d\,t-A\,q\,r\,.\,d\,t+C\,d\,p$, and from [233], we get

[234*f*] $$p=\frac{p'}{C};\qquad q=\frac{q'}{A};\qquad r=\frac{r'}{B};$$

therefore, the equation [234*e*] finally becomes,

$$\frac{(B-A)}{A\,B}\,.\,q'\,r'\,.\,d\,t+d\,p'=d\,N\,.\,\cos.\theta-d\,N'\,.\,\sin.\theta,$$

as in the first of the equations [234],

Again, multiplying the expressions of $-d\,N$, $-d\,N'$, [232*c*, *f*], by $\sin.\theta$, $\cos.\theta$, respectively, and adding the products, we shall obtain by reduction,

[234g] $$d\,L'+d\psi\,.\,L\,.\,\cos.\theta-\sin.\theta\,.\,d\,.(C\,p\,.\,\cos.\theta)+\cos.\theta\,.\,d\,.(C\,p\,.\,\sin.\theta)=-d\,N.\sin.\theta-d\,N'.\cos.\theta,$$

but,

[234*h*] $$-\sin.\theta\,.\,d\,.(C\,p\,.\,\cos.\theta)=-C\,.\,\sin.\theta\,.(d\,p\,.\cos.\theta-p\,d\theta.\sin.\theta)=-C\,dp.\sin.\theta.\cos.\theta+C\,p.d\theta.\sin.^2\theta,$$
$$\cos.\theta\,.\,d\,.(C\,p\,.\,\sin.\theta)=C\,.\,\cos.\theta\,.(d\,p\,.\,\sin.\theta+p\,d\theta\,.\cos.\theta)=C\,d\,p\,.\,\sin.\theta\,.\,\cos.\theta+C\,p\,.\,d\theta\,\cos.^2\theta,$$

whose sum is $-\sin.\theta\,.\,d\,.\,(C\,p\,.\,\cos.\theta)+\cos.\theta\,.\,d\,.\,(C\,p\,.\,\sin.\theta)=C\,p\,.\,d\,\theta,$

hence the equation [234*g*], becomes,

[234*i*] $$d\,L'+d\psi\,.\,L\,.\,\cos.\theta+d\,\theta\,.\,C\,p=-(d\,N\,.\,\sin.\theta+d\,N'\,.\,\cos.\theta).$$

Multiply this by $\sin.\varphi$, and $-d\,N''$, [232*f*], by $-\cos.\varphi$, and take the sum of the products, the second member is evidently equal to the second member of the second equation, [234], and the first member becomes,

[234*k*] $$(d\,L'.\sin.\varphi-d\,L.\cos.\varphi)+d\psi.\cos.\theta.(L.\sin.\varphi+L'\,.\cos.\varphi)+C\,p\,.\,(d\psi.\sin.\theta\,.\,\cos.\varphi+d\,\theta\,.\,\sin.\varphi).$$

The differentials of L, L', [232*a*], are,

[234*l*] $$d\,L=B\,d\,r\,.\,\sin.\varphi-A\,d\,q\,.\,\cos.\varphi+d\,\varphi\,.\,(B\,r\,.\,\cos.\varphi+A\,q\,.\,\sin.\varphi),$$
$$d\,L'=B\,d\,r\,.\,\cos.\varphi+A\,d\,q\,.\,\sin.\varphi+d\,\varphi\,.\,(-B\,r\,.\,\sin.\varphi+A\,q\,.\,\cos.\varphi),$$

which being multiplied by $\sin.\varphi$, $\cos.\varphi$, we shall obtain by reduction,

[234*m*] $$d\,L'\,.\,\sin.\varphi-d\,L\,.\,\cos.\varphi=A\,d\,q-B\,r\,.\,d\,\varphi,$$
$$d\,L'\,.\,\cos.\varphi+d\,L\,.\,\sin.\varphi=B\,d\,r+A\,q\,.\,d\,\varphi;$$

moreover the values of L, L', [232*a*], give by reduction,

[234*n*] $$L\,.\,\sin.\varphi+L'\,.\,\cos.\varphi=B\,r,$$
$$L\,.\,\cos.\varphi-L'\,.\,\sin.\varphi=-A\,q.$$

27. The three principal axes to which we have referred the angles θ, ψ, and φ, deserve particular attention; we shall now proceed to determine their position in any solid body. The values of x', y', z' of the preceding article give, by § 21, the following:*

$$x'' = x' . (\cos.\theta . \sin.\psi . \sin.\varphi + \cos.\psi . \cos.\varphi) + y' . (\cos.\theta . \cos.\psi . \sin.\varphi - \sin.\psi . \cos.\varphi) - z' . \sin.\theta . \sin.\varphi ;$$

$$y'' = x' . \cos.\theta . \sin.\psi . \cos.\varphi - \cos.\psi . \sin.\varphi) + y' . (\cos.\theta . \cos.\psi . \cos.\varphi + \sin.\psi . \sin.\varphi) - z' . \sin.\theta . \cos.\varphi ;$$ [235]

$$z'' = x' . \sin.\theta . \sin.\psi + y' . \sin.\theta . \cos.\psi + z' . \cos.\theta .$$

Whence we deduce†

$$x'' . \cos.\varphi - y'' \sin.\varphi = x' . \cos.\psi - y' . \sin.\psi ;$$

$$x'' . \sin.\varphi + y'' . \cos.\varphi = x' . \cos.\theta . \sin.\psi + y' . \cos.\theta . \cos.\psi - z' . \sin.\theta .$$ [236]

Substitute the first equations of [234*m*, *n*], and the last of [230] in [234*k*], it becomes,

$$A\,dq - Br . d\varphi + d\psi . \cos.\theta . Br + Cpr . dt,$$

in which the coefficient of Br is $-(d\varphi - d\psi . \cos.\theta)$, or $-p\,dt$, [230], so that the preceding expression becomes $A\,dq - Brp . dt + Cpr . dt$, and this, by means of [234*f*], is $dq' + \frac{(C-B)}{CB} . r'p' . dt$, which agrees with the first member of the second equation [234], which is therefore correct.

Lastly, multiply [234*i*], by cos. φ, and $-dN''$, [232*f*], by sin. φ, and add the products, the second member will be like that of the last of the equations [234], and the first member of the sum will be,

$$(dL' \cos.\varphi + dL.\sin.\varphi) + d\psi.\cos.\theta.(L.\cos.\varphi - L'.\sin.\varphi) - Cp.(d\psi.\sin.\theta.\sin.\varphi - d\theta.\cos.\varphi),$$

and this by means of the second equations [234*m*, *n*, 230], becomes,

$$B\,dr + Aq . d\varphi - d\psi . \cos.\theta . Aq - Cpq . dt,$$

and the coefficient of Aq being $d\varphi - d\psi . \cos.\theta = p\,dt$, [230], it may be put equal to $B\,dr + Aqp . dt - Cpq . dt$, and this, by means of [234*f*], is $dr' + \frac{(A-C)}{AC} . p'q' . dt$, as in the last of the equations [234], which is therefore correct.

* (130) These values of x'', y'', z'', may be deduced from those of $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, [172, 173, 174], by writing, as in [227*a*], x', y', z', x'', y'', z'', for x, y, z, $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, $z_{\prime\prime\prime}$, respectively.

† (131) Substituting the values of x'', y'', [235], in the first members of [236], and reducing by putting $\cos.^2\varphi + \sin.^2\varphi = 1$, we obtain the second members of [236].

Put

[237]
$$S.x'^2.dm=a^2;\qquad S.y'^2.dm=b^2;\qquad S.z'^2.dm=c^2;$$
$$S.x'y'.dm=f;\qquad S.x'z'.dm=g;\qquad S.y'z'.dm=h;$$

we shall have*

$$\cos.\varphi.S.x''z''.dm-\sin.\varphi.S.y''z''.dm=(a^2-b^2).\sin.\theta.\sin.\psi.\cos.\psi$$
$$+f.\sin.\theta.(\cos.^2\psi-\sin.^2\psi)+\cos.\theta.(g.\cos.\psi-h.\sin.\psi);$$

[238]
$$\sin.\varphi.S.x''z''.dm+\cos.\varphi.S.y''z''.dm$$
$$=\sin.\theta.\cos.\theta.(a^2.\sin.^2\psi+b^2.\cos.^2\psi-c^2+2f.\sin.\psi.\cos.\psi)$$
$$+(\cos.^2\theta-\sin.^2\theta).(g.\sin.\psi+h.\cos.\psi);$$

by putting the second members of these two equations equal to nothing, we shall have,†

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* (132) The first of these equations is found by multiplying the value of z'', [235], by that of $x''.\cos.\varphi-y''.\sin.\varphi$, [236], and by dm; then integrating relative to S. The first member will evidently agree with [238]; the second member is

$$S.\{x'.\sin.\theta.\sin.\psi+y'.\sin.\theta.\cos.\psi+z'.\cos.\theta\}.\{x'.\cos.\psi-y'.\sin.\psi\}.dm,$$

or by reduction

$$S.\{(x'^2-y'^2).\sin.\theta.\sin.\psi.\cos.\psi+x'y'.\sin.\theta.(\cos.^2\psi-\sin.^2\psi)+\cos.\theta.(x'z'.\cos.\psi-y'z'.\sin.\psi)\}.dm,$$

and as θ, ψ, are not affected by the characteristic S, they may be brought from under that sign; and putting $S.x'^2.dm=a^2$; $S.y'^2.dm=b^2$, &c., as in [237], we shall obtain the second member of that equation under the required form. In a similar way the second equation [238], is found by multiplying the value of z'', [235], by that of

$$x''.\sin.\varphi+y''.\cos.\varphi,\ [236],$$

and by dm, which produces as above, by prefixing the sign S,

$$S.\{x'.\sin.\theta.\sin.\psi+y'.\sin.\theta.\cos.\psi+z'.\cos.\theta\}.\{x'.\cos.\theta.\sin.\psi+y'.\cos.\theta.\cos.\psi-z'.\sin.\theta\}.dm,$$

equal to

$$S.\{\sin.\theta.\cos.\theta.(x'^2.\sin.^2\psi+y'^2.\cos.^2\psi-z'^2+2x'y'.\sin.\psi.\cos.\psi)$$
$$+(\cos.^2\theta-\sin.^2\theta).(x'z'.\sin.\psi+y'z'.\cos.\psi)\}.dm,$$

which is reduced to the required form, by putting $S.x^2.dm=a^2$, &c.

† (133) The second member of the first equation [238], put equal to nothing, and divided by $\cos.\theta$, using $\text{tang}.\theta$ for $\frac{\sin.\theta}{\cos.\theta}$, will give $\text{tang}.\theta$, [239]. The second member of the second equation, [238], put equal to nothing, substituting $\sin.\theta.\cos.\theta=\tfrac{1}{2}\sin.2\theta$; $\cos.^2\theta-\sin.^2\theta=\cos.2\theta$, (31, 32 Int.), then dividing by $\cos.2\theta$, and putting $\text{tang}.2\theta$, for $\frac{\sin.2\theta}{\cos.2\theta}$, will give the value of $\tfrac{1}{2}\text{tang}.2\theta$, [239],

$$\text{tang.}\,\theta=\frac{h\,.\,\sin.\psi-g\,.\,\cos.\psi}{(a^2-b^2)\,.\,\sin.\psi\,.\,\cos.\psi+f\,.\,(\cos.^2\psi-\sin.^2\psi)};\qquad[239]$$

$$\tfrac{1}{2}\,\text{tang.}\,2\theta=\frac{g\,.\,\sin.\psi+h\,.\,\cos.\psi}{c^2-a^2\,.\,\sin.^2\psi-b^2\,.\,\cos.^2\psi-2f\,.\,\sin.\psi\,.\,\cos.\psi};$$

but we have [29 Int.]

$$\tfrac{1}{2}\,\text{tang.}\,2\theta=\frac{\text{tang.}\,\theta}{1-\text{tang.}^2\theta};\qquad[240]$$

Making these two expressions of $\frac{1}{2}$ tang. 2θ equal to each other, and substituting in the last, for tang. θ, its preceding value in ψ; then putting, for brevity, tang. $\psi=u$; we shall obtain, by reduction, the following equation of the third degree:*

$$0=(g\,u+h)\,.\,(h\,u-g)^2$$
$$+\{(a^2-b^2)\,.\,u+f\,.\,(1-u^2)\}\,.\,\{(h\,c^2-h\,a^2+fg)\,.\,u+g\,b^2-g\,c^2-h\,f\}.\qquad[241]$$

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* (134) Put for brevity, n, N, for the numerators, and e, E, for the denominators of the values of tang. θ, and $\frac{1}{2}$ tang. 2θ, [239], we shall have tang. $\theta=\frac{n}{e}$; $\frac{1}{2}$ tang. $2\theta=\frac{N}{E}$. These being substituted in [240], it becomes $\frac{N}{E}=\frac{\frac{n}{e}}{1-\frac{n^2}{e^2}}=\frac{n\,e}{e^2-n^2}$. Multiplying by $E\,.\,(e^2-n^2)$, and reducing, we get $0=N n^2+e\,.\,(n\,E-N e)$; but from the values of n, e, N, E, we obtain,

$$n\,E=h\,.\,\sin.\psi\,.\,\{c^2-a^2\,.\,\sin.^2\psi-b^2\,.\,\cos.^2\psi-2f\,.\,\sin.\psi\,.\,\cos.\psi\}$$
$$-g\,.\,\cos.\psi\,.\,\{c^2-a^2\,.\,\sin.^2\psi-b^2\,.\,\cos.^2\psi-2f\,.\,\sin.\psi\,.\,\cos.\psi\}$$
$$-N e=-g\,.\,\sin.\psi\,.\,\{(a^2-b^2)\,.\,\sin.\psi\,.\,\cos.\psi+f\,.\,(\cos.^2\psi-\sin.^2\psi)\}$$
$$-h\,.\,\cos.\psi\,.\,\{(a^2-b^2)\,.\,\sin.\psi\,.\,\cos.\psi+f\,.\,(\cos.^2\psi-\sin.^2\psi)\},$$

adding these together we get $n\,E-N e$, and, by connecting the similar terms, it becomes,

$$h\,c^2\,.\,\sin.\psi-h\,a^2\,.\,\sin.\psi\,.\,(\sin.^2\psi+\cos.^2\psi)+h\,b^2\,.\,\sin.\psi\,.\,(-\cos.^2\psi+\cos.^2\psi)$$
$$+fg\,.\,\sin.\psi\,.\,(2\cos.^2\psi-\cos.^2\psi+\sin.^2\psi)+g\,a^2\,.\,\cos.\psi\,.\,(\sin.^2\psi-\sin.^2\psi)$$
$$+g\,b^2\,.\,\cos.\psi\,.\,(\cos.^2\psi+\sin.^2\psi)-g\,c^2\,.\,\cos.\psi+hf\,.\,\cos.\psi\,.\,(-2\sin.^2\psi-\cos.^2\psi+\sin.^2\psi),$$

which is easily reduced to

$$h\,c^2\,.\,\sin.\psi-h\,a^2\,.\,\sin.\psi+fg\,.\,\sin.\psi+g\,b^2\,.\,\cos.\psi-g\,c^2\,.\,\cos.\psi-hf\,.\,\cos.\psi,$$

and by putting for sin. ψ its value cos. ψ . tang. ψ, or u . cos. ψ, it becomes,

$$n\,E-N e=\cos.\psi\,.\{(h\,c^2-h\,a^2+fg)\,.\,u+g\,b^2-g\,c^2-hf\}.$$

Again, the value $\sin.\psi=u\,.\,\cos.\psi$, substituted in n, e, N, they become,

$$n=\cos.\psi\,.\,\{h\,u-g\};\qquad e=\cos.^2\psi\,.\,\{(a^2-b^2)\,.\,u+f\,.\,(1-u^2)\};$$

$N=\cos.\psi\,.\Big\{g\,u+h\Big\}$; substituting these in $0=N n^2+e\,.\,(n\,E-N e)$, and rejecting the common factor $\cos.^3\psi$, we obtain the equation [241].

This equation having at least one real root, it is evidently always possible to render the two following quantities at the same time equal to nothing,*

[242]
$$\cos.\varphi\,.\,S\,.\,x''\,z''\,.\,d\,m - \sin.\varphi\,.\,S\,.\,y'\,z''\,.\,d\,m\,;$$
$$\sin.\varphi\,.\,S\,.\,x''\,z''\,.\,d\,m + \cos.\varphi\,.\,S\,.\,y'\,z''\,.\,d\,m\,;$$

consequently, the sum of their squares $(S.x''z''\,.\,d\,m)^2+(S\,.\,y'\,z''\,.\,d\,m)^2=0$, which requires that we should have separately,

[243]
$$S\,.\,x''\,z''\,.\,d\,m=0\,; \qquad S\,.\,y'\,z''\,.\,d\,m=0.$$

The value of u gives that of the angle ψ, consequently that of $\operatorname{tang.}\theta$, and thence the angle θ, [239]. We must now determine the angle φ, which is to be found by means of the condition $S.x''y'.dm=0$; which yet remains to be satisfied. For that purpose we shall observe, that if we substitute in $S\,.\,x''\,y'\,.\,d\,m$, for x'', y', their preceding values; it will become of this form† $H\,.\sin.2\,\varphi+L\,.\cos.2\,\varphi$, H and L being functions of the angles θ and ψ, and of the constant quantities a^2, b^2, c^2, f, g, h; putting this expression equal to nothing, we shall have

[244]
$$\operatorname{tang.}2\,\varphi=\frac{-L}{H}.$$

Equations to determine the Principal Axes of Rotation.

The three axes determined by means of the preceding values of θ, ψ, and φ, satisfy the three equations [228],

[245]
$$S\,.\,x''\,y'\,.\,d\,m=0\,; \qquad S\,.\,x''\,z''\,.\,d\,m=0\,; \qquad S\,.\,y'\,z''\,.\,d\,m=0\,;$$

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* (134*a*) As this value of u renders the second members of [238] equal to nothing, their first members [242], must also be equal to nothing. The squares of these last added together, putting $\cos.^2\varphi+\sin.^2\varphi=1$, give the equation

$$(S\,.\,x''\,z''\,.\,d\,m)^2+S\,.\,(y'\,z''\,.\,d\,m)^2=0,$$

abovementioned; both terms being squares, their sum cannot be nothing, unless we have separately the two equations [243].

† (135). If in the expressions of x'', y', [235], we connect together the terms multiplied by $\sin.\varphi$, and those multiplied by $\cos.\varphi$, and put for brevity,

$$H'=x'\,.\cos.\theta\,.\sin.\psi+y'\,.\cos.\theta\,.\cos.\psi-z'\,.\sin.\theta, \qquad L'=x'\,.\cos.\psi-y'\,.\sin.\psi,$$

they will become $x''=H'\,.\sin.\varphi+L'\,.\cos\varphi\,;$ $y'=H'\,.\cos.\varphi-L'\,.\sin.\varphi$, which, multiplied together, give $x''\,y'=(H'^2-L'^2)\,.\sin.\varphi\,.\cos.\varphi+H'\,L'\,.(\cos.^2\varphi-\sin.^2\varphi)$, or (31, 32 Int.), $x''\,y'=\tfrac{1}{2}\,.(H'^2-L'^2)\,.\sin.2\,\varphi+H'L'\,.\cos.2\,\varphi$. Multiplying this by $d\,m$, and prefixing the sign of integration S, then putting $S\,.\,\tfrac{1}{2}\,(H'^2-L'^2)\,.\,d\,m=H$; $S\,.\,H'\,L'\,.\,d\,m=L$, we get $S\,.\,x''\,y'\,.\,d\,m=H\,.\sin.2\,\varphi+L\,.\cos.2\,\varphi$, as above.

The equation of the third degree in u, seems to indicate three systems of principal axes, similar to the preceding; but we ought to observe that u is the tangent of the angle formed by the axis of x' and by the line of intersection of the plane of x', y', with that of x'', y''; now it is evident that we may change any one of the three axes x'', y'', z'', into any other of them, and still the three preceding equations [245], will be satisfied; the equation in u, ought therefore also to determine the tangent of the angle formed by the axis of x' with the line of intersection of the plane of x', y', either with that of x'', y'', or that of x'', z'', or y'', z''. Therefore the three roots of the equation in u are real, and appertain to the same system of axes. [245']

It follows, from what has been said, that in general a solid has but one system of axes which has the property treated of. These axes have been called *principal axes of rotation*, on account of a property peculiar to them, of which we shall hereafter treat.* [245'']

Momentum of Inertia, or Rotatory Inertia.

The *momentum of inertia*, or *rotatory inertia* of a body, relative to any axis, is the sum of the products of each particle of the body, by the square of its distance from that axis. Thus the quantities A, B, C, are the momenta of inertia of the solid we have just considered, relative to the axes x'', y'', and z'', [227c]. [245'''] Let us now put C' for the momentum of inertia of the same solid relative to the axis of z'; we shall find, by means of the values of x' and y' of the preceding article,†

$$C' = A \,.\, \sin.^2\theta \,.\, \sin.^2\varphi + B \,.\, \sin.^2\theta \,.\, \cos.^2\varphi + C \,.\, \cos.^2\theta \,; \qquad [246]$$

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* (135a) The property here mentioned is treated of in [280^{iv}], where it is proved that if the body begin to turn about one of the principal axes, it will continue to move uniformly about that axis, and from this property they are called principal axes.

† (136) Square the values of x', y', [230a], multiply the sum by dm, prefix the sign S, neglect the products depending on $x''y''$, $x''z''$, $y''z''$, on account of the equations [245], we shall get

$$C' = S\,.\,(x'^2 + y'^2)\,.\,dm = S\,.\,dm\,.\{(A_0^2 + A_1^2)\,.\,x''^2 + (B_0^2 + B_1^2)\,.\,y''^2 + (C_0^2 + C_1^2)\,.\,z''^2\},$$

which by means of the three first equations, [172b], $A_0^2 + A_1^2 = 1 - A_2^2$, &c. becomes,

$$C' = S\,.\,dm\,\{(1 - A_2^2)\,.\,x''^2 + (1 - B_2^2)\,.\,y''^2 + (1 - C_2^2)\,.\,z''^2\}. \qquad [246a]$$

Put $2s = A + B + C$, and from [229], we shall obtain $s = S\,.\,(x''^2 + y''^2 + z''^2)\,.\,dm$, [246b] subtracting from this each of the equations [229], we get,

$$S\,.\,x''^2\,.\,dm = s - A, \qquad S\,.\,y''^2\,.\,dm = s - B, \qquad S\,.\,z''^2\,.\,dm = s - C, \qquad [246c]$$

The quantities $\sin.^2\theta\,.\,\sin.^2\varphi$, $\sin.^2\theta\,.\,\cos.^2\varphi$, and $\cos.^2\theta$, are the squares[*] of the cosines of the angles which the axes of x'', y'', z'', make with the axis of z'; whence it follows, in general, that if we multiply the momentum of inertia relative to each principal axis of rotation, by the square of the cosine
[246'] of the angle it makes with any other axis whatever, the sum of these three products will be the momentum of inertia of the solid, relative to this last axis.

The quantity C' is less than the greatest of the three quantities A, B, C,
[246''] and exceeds the least of them; the greatest and the least momenta of inertia appertain therefore to the principal axes.[†]

[246'''] Let X, Y, Z, be the co-ordinates of the centre of gravity of the solid, referred to the origin of the co-ordinates, which we shall fix at the point about which the body is forced to turn, if it be not free; $x'-X$, $y'-Y$, $z'-Z$, will be the co-ordinates of the particle dm of the body, referred to

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and from the substitution of these in [246a], we find,

$$C' = (1-A_2^2)\,.\,(s-A) + (1-B_2^2)\,.\,(s-B) + (1-C_2^2)\,.\,(s-C)$$
$$= (3-A_2^2-B_2^2-C_2^2)\,.\,s - (A+B+C) + A\,.\,A_2^2 + B\,.\,B_2^2 + C\,.\,C_2^2.$$

But from [172d], $(3-A_2^2-B_2^2-C_2^2)\,.\,s = (3-1)\,.\,s = 2s$, and $(A+B+C) = 2s$, therefore the two first terms of the preceding expression destroy each other, and we finally get,

[246d] $$C' = A\,.\,A_2^2 + B\,.\,B_2^2 + C\,.\,C_2^2 = S\,.\,(x'^2+y'^2)\,.\,dm,$$

which, by using the values of A_2, B_2, C_2, [171a], becomes as in [246].

[*] (137) This is evident from [246d], observing that the values of A_2, B_2, C_2, [172f, 171a], became, by changing $x, y, z, x_{\prime\prime\prime}, y_{\prime\prime\prime}, z_{\prime\prime\prime}$, into $x', y', z', x'', y'', z''$, respectively, [227a],

[246e] $$A_2 = \cos.\,(z', x'') = -\sin.\theta\,.\,\sin.\varphi; \qquad B_2 = \cos.\,(z', y'') = -\sin.\theta\,.\,\cos.\varphi;$$
$$C_2 = \cos.\,(z', z'') = \cos.\theta.$$

[†] (138) For, if in the general expression of C', [246], we substitute for A, B, C, the greatest of those quantities, for example A, the result would evidently exceed C', because each of its terms is positive. Now this result would be,

[247a] $$A\,.\,\left\{\sin.^2\theta\,.\,(\sin.^2\varphi+\cos.^2\varphi)+\cos.^2\theta\right\} = A\,.\,\left\{\sin.^2\theta+\cos.^2\theta\right\} = A.$$

Hence C' is less than the greatest of the quantities A, B, C. And in a similar way, by taking A for the least of the quantities A, B, C, we may prove that C' exceeds the least of the quantities A, B, C.

its centre of gravity; the momentum of inertia, relative to an axis parallel to the axis of z', and passing through the centre of gravity, will therefore be,

$$S.\{(x'-X)^2+(y'-Y)^2\}.dm;$$ [247]

now we have, by the nature of the centre of gravity, $S.x'dm=mX$; $S.y'dm=mY$ [154]; the preceding momentum therefore becomes*

$$-m.(X^2+Y^2)+S.(x'^2+y'^2).dm.$$ [248]

Thus we shall have the momentum of inertia of the solid, relative to the axis which passes through any point whatever, when the momenta are known with respect to the axes which pass through the centre of gravity. It is evident also, that the least of all the momenta of inertia corresponds to one of the three principal axes passing through that centre.† [248']

Suppose that by the nature of the body, the two momenta of inertia A and B are equal, we shall have,‡

$$C'=A.\sin.^2\theta+C.\cos.^2\theta;$$ [249]

making therefore θ equal to a right angle, which renders the axis of z' perpendicular to that of z'' [246e], we shall have $C'=A$. The momenta of inertia relative to all the axes situated in the plane perpendicular to the axis [249']

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* (139) Developing the expression $S.\{(x'-X)^2+(y'-Y)^2\}.dm$, it may be put under the form,

$$S.(x'^2+y'^2).dm+S.(X^2+Y^2).dm-2S.(x'X+y'Y).dm.$$ [247b]

Now

$$S.(X^2+Y^2).dm=(X^2+Y^2).S.dm=(X^2+Y^2).m;$$
$$-2S.x'Xdm=-2X.S.x'dm,$$

and by [154], $S.x'dm=mX$, hence $-2S.x'Xdm=-2mX^2$; in like manner $-2S.y'Ydm=-2mY^2$; these being substituted in [247b], the whole expression becomes, $-m.(X^2+Y^2)+S.(x'^2+y'^2).dm$, as in [248].

† (140) For the momentum of inertia about the axis of z' is $S.(x'^2+y'^2).dm$, and about a parallel axis passing through the centre of gravity is

$$S.(x'^2+y'^2).dm-m.(X^2+Y^2),\quad [248].$$

Hence the latter must generally be the least, and as the least momentum corresponds to one of the principal axes [246''], the proposition becomes manifest.

‡ (141) Put $B=A$, in the general expression, [246], and $\cos.^2\varphi+\sin.^2\varphi=1$, we shall get [249]. An ellipsoid of revolution, about the axis of z'', is a figure of this kind.

of z'', will then be equal to each other. But it is easy to prove that we shall have, in this case, for the system of the axis of z'', and of any two axes whatever, perpendicular to each other and to this axis, the following system of equations.

[250] $$S.x'y'.dm=0; \qquad S.x'z''.dm=0; \qquad S.y'z''.dm=0;$$

for, if we denote by x'', y'', the co-ordinates of a particle dm of the body, referred to the two principal axes, taken in the plane perpendicular to the axis of z'', and with respect to which the momenta of inertia are supposed to be equal, we shall have*

[251] $$S.(x''^2+z''^2).dm=S.(y''^2+z''^2).dm;$$

[251′] or simply $S.x''^2.dm=S.y''^2.dm$; now by putting ε equal to the angle which the axis of x' makes with the axis of x'', we shall have†

[252] $$x'=x''.\cos.\varepsilon+y''.\sin.\varepsilon;$$
$$y'=y''.\cos.\varepsilon-x''.\sin.\varepsilon;$$

we therefore have

[253] $$S.x'y'.dm=S.x''y''.dm.(\cos.^2\varepsilon-\sin.^2\varepsilon)+S.(y''^2-x''^2).dm.\sin.\varepsilon.\cos.\varepsilon=0;$$

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* (142) The momentum of inertia relative to the axis of y'' is $S.(x''^2+z''^2).dm$, [229, 227c], and relative to the axis of x'', is $S.(y''^2+z''^2).dm$; putting these equal evidently gives $S.x''^2.dm=S.y''^2.dm$.

† (143) The values of x', y', are found, by using the figure in page 112, changing the co-ordinates x, y, $x_{,}$, $y_{,}$, of the point K, into x', y', x'', y'', respectively, and putting $\psi=\varepsilon$, by which means the two first equations, [168], become as in [252]. Multiply these values of x', y', together, and the product by dm, prefixing the sign S, we obtain [253],

$$S.x'y'.dm=(\cos.^2\varepsilon-\sin.^2\varepsilon).S.x''y''.dm+\sin.\varepsilon.\cos.\varepsilon.S.(y''^2-x''^2).dm;$$

observing that $\sin.\varepsilon$, $\cos.\varepsilon$, being common to all points of the body, may be placed without the sign S. The second member of this equation becomes nothing in consequence of the equations $S.x''y''.dm=0$, [245], and $S.x''^2.dm=S.y''^2.dm$, [251′], so that we shall [253a] have $S.x'y'.dm=0$, as in [250]. Again, multiply the values [252] by $z''.dm$, and prefix the sign S, we shall find,

$$S.x'z''.dm=\cos.\varepsilon.S.x''z''.dm+\sin.\varepsilon.S.y''z''.dm,$$
$$S.y'z''.dm=\cos.\varepsilon.S.y''z''.dm-\sin.\varepsilon.S.x''z''.dm,$$

the second members of these equations become nothing, by means of the equations [245], therefore we shall have $S.x'z''.dm=0$, $S.y'z''.dm=0$, as in [250].

we shall find in a similar manner $S.x'z''.dm=0$; $S.y'z''.dm=0$; [253′]
therefore all the axes perpendicular to that of z'' are then principal axes, and in this case the solid has an infinite number of principal axes.

If we have, at the same time, $A=B=C$ we shall have in general* [253″]
$C'=A$; that is, all the momenta of inertia of the solid will be equal; but then we shall have generally†

$$S.x'y'.dm=0; \qquad S.x'z'.dm=0; \qquad S.y'z'.dm=0; \qquad [254]$$

whatever be the position of the plane of x', y', so that all the axes will then be principal axes. This is the case of the sphere; we shall see hereafter that this property appertains to an infinite number of solids, of which we shall give the general equation.‡ [254′]

28. The quantities p, q, r, which we have introduced in the equations [254″]
(C) § 26, [231], have this remarkable property, that they determine the position of the real but momentary axis of rotation of the body, with respect

Momentary Axis of Rotation.

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* (144) Putting $A=B=C$, in the general expression of C, [246], it becomes equal to A, as in [247a].

† (145) From $A=B$, we deduced $S.x'y'.dm=0$, in [253a]. In like manner, by putting $A=C$, we should get $S.x'z'.dm=0$; and $B=C$ would give

$$S.y'z'.dm=0.$$

We might also prove this by means of x', y', z', [230a]. For, if we multiply those values of x', y', together, and their product by dm, prefixing the sign S, neglecting the quantities [228], we shall get,

$$S.x'y'.dm=A_0A_1.S.x''^2.dm+B_0B_1.S.y''^2.dm+C_0C_1S.z''^2.dm, \qquad [254a]$$

but from $A=B=C$, [229], we get $S.x''^2.dm=S.y''^2.dm=S.z''^2.dm$. Substituting these in [254a], we find,

$$S.x'y'.dm=(A_0A_1+B_0B_1+C_0C_1).S.x''^2.dm,$$

and this, by means of the fourth equation, [172d], becomes $S.x'y'.dm=0$. In like manner we might find the other two equations [254].

‡ (146) In Book V, § 2, [2940].

to the three principal axes. For we have, relative to the points situated in the axis of rotation,*

[254‴] $$dx'=0; \qquad dy'=0; \qquad dz'=0;$$

and by taking the differentials of the values of x', y', z', § 26 [227], making $\sin.\psi=0$, after taking the differential,† which can be done, since we may fix at pleasure the position of the axis of x' in the plane of x', y', we shall have

[255]
$$\begin{aligned}dx'&=x''.\{d\psi.\cos.\theta.\sin.\varphi-d\varphi.\sin.\varphi\}+y''.\{d\psi.\cos.\theta.\cos.\varphi-d\varphi.\cos.\varphi\}\\&\quad+z''.d\psi.\sin.\theta=0;\\dy'&=x''.\{d\varphi.\cos.\theta.\cos.\varphi-d\theta.\sin.\theta.\sin.\varphi-d\psi.\cos.\varphi\}\\&\quad+y''.\{d\psi.\sin.\varphi-d\varphi.\cos.\theta.\sin.\varphi-d\theta.\sin.\theta.\cos.\varphi\}+z''.d\theta.\cos.\theta=0;\\dz'&=-x''.\{d\theta.\cos.\theta.\sin.\varphi+d\varphi.\sin.\theta.\cos.\varphi\}\\&\quad-y''.\{d\theta.\cos.\theta.\cos.\varphi-d\varphi.\sin.\theta.\sin.\varphi\}-z''.d\theta.\sin.\theta=0;\end{aligned}$$

If we multiply the first of these equations by $-\sin.\varphi$; the second by $\cos.\theta.\cos.\varphi$; and the third by $-\sin.\theta.\cos.\varphi$; we shall have by adding them‡

[256] $$0=p\,x''-q\,z'';$$

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* (147) If a point, whose co-ordinates are x', y', z', be at rest, the quantities $\frac{dx'}{dt}$, $\frac{dy'}{dt}$, $\frac{dz'}{dt}$, which represent the velocities in directions of the axes x, y, z, respectively, must be nothing, consequently, $dx'=0$, $dy'=0$, $dz'=0$. Substituting in these, the values of x', y', z', [227], we must suppose θ, ψ, φ, to be variable, and x'', y'', z'', constant, because any particle of the body retains always the same relative position to the principal axes x'', y'', z'', as was observed in [227*b*].

† (148) By this substitution, the differential of any expression of the form

$$M.\cos.\psi+L.\sin.\psi,$$

becomes $dM+L.d\psi$, as is shown in [232*d*], therefore, in finding the differential, we may change $\sin.\psi$ into $d\psi$, and in the terms multiplied by $\cos,\psi$, put $\cos.\psi=1$, and take the differential of the terms in which this substitution is made.

‡ (149) The equations [255], are the same as [230*b*] which are to be put equal to nothing, [254‴], and then the values [230*d*], are to be substituted. The first of these equations arising from $dx'=0$, is,

$$0=x''.(B_0p-C_0r)+y''.(C_0q-A_0p)+z''.(A_0r-B_0q)$$

If we multiply the first of the same equations by $\cos.\varphi$; the second by $\cos.\theta.\sin.\varphi$; and the third by $-\sin.\theta.\sin.\varphi$; we shall have by adding them

$$0 = p y'' - r z'';$$ [257]

Lastly, if we multiply the second of the same equations by $\sin.\theta$; and the third by $\cos.\theta$; we shall have by adding them

$$0 = q y'' - r x''.$$ [258]

This last equation evidently results from the two preceding;* therefore the three equations $dx' = 0$, $dy' = 0$, $dz' = 0$, are reduced to these two equations, which appertain to a right line forming with the axes of x'', y'', z'', the angles whose cosines are†

$$\frac{q}{\sqrt{p^2 + q^2 + r^2}}; \qquad \frac{r}{\sqrt{p^2 + q^2 + r^2}}; \qquad \frac{p}{\sqrt{p^2 + q^2 + r^2}};$$ [259]

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the values $dy' = 0$, $dz' = 0$, give similar expressions, and by arranging them according to the order of the letters A, B, C, we get the following system of equations,

$$\begin{aligned} 0 &= A_0 . (r z'' - p y'') + B_0 . (p x'' - q z'') + C_0 . (q y'' - r x''), \\ 0 &= A_1 . (r z'' - p y'') + B_1 . (p x'' - q z'') + C_1 . (q y'' - r x''), \\ 0 &= A_2 . (r z'' - p y'') + B_2 . (p x'' - q z'') + C_2 . (q y'' - r x''), \end{aligned}$$ [256a]

Multiply these equations by A_0, A_1, A_2, and add the products; the coefficient of $r z'' - p y''$, becomes 1, and the others vanish, in consequence of the equations, [172b], hence $r z'' - p y'' = 0$, as in [257]. Multiplying the same equations by B_0, B_1, B_2, the sum of the products, reduced in the same manner, becomes $p x'' - q z'' = 0$, as in [256]. Lastly, the same equations, being multiplied by C_0, C_1, C_2, and the sum of the products taken, is $q y'' - r x'' = 0$, as in [258]. It may be observed that the factors given above by the author are the same as A_0, A_1, A_2, B_0, &c. [171a], putting $\psi = 0$.

* (149a) From [256], we get $z'' = \frac{p x''}{q}$, and from [257], $z'' = \frac{p y''}{r}$, hence $\frac{p x''}{q} = \frac{p y''}{r}$, which is easily reduced to the form $0 = q y'' - r x''$, [258].

† (150) The equations [256, 257, 258], may be easily reduced to the form of the equations [19b′], which correspond to those of a right line passing through the origin of the co-ordinates. Let this line be Ac, (Fig. page 7), whose origin is A, the co-ordinates of the point c being $AD = x''$, $Dd = Aa = y''$, $dc = AB = z''$, and the cosines of the angles, which the line Ac makes with the axes of x'', y'', z'', will be represented by

this right line is therefore at rest, and forms the real axis of rotation of the body.

Rotatory Velocity. To obtain the rotatory velocity of the body; let us consider the point of the axis of z'', which is distant from the origin of the co-ordinates by a quantity equal to unity. We shall have its velocities parallel to the axes of

[259'] x', y', and z', by making $x''=0$, $y''=0$, $z''=1$, in the preceding expressions of dx', dy', dz', [255], and dividing them by dt; which gives for these partial velocities,

[260] $$\frac{d\psi}{dt}.\sin.\theta; \qquad \frac{d\theta}{dt}.\cos.\theta; \qquad \frac{-d\theta}{dt}.\sin.\theta;$$

therefore the whole velocity of that point is*

[260'] $$\frac{\sqrt{d\theta^2+d\psi^2.\sin.^2\theta}}{dt}=\sqrt{q^2+r^2}.$$

[258a] $\frac{AD}{Ac}$, $\frac{Aa}{Ac}$, $\frac{AB}{Ac}$, and since $Ac^2=AD^2+Dd^2+dc^2=x''^2+y''^2+z''^2$, [11a], these cosines will be represented by

[259a] $$\frac{x''}{\sqrt{x''^2+y''^2+z''^2}}, \qquad \frac{y''}{\sqrt{x''^2+y''^2+z''^2}}, \qquad \frac{z''}{\sqrt{x''^2+y''^2+z''^2}},$$

respectively. Now the equations [256, 257], give $x''=\frac{qz''}{p}$, $y''=\frac{rz''}{p}$, hence

$$\sqrt{x''^2+y''^2+z''^2}=\frac{z''.\sqrt{p^2+q^2+r^2}}{p},$$

therefore the preceding cosines will be

[259b] $$\frac{q}{\sqrt{p^2+q^2+r^2}}, \qquad \frac{r}{\sqrt{p^2+q^2+r^2}}, \qquad \frac{p}{\sqrt{p^2+q^2+r^2}},$$

as in [259]. Hence we find that the sines of the same angles will be,

[259c] $$\frac{\sqrt{p^2+r^2}}{\sqrt{p^2+q^2+r^2}}, \qquad \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}}, \qquad \frac{\sqrt{q^2+r^2}}{\sqrt{p^2+q^2+r^2}},$$

and the tangents of the same angles will be

[259d] $$\frac{\sqrt{p^2+r^2}}{q}, \qquad \frac{\sqrt{p^2+q^2}}{r}, \qquad \frac{\sqrt{q^2+r^2}}{p},$$

as is evident from the common rules of trigonometry.

* (151) This expression is found as in [40a], by taking the square root of the sum of the squares of the partial velocities, [260], and reducing by putting $\cos.^2\theta+\sin.^2\theta=1$, by which means it becomes, $\frac{\sqrt{d\theta^2+d\psi^2.\sin.^2\theta}}{dt}$. But if we take the sum of the squares of

Dividing this velocity by the distance of the point from the momentary axis of rotation, we shall have the angular velocity of rotation of the body; now this distance is evidently equal to the sine of the angle which the real axis of rotation makes with the axis of z'', the cosine of which angle is* $\frac{p}{\sqrt{p^2+q^2+r^2}}$; we shall therefore have $\sqrt{p^2+q^2+r^2}$ for the angular velocity of rotation. [260'']

Hence we perceive that whatever be the motion of rotation of a body, about a fixed point, or a point considered as fixed; this motion can be no other than rotatory motion about an axis fixed during an instant, but which may vary from one instant to another. The position of this axis, with respect to the three principal axes, and the angular velocity of rotation, depend upon the variable quantities p, q, r, the determination of which is very important in these researches, and as they denote quantities independent of the situation of the plane of† x', y', they must be independent of that situation. [260'''] [260^{iv}]

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the values of $q\,dt$, $r\,dt$, [230], and reduce it by putting $\sin.^2\varphi+\cos.^2\varphi=1$, we shall get $d\theta^2+d\psi^2.\sin.^2\theta=(q^2+r^2).dt^2$, hence,

$$\frac{\sqrt{d\theta^2+d\psi^2.\sin.^2\theta}}{dt}=\sqrt{q^2+r^2}.$$

[*] (152) The sine corresponding to this cosine is equal to $\frac{\sqrt{q^2+r^2}}{\sqrt{p^2+q^2+r^2}}$, [259c], and as the distance of the point, assumed in [259'], from the origin of the co-ordinates is 1, its distance from the momentary axis of rotation will be $\frac{\sqrt{q^2+r^2}}{\sqrt{p^2+q^2+r^2}}$. If we divide the velocity $\sqrt{q^2+r^2}$ [260'], by this distance, we shall obtain the angular velocity of rotation, $\sqrt{p^2+q^2+r^2}$, which for brevity we shall call a. [260a]

[†] (153) The terms p, q, r, have been shown, [259], to be proportional to the cosines of the angles, which the momentary axis of rotation forms with the principal axes, and as these axes are wholly independent of the plane arbitrarily assumed, for that of x', y', it is evident that p, q, r, cannot depend on the position of this plane.

29. Let us now determine these variable quantities, in functions of the time t, in the case where the body is not urged by any external forces. For this purpose, let us resume the equations (D) § 26 [234] between the variable quantities p', q', r', which are in a constant ratio to the preceding.
[260v] The differentials dN, dN', dN'', are in this case nothing;[*] and these equations give, by adding them together, after having multiplied them respectively by p', q', r',

[261] $$0=p'\,dp'+q'\,dq'+r'\,dr';$$

and by integration,

[262] $$p'^2+q'^2+r'^2=k^2;$$

k being an arbitrary constant quantity.

The equations (D) [234], multiplied respectively by $2AB.p'$, $2BC.q'$, and $2AC.r'$, and these products added, give, by integrating their sum,[†]

[263] $$AB.p'^2+BC.q'^2+AC.r'^2=H^2;$$

H being an arbitrary constant quantity; this equation comprises the principle of the preservation of the living forces.[‡] We may deduce from these two integrals,

[264] $$q'^2=\frac{AC.k^2-H^2+A.(B-C).p'^2}{C.(A-B)};$$
$$r'^2=\frac{H^2-BC.k^2-B.(A-C).p'^2}{C.(A-B)};$$

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[*] (154) As is evident by putting the forces [212iv], $P=0$, $Q=0$, $R=0$, in the
[260b] differentials of the expressions [225], which makes $d\mathcal{N}=0$, $d\mathcal{N}'=0$, $d\mathcal{N}''=0$, and then the equations, [234], become

[261a] $$dp'+\frac{(B-A)}{AB}.q'r'.dt=0,\quad dq'+\frac{(C-B)}{CB}.r'p'.dt=0,\quad dr'+\frac{(A-C)}{AC}.p'q'.dt=0,$$

which being multiplied by p', q', r', respectively, and the products added, the coefficient of $p'q'r'$, becomes nothing, and the sum is $0=p'\,dp'+q'\,dq'+r'\,dr'$, as in [261].

[†] (155) Using the equations [261a], of the last note, and multiplying them as above directed, the coefficient of $p'q'r'$, becomes 0, and the sum of these products is

$$2AB.p'\,dp'+2BC.q'\,dq'+2AC.r'\,dr'=0,$$

its integral is [263].

[‡] (156) To prove this, we shall use the figure page 112, taking CG for the axis of x'', CH for the axis of y'', CX for the axis of x', CY for the axis of y', and we shall suppose the momentary axis of rotation to be the axis of z', perpendicular to the plane of the

thus we shall have q' and r' in functions of the time t, when p' shall be determined; now the first of the equations (D) [234] gives*

$$d\,t=\frac{A\,B\,.\,d\,p'}{(A-B)\,.\,q'\,r'}\,; \qquad [265]$$

therefore

$$d\,t=\frac{A\,B\,C\,.\,d\,p'}{\sqrt{\{A\,C\,.\,k^2-H^2+A\,.\,(B-C)\,.\,p'^2\}\,.\,\{H^2-B\,C\,.\,k^2-B\,.\,(A-C)\,.\,p'^2\}}}\,; \qquad [266]$$

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figure $B\,A\,P$; and since $a=\sqrt{p^2+q^2+r^2}$, [260a], is the velocity of a particle revolving about the axis of z', at the distance 1, the velocity of the particle $d\,m$, whose distance from that axis is $\sqrt{x'^2+y'^2}$, will be $a\,.\sqrt{x'^2+y'^2}$; therefore the expression of the living force, corresponding to this particle will be $a^2\,.\,(x'^2+y'^2)\,.\,d\,m$. Its integral relative to the characteristic S, gives the whole living force,

$$a^2\,.\,S\,.\,(x'^2+y'^2)\,.\,d\,m=a^2\,.\,\{A\,.\,A_2^2+B\,.\,B_2^2+C\,.\,C_2^2\}, \qquad [246d],$$

in which A_2, B_2, C_2, represent, as in [172f, 227a], the cosines of the angles which the axis of z' makes with the axes of x'', y'', z'', respectively, so that we shall have, from [259, 260a],

$$A_2=\frac{q}{a}, \qquad B_2=\frac{r}{a}, \qquad C_2=\frac{p}{a},$$

consequently,

$$a^2\,.\,S\,.\,(x'^2+y'^2)\,.\,d\,m=a^2\,.\left(\frac{A\,q^2}{a^2}+\frac{B\,r^2}{a^2}+\frac{C\,p^2}{a^2}\right)=A\,q^2+B\,r^2+C\,p^2.$$

Substituting the values of p, q, r, [234f], it becomes,

$$\frac{q'^2}{A}+\frac{r'^2}{B}+\frac{p'^2}{C}=\frac{A\,B\,.\,p'^2+B\,C\,.\,q'^2+A\,C\,.\,r'^2}{A\,B\,C},$$

and by the principle of the living forces this ought to be a constant quantity as $\frac{H^2}{A\,B\,C}$; hence this becomes as in [263]. Lastly, from [262, 263], we obtain the values of q'^2, r'^2, by the usual rules of algebra.

* (156a) The equations [234], are reduced to the form [261a]. The first of these gives $d\,t$, [265], and this, by means of q', r', [264], becomes as in [266].

We may also find $d\,t$ in terms of q' or r', by using the second or third of the equations [261a]. By comparing these three equations together, it appears that they will not be altered if we change in the two series of letters, p', q', r', A, B, C, each letter into the following one of the series, beginning the series again when we come to the last terms r', C.

[266′] an equation which is integrable only in the three following cases, $B=A$, $B=C$, $A=C$.*

And the same takes place in the formulas [263—266], derived from [261*a*]; hence we get from [266] the two following expressions,

[266*a*]
$$dt=\frac{ABC.dq'}{\sqrt{\{BAk^2-H^2+B.(C-A).q'^2\}.\{H^2-CAk^2-C.(B-A).q'^2\}}}$$

[266*b*]
$$dt=\frac{ABC.dr'}{\sqrt{\{CBk^2-H^2+C.(A-B).r'^2\}.\{H^2-ABk^2-A.(C-B).r'^2\}}}.$$

These may be used when $A=B$, or $B=C$, to prevent the formulas [264, 265] becoming $\frac{0}{0}$, or indeterminate.

* (157) The general integral of the equation [266] depends on Le Gendre's elliptical functions [82*a*]. In the three cases mentioned by the author, [266′], this integration can be done by means of circular arcs and logarithms. There are also three other cases, not mentioned by him, in which this integration is possible, by the same method; namely, when $H^2=AC.k^2$, $H^2=AB.k^2$, or $H^2=BC.k^2$. We shall examine these cases separately. *First*, when $B=A$, the expression [266*a*], becomes,

$$dt=\frac{ABC.dq'}{\sqrt{\{BA.k^2-H^2+B.(C-A).q'^2\}}\,.\sqrt{H^2-CA.k^2}}.$$

This formula is used in preference to [266], because the denominators of [264, 265] vanish. Putting $\frac{ABC}{\sqrt{(H^2-CA.k^2).B.(C-A)}}=a$, and $\frac{BA.k^2-H^2}{B.(C-A)}=b^2$, it becomes $dt=\frac{a\,dq'}{\sqrt{b^2+q'^2}}$. If a and b be both real quantities, we shall have for its integral

$$t=a\,.\,\text{hyp. log.}\,(q'+\sqrt{b^2+q'^2}),$$

as is easily proved by differentiation and reduction. If the coefficient of q'^2, in [266*a*], should be negative, it might then depend on circular arcs, but it is not necessary to go into this investigation. *Second*, When $B=C$, or $A=C$, the coefficient of p'^2 in one of the factors of the denominator, [266], vanishes, and the equation may be put under the form $\frac{a\,dp'}{\sqrt{b^2-p'^2}}$, a and b being constant quantities, differing from the preceding. Its integral, when a is real, and bb positive, is $t=a\,.\,\text{arc.}\left(\sin.\frac{p'}{b}\right)+\text{constant}$. The other cases where bb is negative, or a imaginary, which may depend on logarithms, are easily computed. *Third*, When $H^2=AC.k^2$, or $H^2=BC.k^2$, the terms independent of p'^2, in one of the factors of the denominator, [266], vanishes, and the term p'^2 comes from under the radical,

The determination of the three quantities p', q', r', introduces three
arbitrary constant quantities, namely, H^2, k^2, and that introduced by the [266'']
integration of the preceding equation. But these quantities give only the
position of the momentary axis of rotation upon its surface, or relative to the
three principal axes, and its angular velocity of rotation. To obtain the real
motion of the body about the fixed point, we must also find the position of
the principal axes in space; which would introduce three new arbitrary
constant quantities, depending on the primitive position of these axes, and [266''']
this would require three new integrations, which, combined with the
preceding, would give the complete solution of the problem. The equations
(*C*) § 26 [231] contain three arbitrary constant quantities, N, N', N''; but
they are not wholly distinct from the arbitrary constant quantities H and k. [266^iv]
For, if we add together the squares of the first members of the equations
(*C*) [231], we shall have*

$$p'^2+q'^2+r'^2=N^2+N'^2+N''^2;$$ [267]

which gives $k^2=N^2+N'^2+N''^2$.

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and that equation may be put either in the form $d\,t=\frac{2ab.dp'}{p'\sqrt{b^2+p'^2}}$, whose integral is

$$t=a\,.\,\text{hyp. log.}\,\frac{\sqrt{b^2+p'^2}-b}{\sqrt{b^2+p'^2}+b};$$

or, in the form $d\,t=\frac{2ab.dp'}{p'\sqrt{b^2-p'^2}}$, whose integral is $t=\text{hyp. log.}\,\frac{b-\sqrt{b^2-p'^2}}{b+\sqrt{b^2-p'^2}}$, b being a real quantity. It was not thought necessary to notice the different cases arising from the negative values of b^2. When $H^2=AB\,.\,k^2$, the two factors of the denominator become divisible by $k^2-p'^2$, the one being

$$(AC-AB)\,.\,(k^2-p'^2),\quad\text{the other}\quad(AB-BC)\,.\,(k^2-p'^2),$$

and by putting $2fk=\frac{ABC}{\sqrt{(AC-AB)\,.\,(AB-BC)}}$, the value of dt becomes $d\,t=\frac{2fk.dp'}{k^2-p^2}$, whose integral is $t=f\,.\,\text{hyp. log.}\,\frac{k+p}{k-p}$.

* (158) The sum of the squares of N', N'', [232*b*], reduced by putting $\cos.^2\psi+\sin.^2\psi=1$, is

$$N'^2+N''^2=L^2+M^2,$$ [267*a*]

consequently, $N^2+N'^2+N''^2=L^2+M^2+N^2$. Also, the sum of the squares of

The constant quantities N, N', N'', correspond to c, c', c'' of § 21;* and [267′] the function $\frac{1}{2}t.\sqrt{p'^2+q'^2+r'^2}$ expresses the sum of the areas described during the time t, by the projection of each particle of the body upon the plane on which this sum is a maximum, multiplied respectively by the mass of each particle. N' and N'' are nothing relative to this plane; by putting therefore the values found in § 26 [231], equal to nothing, we shall have†

$M = L'.\cos.\theta + Cp.\sin.\theta$, $N = -L'.\sin.\theta + Cp.\cos.\theta$, [232*a*, *b*], reduced by putting $\cos.^2\theta + \sin.^2\theta = 1$, is $M^2 + N^2 = L'^2 + C^2p^2$, hence

$$N^2 + N'^2 + N''^2 = L^2 + L'^2 + C^2p^2.$$

Again, the sum of the squares of L, L', [232*a*), reduced by putting $\cos.^2\varphi + \sin.^2\varphi = 1$, is $L^2 + L'^2 = A^2q^2 + B^2r^2$, therefore, $N^2 + N'^2 + N''^2 = A^2q^2 + B^2r^2 + C^2p^2$, and this, by the substitution of the values [233], becomes $p'^2 + q'^2 + r'^2$, as in [267], and then from [262] we get,

[267*b*]

$$k^2 = p'^2 + q'^2 + r'^2 = N^2 + N'^2 + N''^2.$$

* (159) The forms of the expressions N, N', N'', [226], are precisely similar to those of c, c', c'', [167], and we may derive from the former, similar results to those derived from the latter, in [180, 181′], namely, that the plane corresponding to $N' = 0$, $N'' = 0$, will be that of the maximum of the areas mentioned in [267′]. Moreover the quantity

$$\tfrac{1}{2}.dt.\sqrt{c^2 + c'^2 + c''^2},$$

deduced from [181′], will become, for this case,

$$\tfrac{1}{2}.dt.\sqrt{N^2 + N'^2 + N''^2} = \tfrac{1}{2}.dt.\sqrt{p'^2 + q'^2 + r'^2}, \ [267].$$

This expression represents the sum of the areas described in the time dt, by each particle of the body, projected upon the plane of maximum areas, and multiplied by the particle. Its integral relative to the time t, is $\frac{1}{2}t.\sqrt{p'^2 + q'^2 + r'^2}$, observing that the terms under the [267*c*] radical are, in the case under consideration, equal to the constant quantity k, [267*b*]. It may be observed that the words, "multiplied respectively by the mass of each particle," were accidentally omitted in the original.

† (160) Put $N' = 0$, $N'' = 0$, in the two last equations [232*b*], and they will become, $L.\sin.\psi + M.\cos.\psi = 0$, $L.\cos.\psi - M.\sin.\psi = 0$. Multiply these by $\sin.\psi$, $\cos.\psi$, respectively; the sum of the products, reduced by putting $\sin.^2\psi + \cos.^2\psi = 1$, becomes $L = 0$; substituting this in either of the equations, we shall get, $M = 0$. Using the values [232*a*], these equations, $L = 0$, $M = 0$, become as in [268]. We might also have deduced these equations, by putting $N' = 0$, $N'' = 0$, in [267*a*], which becomes $L^2 + M^2 = 0$, and when L or M is not imaginary, this must give, $L = 0$, $M = 0$.

$$0 = Br\,.\sin.\varphi - Aq\,.\cos.\varphi\,;$$

$$0 = Aq\,.\cos.\theta\,.\sin.\varphi + Br\,.\cos.\theta\,.\cos.\varphi + Cp\,.\sin.\theta,{}^{*}$$ [268]

whence we deduce

$$\cos.\theta = \frac{p'}{\sqrt{p'^2 + q'^2 + r'^2}}\,;$$

$$\sin.\theta\,.\sin.\varphi = \frac{-q'}{\sqrt{p'^2 + q'^2 + r'^2}}\,;$$ [269]

$$\sin.\theta\,.\cos.\varphi = \frac{-r'}{\sqrt{p'^2 + q'^2 + r'^2}}.$$

By means of these equations, we shall know, in functions of the time, the values of θ and φ, referred to the fixed plane, which we have considered. It only remains to find the angle ψ, which the intersection of this plane and that of the two first principal axes, makes with the axis of x'; which requires another integration.

The values of q and r § 26 [230] give[†] [270]

$$d\psi\,.\sin.^2\theta = q\,dt\,.\sin.\theta\,.\sin.\varphi + r\,dt\,.\sin.\theta\,.\cos.\varphi\,;$$

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[*] (160a) Substitute the values [233], in [268], and they will become,

$$r'\,.\sin.\varphi - q'\,.\cos.\varphi = 0,$$ [268a]

$$(r'\,.\cos.\varphi + q'\,.\sin.\varphi)\,.\cos.\theta = -p'\,.\sin.\theta.$$ [268b]

To the square of [268b] add the square of [268a], multiplied by $\cos.^2\theta$, reducing the sum by putting $\sin.^2\varphi + \cos.^2\varphi = 1$, we get

$$(r'^2 + q'^2)\,.\cos.^2\theta = p'^2\,.\sin.^2\theta = p'^2\,.(1 - \cos.^2\theta),$$

or $(p'^2 + q'^2 + r'^2)\,.\cos.^2\theta = p'^2$, whence we get $\cos.\theta$, as in the first of the equations [269], also $p' = \cos.\theta\,.\sqrt{p'^2 + q'^2 + r'^2}$. This being substituted in [268b], we get from division by $\cos.\theta$,

$$r'\,.\cos.\varphi + q'\,.\sin.\varphi = -\sin.\theta\,.\sqrt{p'^2 + q'^2 + r'^2}.$$ [268c]

Multiply this by $-\sin.\varphi$, and [268a] by $\cos.\varphi$, add the products and reduce, by putting $\sin.^2\varphi + \cos.^2\varphi = 1$, we get $-q' = \sin.\theta\,.\sin.\varphi\,.\sqrt{p'^2 + q'^2 + r'^2}$, which is the same as the second of the equations [269]. Substitute this in [268a], and divide by $\sin.\varphi$, we find $-r' = \sin.\theta\,.\cos.\varphi\,.\sqrt{p'^2 + q'^2 + r'^2}$, which is the last of the equations [269].

[†] (161) Multiply the values of $q\,dt$, $r\,dt$, [230], by $\sin.\theta\,.\sin.\varphi$, $\sin.\theta\,.\cos.\varphi$, respectively, add the products and put $\cos.^2\varphi + \sin.^2\varphi = 1$, we shall obtain [270]. Substitute in this $q = \frac{q'}{A}$, $r = \frac{r'}{B}$, [234f], also the values of $\sin.\theta\,.\sin.\varphi$, $\sin.\theta\,.\cos.\varphi$,

whence we deduce

[271] $$d\psi=\frac{-k\,d\,t.(B\,q'^2+A\,r'^2)}{AB.(q'^2+r'^2)}.$$

Now we have by what precedes [262, 263].

[272] $$q'^2+r'^2=k^2-p'^2\,; \qquad B\,q'^2+A\,r'^2=\frac{H^2-AB.p'^2}{C}\,;$$

we shall therefore have

[273] $$d\psi=\frac{-k.d\,t.\{H^2-AB.p'^2\}}{ABC.(k^2-p'^2)}.$$

If we substitute instead of $d\,t$ its value found above [266], we shall have ψ in a function of p'; the three angles θ, φ, ψ, will therefore be determined in functions of the variable quantities p', q', r', which are themselves determined in functions of the time t. We shall know therefore, at any instant whatever, the values of these angles with respect to the plane of x', y', which we have considered, and it will be easy, by the formulas of spherical trigonometry, [273′] to deduce from them the values of the same angles referred to any other plane, which will introduce two more arbitrary quantities, which, combined with the four preceding,* will make the six arbitrary quantities which the complete solution of this problem requires. But it is evident that the consideration of the plane just mentioned simplifies the problem.

The position of the three principal axes upon the surface of the body being supposed known, if we know also, at any instant whatever, the position of the real axis of rotation upon that surface, and the angular velocity of [273″] rotation, we shall have, at that instant, the values of p, q, r, because these values, divided by the angular velocity of rotation, express the cosines of the

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[269], and for $\sin.^2\theta$, its value $\frac{q'^2+r'^2}{p'^2+q'^2+r'^2}$, deduced from that of $\cos.\theta$ [269], and it will become,

$$d\psi.\frac{q'^2+r'^2}{p'^2+q'^2+r'^2}=\frac{-B\,q'^2.dt-A\,r'^2.dt}{AB.\sqrt{p'^2+q'^2+r'^2}}.$$

The substitution of $k=\sqrt{p'^2+q'^2+r'^2}$, [267b], will give [271].

* (162) These four quantities are H, k, and the two constant quantities introduced by the integration of $d\,t$, $d\,\psi$, [266, 273].

angles which the real axis of rotation forms with the three principal axes;* we shall therefore have the values of p', q', r'; now these last values are proportional to the sines† of the angles which the three principal axes form with that plane of x', y', upon which the sum of the areas of the projections of the particles of the body, multiplied respectively by their masses, is a maximum; we can therefore determine, at every instant, the intersection [273'''] of the surface of the body by this invariable plane; therefore the position of this plane can be found by the actual conditions of the motion of the body.

Suppose that the rotatory motion of the body was caused by a force striking it in a direction not passing through its centre of gravity. It will follow from what we have demonstrated in § 20, 22, [155', 188'], that the centre of gravity will acquire the same motion, as if this force was applied [273$^{\text{iv}}$] directly to it, and that the body will have, about this centre, the same rotatory motion, as if the centre was immoveable. The sum of the areas described about this point, by the radius vector of each particle, projected upon a fixed plane, and multiplied respectively by these particles, will be proportional to the momentum of the impressed force projected on the same plane; now this momentum is the greatest when referred to the plane which passes through the direction of the force and the centre of gravity; this plane is therefore the invariable plane. If we put f for the distance of the [273$^{\text{v}}$]

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* (163) This angular velocity is $\sqrt{p^2+q^2+r^2}=a$, [260a], which is supposed to be given. The cosines of the angles abovementioned are $\frac{q}{a}$, $\frac{r}{a}$, $\frac{p}{a}$, [259b]. These being known, we may obtain from them the values of p, q, r, and thence by [233], the values of p', q', r'.

† (164) The sines of the angles which the three principal axes x'', y'', z'', make with the plane of x', y', are evidently the same as the cosines of the angles which the same axes x'', y'', z'', respectively make with the axis of z', and by [246e], these are respectively, $-\sin.\theta.\sin.\varphi$; $-\sin.\theta.\cos.\varphi$; and $\cos.\theta$; but in [269], it is shown that these quantities [273a] are represented by

$$\frac{q'}{\sqrt{p'^2+q'^2+r'^2}},\qquad \frac{r'}{\sqrt{p'^2+q'^2+r'^2}},\qquad \frac{p'}{\sqrt{p'^2+q'^2+r'^2}},$$

which are to each other in the same proportion as the quantities q', r', p', as above.

line of the primitive impulse from the centre of gravity; and v the velocity
[273$^{\rm vi}$] it impresses on this point; m being the mass of the body, mfv, will be the momentum of this force, and by multiplying it by $\frac{1}{2}t$, the product will be equal to the sum of the areas described during the time t;* but this sum, by what precedes, is $\frac{1}{2}t.\sqrt{p'^2+q'^2+r'^2}$ [267'], therefore we shall have

[274]
$$\sqrt{p'^2+q'^2+r'^2}=m.fv;$$

If at the commencement of the motion, we know the position of the principal axes with respect to the invariable plane, or the angles θ and φ; we shall have, at that time, the values of p', q', r',† consequently those of
[274'] p, q, r; we shall therefore have, at any instant whatever, the values of the same quantities.

Progressive and Rotatory Motions may be produced by one force.

This theory may serve to explain the double motion of rotation and revolution of the planets by a single primitive impressed force. Suppose the

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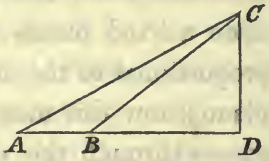
* (165) Let the force exerted, in giving motion to the body m, be represented by a mass m', moving with the velocity v', and therefore with the force $m'v'$, in the direction ABD, which line is perpendicular to the line $CD=f$, drawn from the centre of gravity C, of the body m. Then, by the nature of the centre of gravity [155'], this must cause the same progressive velocity v, in the centre of gravity C, of the body m, as if the whole force was applied at that centre, and as the whole force $m'v'$, is supposed to be exerted upon the body m, we shall have $m'v'=mv$, [273$^{\rm iv}$]. Again, if we put $AB=v't$, for the space passed over by the mass m', in the time t, before impact, the corresponding area described about the point C, would be the triangle $ABC=\frac{1}{2}v'tf$, and this multiplied by the mass m', produces $\frac{1}{2}m'v'.tf$, or $\frac{1}{2}mv.tf$, and by the principle of the preservation of areas, [167'''], this must remain the same after impact, or be equal to the quantity $\frac{1}{2}t.\sqrt{p'^2+q'^2+r'^2}$, [267'], corresponding to the plane of maximum areas, it being evident that this plane must be the same as that passing through the centre of gravity of the body m, and the line of direction of the primitive force [273$^{\rm v}$]. Putting therefore these two expressions equal to each other, and dividing by the common factor $\frac{1}{2}t$, we get [274].



† (166) The values p', q', r', may be deduced from the known values of θ, φ, m, f, v, by means of the formulas [269, 274], and from the formulas [233], we may find p, q, r. These values of p', q', r', at the beginning of the motion, which serve to determine the constant quantities of the integrals of [266, 266a, 266b].

planet to be a homogeneous sphere, whose radius is R, and that it revolves about the sun with an angular velocity U; r being supposed to express its [274″] distance from the sun, we shall have $v = r\,U$; moreover, if we suppose that the planet moves in consequence of a primitive impressed force, whose direction passes at the distance f from the centre of gravity, it is evident that the body will also revolve about an axis perpendicular to the invariable plane; considering this axis as the third principal axis,* we shall have $\theta = 0$; consequently $q' = 0$, $r' = 0$; we shall therefore have $p' = m\,f\,v$, or

$$C\,p = m\,f\,r\,U. \qquad [274'''']$$

But in the sphere, we have† $C = \tfrac{2}{5}\,.\,m\,R^2$; therefore

$$f = \frac{2}{5}\,.\,\frac{R^2}{r}\,.\,\frac{p}{U}; \qquad [275]$$

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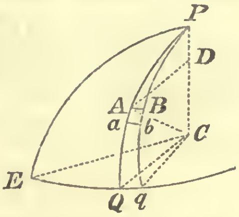
* (167) This supposition may be made, because all the axes of a sphere are principal axes, [254′]. In this case the axis of revolution z'', being perpendicular to the invariable plane x', y', the inclination of this plane to that of x'', y'', or θ, [226‴], must be equal to nothing. This being substituted in the two last of the equations [269], gives $q' = 0$, $r' = 0$; hence $p' = m\,f\,v$, [274], and by substituting $p' = C\,p$, [233], and $v = r\,U$, [274″] we shall get $C\,p = m\,f\,r\,U$, [274‴].

† (168) Let C be the centre of the sphere, $C\,P$ the axis of z'', $C\,E\,q$ the plane of $x''\ y''$, $P\,E\,Q\,q\,P$ a spherical surface described about the centre C with the radius $C\,P = C\,E = R$, $P\,E$, $P\,Q$, $P\,q$, quadrantal arches drawn through P, perpendicular to $E\,Q$, the two last being infinitely near to each other; $A\,B$, $a\,b$, arcs of circles parallel to $Q\,q$, forming the infinitely small parallelogram $A\,B\,a\,b$, whose area is $A\,B \times A\,a$, and if on this base we erect a parallelopiped whose height, taken in the direction of $C\,A$, is $d\,R$, its solidity will be equal to the element $d\,m$ of the mass of the sphere, hence $d\,m = A\,B \times A\,a \times d\,R$. Put the angles $E\,C\,Q = s$, $A\,C\,P = p'$, then the arc $E\,Q = R\,.\,s$, $Q\,q = R\,.\,d\,s$, $A\,B = Q\,q\,.\,\sin.\,A\,C\,P = R\,.\,d\,s\,.\,\sin.\,p'$, $A\,a = R\,.\,d\,p'$. Substituting these we get $d\,m = R^2\,d\,R\,.\,d\,s\,.\,d\,p'\,.\,\sin.\,p'$, whence $m = \int\!\!\int\!\!\int R^2\,d\,R\,.\,d\,s\,.\,d\,p'\,.\,\sin.\,p'$. The [275a] integral relative to s being taken from $s = 0$, to s equal to the whole circumference $2\,\pi$, gives $m = 2\,\pi\,.\int\!\!\int R^2\,d\,R\,.\,d\,p'\,\sin.\,p'$. Again, $\int d\,p'\,.\,\sin.\,p'$, taken from $p' = 0$, to $p' = \pi$, [275b] is $\int d\,p'\,.\,\sin.\,p' = 1 - \cos.\,p' = 2$, hence $m = 4\,\pi\,.\int R^2\,d\,R$, this, integrated from R equal 0, to $R = R$, gives $m = \frac{4\,\pi}{3}\,.\,R^3$, which is the well known theorem for the solidity



which gives the distance f of the direction of the impressed force from the centre of the planet, corresponding to the observed ratio between the angular velocity of rotation p, and the angular velocity of revolution about the sun U. As it respects the earth, we have $\frac{p}{U} = 366{,}25638$; the parallax of the sun*
[275′] gives $\frac{R}{r} = 0{,}000042665$, consequently $f = \frac{1}{160} \cdot R$ very nearly.

The planets not being homogeneous, we may consider them as being formed of spherical and concentric strata of different densities. Let ρ be the density of one of these strata, whose radius is R, ρ being a function of R; we shall have†

[276] $$C = \frac{2m}{3} \cdot \frac{\int \rho \,.\, R^4 \,.\, dR}{\int \rho \,.\, R^2 \,.\, dR},$$

m being the whole mass of the planet, and the integrals being taken from $R = 0$, to its value at the surface; we shall therefore have

[277] $$f = \frac{2}{3} \cdot \frac{p}{ru} \cdot \frac{\int \rho \,.\, R^4 \,.\, dR}{\int \rho \,.\, R^2 \,.\, dR}.$$

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of a sphere. To obtain the momentum of inertia C, [229], about the axis CP, we must multiply the particle dm, by the square of its distance AD from the axis CP, [245‴], and take the integrals as before, and as $AD = R \,.\, \sin.\, p'$,

[275c] $C = \int\int\int dm \,.\, AD^2 = \int\int\int R^4\, dR \,.\, ds \,.\, dp' \,.\, \sin.^3 p' = 2\pi \,.\, \int\int R^4\, dR \,.\, dp' \,.\, \sin.^3 p'$. Now by [84a], $\int dp' \,.\, \sin.^3 p' = -\frac{1}{3} \,.\, \cos. p' \,.\, \sin.^2 p' + \frac{2}{3} \,.\, \int dp' \,.\, \sin. p'$, consequently $\int dp' \,.\, \sin.^3 p' = -\frac{1}{3} \,.\, \cos. p' \,.\, \sin.^2 p' - \frac{2}{3} \,.\, \cos. p' + \frac{2}{3}$, the integral being supposed to commence with $p' = 0$, this when $p' = \pi$, becomes $\frac{4}{5}$, consequently

[275d] $$C = \frac{8\pi}{3} \,.\, \int R^4\, dR = \frac{8\pi}{15} \,.\, R^5,$$

and as $m = \frac{4\pi}{3} \,.\, R^3$, this becomes $C = \frac{2}{5} \,.\, mR^2$.

* (169) The parallax here used is $8^s{,}8$, whose natural sine is nearly equal to 0,00004266.

† (170) The elements of the integrals of m, C, found in the note [168], are to be multiplied by ρ, and the integrations being made relative to s, p', which are independent of R,
[276a] we shall get as in [275b, d], $m = 4\pi \,.\, \int \rho \,.\, R^2 \,.\, dR$, $C = \frac{8\pi}{3} \,.\, \int \rho \,.\, R^4 \,.\, dR$. Dividing this last by the former, and multiplying the quotient by m, gives C, [276]. Substitute this in [274‴], and it will give f, [277].

If the strata nearest the centre are the most dense, as it is natural to suppose is the case, the function $\frac{\int \rho . R^4 . dR}{\int \rho . R^2 . dR}$, will be less than $\frac{3}{5} R^2$,* the value of f will therefore be less than in the case of homogeneity. [277']

30. We shall now compute the oscillations of the body, in the case where it turns very nearly about the third principal axis. We may deduce them from the integrals we have obtained in the preceding article; but it is more simple to deduce them directly from the equations (*D*) § 26 [234]. The body not being impelled by any forces, these equations become, by substituting, for p', q', r', their values $C p$, $A q$, $B r$,† [277'']

Oscillation of a solid body.

$$dp + \frac{(B-A)}{C} . qr . dt = 0;$$
$$dq + \frac{(C-B)}{A} . rp . dt = 0; \qquad [278]$$
$$dr + \frac{(A-C)}{B} . pq . dt = 0.$$

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* (171) When $\rho = 1$, $\int \rho . R^4 . dR = \frac{1}{5} R^5$, and $\int \rho . R^2 . dR = \frac{1}{3} R^3$,

$$\frac{\int \rho . R^4 dR}{\int \rho . R^2 dR} = \frac{3}{5} R^2. \qquad [278a]$$

Suppose now the whole mass of the planet to remain unchanged, but to become denser towards the centre, by the removal of some of the particles from the outer parts towards the centre. The differentials of m, C, [276a], being

$$dm = 4\pi . \rho R^2 . dR, \qquad dC = \frac{8\pi}{3} . \rho R^4 . dR, \qquad [278b]$$

we shall have $dC = \frac{2}{3} R^2 . dm$, and $C = \frac{2}{3} \int R^2 . dm$. Now if any particle dm is carried from the surface towards the centre, to increase the density of the parts near the centre, the radius corresponding to this particle must be decreased, consequently, $R^2 dm$, and $\int R^2 . dm$, or C, must be decreased, therefore the quantity $\frac{\int \rho . R^4 dR}{\int \rho . R^2 dR} = \frac{3C}{2m}$, [276$a$], must be decreased, m being constant, and it must be less than $\frac{3}{5} R^2$, found above.

† (172) Put in [234], $dN = 0$, $dN' = 0$, $dN'' = 0$, [260b]. Substitute the values p', q', r', [233], divide the results by C, A, B, respectively, and we shall obtain [278].

The solid being supposed to turn very nearly about its third principal axis; [278′] q and r will be very small quantities,* whose squares and products may be neglected; which gives $dp=0$, consequently p is constant. If in the two other equations we suppose

[279] $$q=M.\sin.(nt+\gamma)\,; \qquad\qquad r=M'.\cos.(nt+\gamma)\,;$$

we shall have†

[280] $$n=p.\sqrt{\frac{(C-A).(C-B)}{AB}}\,; \qquad\qquad M'=-M.\sqrt{\frac{A.(C-A)}{B.(C-B)}}\,;$$

M and γ being two arbitrary constant quantities. The angular velocity of [280′] rotation will be $\sqrt{p^2+q^2+r^2}$ [260″], or simply p, by neglecting the squares of q and r; this velocity will therefore be nearly constant. Lastly, the sine

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* (173) In this case, the angles formed by the axis of rotation and the axes of x'', or y'', must be nearly equal to a right angle, consequently their cosines must be very small; now by [259] these cosines are represented by $\frac{q}{\sqrt{p^2+q^2+r^2}}$, $\frac{r}{\sqrt{p^2+q^2+r^2}}$, consequently q, r, must be much smaller than p. If we therefore neglect the product qr, in the first of the proposed equations, [278], it will become $dp=0$, whose integral is $p=\text{constant}$.

† (174) If we substitute the assumed values q, r, [279], in the two last of the equations [278], supposing p to be constant, we shall get, by rejecting the factors $\cos.(nt+\gamma)$, $\sin.(nt+\gamma)$,

$$Mn+\frac{(C-B)}{A}.M'p=0, \qquad\qquad -M'n+\frac{(A-C)}{B}.Mp=0,$$

whence we may easily deduce the values n, M', [280], which will evidently give for q, r, [279], values satisfying the proposed equations, and containing two arbitrary constant quantities. This solution is only a particular case of a much more general form given in [1089]. We might also have obtained this solution, by taking the differential of the second equation [278], substituting in it the value dr, deduced from the third of the equations [278], and using for brevity the value of n, [280], by which means it would become

$$\frac{ddq}{dt^2}+n^2q=0,$$

which is of the same form as [865′], whose solution, [864a], is as in [279]. This value of q would give r, by means of the second equation [278].

of the angle formed by the real axis of rotation and the third principal axis, will be* $\frac{\sqrt{q^2+r^2}}{p}$. [280'']

Remarkable Property of the Principal Axes of Rotation.

If, at the commencement of the motion, we have $q=0$, and $r=0$, that is, if the real axis of rotation coincides, at that instant, with the third principal axis, we shall have $M=0$, $M'=0$; q and r will therefore always be nothing; and the axis of rotation will always coincide with the third principal axis;† whence it follows, that *if the body begin to turn about one of the principal axes, it will continue to revolve uniformly about that axis.* This [280''']

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* (175) By [259c] this sine is $\frac{\sqrt{q^2+r^2}}{\sqrt{p^2+q^2+r^2}}$, or nearly $\frac{\sqrt{q^2+r^2}}{p}$.

† (176) The radical expressions which occur in [280] being supposed free from imaginary quantities, if $q=0$, and $r=0$, when $t=0$, the expressions [279] will become

$$0=M.\sin.\gamma, \qquad 0=M'.\cos.\gamma=-M.\sqrt{\frac{A.(C-A)}{B.(C-B)}}.\cos.\gamma,$$

which cannot generally be satisfied for all values of A, B, C, except by putting $M=0$, and $M'=0$. These being substituted in the general values of q, r, [279], they become nothing, as in [280''']. The case where n becomes imaginary is treated of in [281$^{\text{v}}$, &c.].

In this demonstration the values of q, r, are supposed absolutely correct, but as they are only approximate values, obtained by supposing p constant, it may not be amiss to give a more accurate demonstration. To do this, we may multiply the two last of the equations [278], respectively, by $2A.(A-C).q$, and $2B.(B-C).r$, and take their sum,

$$A.(A-C).2q\,dq+B.(B-C).2r\,dr=0,$$

whose integral is $A.(A-C).q^2+B.(B-C).r^2=0$, the constant quantity being 0, because at the beginning of the motion $q=0$, and $r=0$. The same equation would also result by the extermination of p'^2, from the equations [262, 263], using the values [233]. Now if C be greater than A and B, the quantities $A-C$, $B-C$, will both be negative, and if q, r, are real, q^2, r^2, will be positive, and the preceding equation,

$$A.(A-C).q^2+B.(B-C).r^2=0,$$

cannot be satisfied except by supposing the general values of q, r, to be $q=0$, $r=0$. In like manner, if C is less than A and B, the quantities $A-C$, $B-C$, will be positive, and the preceding equation cannot be satisfied except by putting $q=0$, $r=0$. Lastly, when C falls between A and B, the quantities $A-C$, $B-C$, will have different signs, and then it will not necessarily follow that we must put $q=0$, $r=0$, to satisfy that equation,

remarkable property of the principal axes, has caused them to be named
[280^{iv}] *principal axes of rotation*: it appertains exclusively to them; for if the real axis of rotation is invariable at the surface of the body, we shall have*
[280^{v}] $dp=0$, $dq=0$, $dr=0$; the preceding values of these quantities then become,

[281] $$\frac{(B-A)}{C}.rq=0; \qquad \frac{(C-B)}{A}.rp=0; \qquad \frac{(A-C)}{B}.pq=0.$$

In the general case where A, B, C, are unequal, two of the three quantities
[$281'$] p, q, r, are nothing in consequence of these equations, which requires that the real axis of rotation should coincide with one of the principal axes.†

If two of the three quantities A, B, C, are equal, for example if $A=B$,
[$281''$] the three preceding equations are reduced to these, $rp=0$, $pq=0$; which may be satisfied by supposing only $p=0$. The axis of rotation is then in a plane perpendicular to the third principal axis;‡ but we have seen, in

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because it would be satisfied by putting $r=q.\sqrt{\frac{A.(A-C)}{B.(C-B)}}$, in which the radical $\sqrt{\frac{A.(A-C)}{B.(C-B)}}$, is a real quantity; this corresponds to the case of *unstable* equilibrium, mentioned in [281^{vii}]. These results agree with those found above, upon the supposition that p is constant.

[*] (177) When the real axis of rotation is *invariable*, the angular velocity of rotation,
[$281a$] which is $\sqrt{p^2+q^2+r^2}=a$, [$260a$], must be constant, by the principle of the preservation of the areas; and the cosines of the angles, which this axis makes with the three principal
[$281b$] axes must also be constant. These cosines are represented by $\frac{q}{a}$, $\frac{r}{a}$, $\frac{p}{a}$, [259], and since they are constant, their differentials must be put $=0$, hence $dp=0$, $dq=0$, $dr=0$. Substituting these in [278], we shall get [281].

[†] (177a] Thus if p, r, were nothing, the expressions of the cosine of the angle, formed by the axis of x'', and the momentary axis of rotation, $\frac{q}{\sqrt{p^2+q^2+r^2}}$, [259], would become 1; therefore these axes would coincide.

[‡] (178) Because by [259], the cosine of the angle, formed by the axis of z'', and the axis of rotation is $\frac{p}{\sqrt{p^2+q^2+r^2}}$, which being equal to nothing, that angle is a right angle.

§ 27 [249'], that all the axes situated in this plane are then principal axes.

[281'''] Lastly, if we have at the same time $A=B=C$, the three preceding equations will be satisfied, whatever be p, q, r; but then by § 27 [254'], all the axes of the body are principal axes.

[281^{iv}] Hence it follows, that the principal axes alone have the property of being invariable axes of rotation; but they do not all possess this property in the same manner. The rotatory motion about that axis whose momentum of inertia falls between the two others, may be sensibly troubled by the slightest cause; so that there is no stability in this motion.

[281^{v}] A system of bodies is said to be in a *stable state of equilibrium*, when an
infinitely small derangement of the system can produce only an infinitely Stable Equilibrium.
small change in the positions of the bodies, by making continual oscillations
about the situation of equilibrium. This being premised, suppose that the
real axis of rotation is infinitely near to the third principal axis; in this case
the constant quantities M and M' [279] are infinitely small; and if n is a
real quantity, the values of q and r will always remain infinitely small, and [281^{vi}]
the real axis of rotation will not deviate from the third principal axis but by
quantities of the same order. But if n be imaginary, sin. $(nt+\gamma)$, and
cos. $(nt+\gamma)$, will then become exponential quantities;* and the values of q [281^{vii}]
and r might then increase indefinitely, and at length cease to be infinitely Unstable Equilibrium.
small; there will then be no stability in the rotatory motion of the body

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* (179) If in

$$\sin.\,(nt+\gamma)=\frac{c^{(nt+\gamma).\sqrt{-1}}-c^{-(nt+\gamma).\sqrt{-1}}}{2\sqrt{-1}},$$

(Form. 11 Int.) we put $\frac{c^{\gamma.\sqrt{-1}}}{2\sqrt{-1}}=a$, or $-\frac{c^{-\gamma.\sqrt{-1}}}{2\sqrt{-1}}=\frac{1}{4a}$, it will become

$$\sin.\,(nt+\gamma)=a.c^{nt.\sqrt{-1}}+\frac{1}{4a}.c^{-nt.\sqrt{-1}}.$$

Now if n be imaginary, and equal to $n'.\sqrt{-1}$, n' being a real quantity, the preceding expression will become $\sin.\,(nt+\gamma)=a.c^{-n't}+\frac{1}{4a}.c^{n't}$, and as the exponent of $c^{n't}$ increases with the time, this quantity may become indefinitely great. The same takes place with cos. $(nt+\gamma)$, as is easily proved in the same manner, using (Form. 12 Int.).

about the third principal axis. The value of n is real, if C is the greatest,
[281$^{\text{viii}}$] or the least, of the three quantities A, B, C; for then the product
$(C-A).(C-B)$ is positive; but this product is negative, if C falls
between A and B; and in this case, n [280] is imaginary; therefore the
[281$^{\text{ix}}$] rotatory motion is stable about the two principal axes whose momenta of
inertia are the greatest and the least; and it is unstable about the other
principal axis.

Now to determine the position of the principal axes in space, we shall
suppose the third principal axis to be nearly perpendicular to the plane of
[281$^{\text{x}}$] x', y', so that θ may be a very small quantity, whose square can be neglected.
We shall have, by § 26,*

[282] $$d\varphi - d\psi = p\,dt;$$

which gives by integration

[283] $$\psi = \varphi - pt - \varepsilon,$$

ε being an arbitrary constant quantity. If we then put

[284] $$\sin.\theta\,.\,\sin.\varphi = s; \qquad\qquad \sin.\theta\,.\,\cos.\varphi = u;$$

the values of q and r of § 26,† will give, by exterminating $d\psi$,

[285] $$\frac{ds}{dt} - pu = r; \qquad\qquad \frac{du}{dt} + ps = -q;$$

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* (180) In the first equation of [230], we may substitute 1 for $\cos.\theta$, neglecting the square and higher powers of θ, (44 Int.) and it becomes as in [282].

† (181) Take the differentials of s and u, [284], substitute 1 for $\cos.\theta$, u for $\sin.\theta.\cos.\varphi$, s for $\sin.\theta\,.\,\sin.\varphi$, and divide by dt, we shall obtain,

$$\frac{ds}{dt} = \frac{d\theta}{dt}\,.\,\sin.\varphi + u\,.\,\frac{d\varphi}{dt}; \qquad\qquad \frac{du}{dt} = \frac{d\theta}{dt}\,.\,\cos.\varphi - s\,.\,\frac{d\varphi}{dt};$$

substract pu from the first, add ps to the second, and put $\frac{d\varphi}{dt} - p = \frac{d\psi}{dt}$, [282], they will become $\frac{ds}{dt} - pu = \frac{d\theta}{dt}\,.\,\sin.\varphi + u\,.\,\frac{d\psi}{dt}$; $\frac{du}{dt} + ps = \frac{d\theta}{dt}\,.\,\cos.\varphi - s\,.\,\frac{d\psi}{dt}$; the second members of which are equal to the values of r and $-q$, deduced from $q\,dt$, $r\,dt$, [230]; substituting these values in the preceding equations, they become as in [285].

whence by integration*

$$s = \beta . \sin. (pt + \lambda) - \frac{AM}{Cp} . \sin. (nt + \gamma) ;$$
$$u = \beta . \cos. (pt + \lambda) - \frac{BM'}{Cp} . \cos. (nt + \gamma) ;$$ [286]

β and λ being two other constant quantities: the problem is thus completely resolved, since the values of s and u give the angles θ and φ in functions of the time,† and ψ is determined in a function of φ and t. If β is nothing, [286]

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* (182) If we substitute in [285], the values of r, q, [279], these equations will be satisfied by the assumed values of s, u, [286], as may be easily proved by substituting in the coefficients of $\cos. (nt + \gamma)$, and $\sin. (nt + \gamma)$, the values of n, M', [280], which renders these coefficients nothing. We may also find the equations [286], by a direct method by means of formula [865], in the following manner.

Take the differential of the first of the equations [285], supposing, as above, p to be constant. Substitute in this, the value of $\frac{du}{dt}$, deduced from the second of these equations, and we shall get $\frac{dds}{dt^2} + p^2 s - \frac{dr}{dt} + pq = 0$. But the values q, r, [279], give

$$-\frac{dr}{dt} + pq = (M'n + Mp) . \sin. (nt + \gamma),$$

and from [280], we get $M'n = -Mp . \frac{(C-A)}{B}$, hence

$$M'n + Mp = \frac{A+B-C}{B} . Mp,$$

which being put equal to αK, the preceding equation will become

$$\frac{dds}{dt^2} + p^2 s + \alpha K . \sin. (nt + \gamma) = 0.$$

which is of the same form as [865], whose solution is given in [865b, 870', 871], changing $y, a, b, m, \varepsilon, \varphi$, into $s, p, \beta, n, \gamma, \lambda$, respectively, so that from [865b, 871], we get,

$$s = \beta . \sin. (pt + \lambda) + \frac{\alpha K}{n^2 - p^2} . \sin. (nt + \gamma).$$

But from the value of n [280], we get $n^2 - p^2 = -p^2 . \frac{C.(A+B-C)}{AB}$, hence $\frac{\alpha K}{n^2 - p^2} = -\frac{AM}{Cp}$, and the preceding value of s becomes as in [286]. Substitute this in $u = \frac{1}{p} . \left\{ \frac{ds}{dt} - r \right\}$, [285], using r [279], and reducing as above, it becomes as in [286].

† (183) Having s and u, [286], we obtain θ, φ, from the equations [284], and then ψ, from [283].

the plane of x', y', becomes the invariable plane, to which we have referred the angles θ, φ and ψ in the preceding article.*

[286''] 31. If the solid body be free, the analysis of the preceding articles will give the motion about its centre of gravity: if the body be forced to move about a fixed point, it will give its motion about that point. It now remains to consider the motion of a body which is forced to move about a fixed axis.

Compound Pendulum.

Let x' be this fixed axis, which we shall suppose to be horizontal. In this case, the last of the equations (B) § 25 [226] will be sufficient to determine the motion of the body. Suppose also the axis of y' to be horizontal, and the axis of z' vertical, and directed towards the centre of the earth. Lastly, suppose the plane which passes through the axes of y', z', passes also through the centre of gravity of the body, and that an axis is drawn from this centre of gravity to the origin of the co-ordinates. Let θ be the angle that this last axis makes with the axis of z'; and if we put y'', z'', for the co-ordinates of any particle referred to this new axis, we shall have† [286''']

$$y'=y''.\cos.\theta+z''.\sin.\theta\,; \qquad z'=z''.\cos.\theta-y''.\sin.\theta\,; \qquad [287]$$

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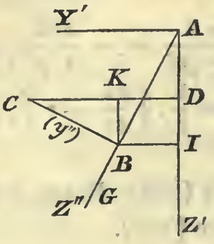
* (184) When $\beta=0$, the general values of s, u, [286], become

$$s=-\frac{A}{Cp}.M.\sin.(nt+\gamma), \qquad u=-\frac{B}{Cp}.M'.\cos.(nt+\gamma),$$

or by substituting the values q, r, [279], $s=\frac{-Aq}{Cp}$, $u=\frac{-Br}{Cp}$; and by using the values s, u, [284], p', q', r', [233], they become $\sin.\theta.\sin.\varphi=\frac{-q'}{p'}$, $\sin.\theta.\cos.\varphi=\frac{-r'}{p'}$, which agree with the values of $\sin.\theta.\sin.\varphi$; $\sin.\theta.\cos.\varphi$, given in [269], for the invariable plane, neglecting the small quantities q'^2, r'^2, in comparison with p'^2.

† (185) In the adjoined figure, let A be the origin of the cordinates, G the centre of gravity of the body, ADZ' the axis of z', AY that of y'; ABG the axis of z'', and BC parallel to that of y''. Then the co-ordinates of a particle C are either $AD=z'$, $DC=y'$, or $AB=z''$, $BC=y''$. Through B draw BK parallel to z', and BI parallel to y'. Then we have



$$BCK=BAI=\theta,$$

whence we deduce

$$S.\left(\frac{y'\,dz'-z'\,dy'}{dt}\right).\,dm=-\frac{d\theta}{dt}.\,S.\,dm.\,(y''^2+z''^2).$$ [288]

$S.\,dm.\,(y''^2+z''^2)$ is the momentum of inertia of the body relative to the axis of x': let this momentum be C. The last of the equations (B) § 25 [226] will give* [288']

$$-C\left(\frac{dd\theta}{dt^2}\right)=\frac{dN''}{dt}.$$ [289]

Suppose that the body is acted upon by gravity only; the values of P, Q, of § 25 will be nothing, and R will be constant, which gives†

$$\frac{dN''}{dt}=S.\,Ry'.\,dm=R.\cos.\theta.\,S.\,y''.\,dm+R.\sin.\theta.\,S.\,z''.\,dm.$$ [290]

The axis of z'' passing through the centre of gravity of the body, makes $S.\,y''.\,dm=0$; and if we put h for the distance of the centre of gravity

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$AI=AB.\cos.BAI=z''.\cos.\theta$; $BI(=KD)=AB.\sin.BAI=z''.\sin.\theta$;
$CK=BC.\cos.BCK=y''.\cos.\theta$; $BK(=DI)=BC.\sin.BCK=y''.\sin.\theta$.

Substitute these in $y'=DC=CK+KD$, $z'=AD=AI-DI$, they become as in [287], whose differentials, considering θ only as variable, are

$dy'=d\theta.(z''.\cos.\theta-y''.\sin.\theta)=z'd\theta$, and $dz'=-d\theta.(y''.\cos.\theta+z''.\sin.\theta)=-y'.d\theta$,

hence $y'dz'-z'dy'=-d\theta.(z'^2+y'^2)=-d\theta.(z''^2+y''^2)$. This multiplied by $\frac{dm}{dt}$, and integrated relative to S, gives the value of $S.\left(\frac{y'dz'-z'dy'}{dt}\right)$, as in [288].

* (186) By taking its differential and dividing by dt, having first substituted the value of $S.\left(\frac{y'dz'-z'dy'}{dt}\right).\,dm$, of [288], and that of C [288'].

† (187) Take the differential of the third of the equations [225], and divide it by dt, putting $Q=0$, we obtain $\frac{dN''}{dt}=S.\,Ry'.\,dm$, [290]. Substitute in this the value of y', [287], and bring the terms R, θ, from under the sign S, because they are the same for all the particles, we get the second expression of $\frac{dN''}{dt}$, [290]. The values $S.\,y''.\,dm=0$, and $S.\,z''.\,dm=mh$, are easily deduced from [127], hence we get the value of $\frac{dN''}{dt}$, [291], which, being substituted in [289], gives [292].

of the body from the axis of x', we shall have $S\,.\,z''\,.\,dm = mh$, m being the whole mass of the body; therefore we shall have

[291] $$\frac{dN''}{dt} = mh\,.\,R\,.\,\sin.\theta\,;$$

consequently

[292] $$\frac{dd\theta}{dt^2} = \frac{-mh\,.\,R\,.\,\sin.\theta}{C}.$$

Let us now consider a second body, all whose parts are united at a single point, at the distance l from the axis of x'; we shall have, with respect to this body, $C = m'l^2$, m' being its mass; moreover, h being equal to l, we shall have* [292']

[293] $$\frac{dd\theta}{dt^2} = -\frac{R}{l}\,.\,\sin.\theta.$$

These two bodies will therefore have exactly the same motion of oscillation, if their initial angular velocities, when their centres of gravity are in the vertical, are equal, and also†

[293'] $$l = \frac{C}{mh}.$$

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* (188) This is found by substituting $h = l$, and $C = m'l^2$, in the general expression of $\frac{dd\theta}{dt^2}$ [292], changing also m into m'.

† (189) Substitute $l = \frac{C}{mh}$, in the equation [293], corresponding to the simple pendulum, and it becomes identical with the expression of $\frac{dd\theta}{dt^2}$, [292], corresponding to the compound pendulum. Multiply this by $2\,d\theta$, and put for brevity $\beta = -\frac{2m.hR}{C}$, it becomes $\frac{2d\theta.dd\theta}{dt^2} = -\beta\,d\theta\,.\sin.\theta$, whose integral is $\frac{d\theta^2}{dt^2} = \alpha + \beta\,.\cos.\theta$; α being an arbitrary constant quantity, which may be determined by means of the angular velocity $\frac{d\theta}{dt}$, when $\theta = 0$, and if this quantity be the same in both pendulums, the angular velocity $\frac{d\theta}{dt}$ will be the same in all situations. Lastly, the value of dt being found from the preceding equation, it will give, by integration $t = \int \frac{d\theta}{\sqrt{\alpha + \beta.\cos.\theta}}$. [293a] [293b]

The second body, we have just mentioned, is a simple pendulum, whose oscillations were computed in § 11 [84, 86]; we can therefore always compute, by this formula, the length l of a simple pendulum, whose vibrations are isochronous with those of the solid just considered, and which may be considered as a compound pendulum. In this manner the length of a simple pendulum vibrating in a second, has been ascertained by observations made with compound pendulums.

CHAPTER VIII.

ON THE MOTION OF FLUIDS.

32. We shall make the laws of motion of fluids depend on those of their equilibrium; in the same manner as we have deduced, in Chapter V, the laws of motion of a system of bodies from those of its equilibrium. Let us therefore resume the general equation of the equilibrium of fluids, given in § 17 [133].

[294] $$\delta p = \rho . \{ P . \delta x + Q . \delta y + R . \delta z \} ;$$

the *characteristic* δ *refers only to the co-ordinates* $x, y, z,$ *of the particle,* [294′] *and does not affect the time t.** When the fluid is in motion, the forces which would retain the particles in equilibrium, are, by § 18 [141, 142], supposing dt constant,

[295] $$P - \left(\frac{ddx}{dt^2}\right) ; \qquad Q - \left(\frac{ddy}{dt^2}\right) ; \qquad R - \left(\frac{ddz}{dt^2}\right) ;$$

we must therefore substitute these forces,† instead of $P, Q, R,$ in the preceding equation of equilibrium. Putting

[295′] $$\delta V = P . \delta x + Q . \delta y + R . \delta z,$$

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* (190) As in [36″], where $\delta x, \delta y, \delta z,$ are arbitrary variations of $x, y, z,$ independent of the time t.

† (190*a*) These forces, as in note 60*a*, are supposed to tend to increase the co-ordinates. [295*a*] Moreover, the quantities $\frac{ddx}{dt^2}, \frac{ddy}{dt^2}, \frac{ddz}{dt^2},$ being partial differentials relative to t, are included in parentheses in [295], the reasons for which are more fully stated in the note 197.

which we shall suppose to be an exact variation, we shall have

$$\delta V - \frac{\delta p}{\rho} = \delta x \,.\left(\frac{ddx}{dt^2}\right) + \delta y \,.\left(\frac{ddy}{dt^2}\right) + \delta z \,.\left(\frac{ddz}{dt^2}\right); \qquad (F)$$ [296]

First important Equation of the Motion of a Fluid.

this equation is equivalent to three distinct equations; for the variations δx, δy, δz, being independent, we may put their coefficients separately equal to nothing. [296']

The co-ordinates x, y, z, are functions of the primitive co-ordinates and of the time t; let a, b, c, be these primitive co-ordinates, we shall have* [296'']

$$\begin{aligned} \delta x &= \left(\frac{dx}{da}\right).\delta a + \left(\frac{dx}{db}\right).\delta b + \left(\frac{dx}{dc}\right).\delta c; \\ \delta y &= \left(\frac{dy}{da}\right).\delta a + \left(\frac{dy}{db}\right).\delta b + \left(\frac{dy}{dc}\right).\delta c; \\ \delta z &= \left(\frac{dz}{da}\right).\delta a + \left(\frac{dz}{db}\right).\delta b + \left(\frac{dz}{dc}\right).\delta c. \end{aligned}$$ [297]

Substituting these values in the equation (F) [296], we may put the coefficients of δa, δb, δc, separately equal to nothing; which will give three equations, of partial differentials between the three co-ordinates x, y, z, of the particle, its primitive co-ordinates a, b, c, and the time t.

It now remains to fulfil the conditions arising from the continuity of the fluid. For that purpose, we shall consider, at the commencement of the motion, a rectangular fluid parallelopiped, whose three dimensions are $\mathrm{d}a$, $\mathrm{d}b$, $\mathrm{d}c$. Denoting by (ρ) the primitive density of this particle, its mass will be $(\rho).\mathrm{d}a.\mathrm{d}b.\mathrm{d}c$. We shall call this parallelopiped (A); it is easy to perceive, that at the end of the time t, it will become an oblique parallelopiped; for all the particles situated at first on any face of the parallelopiped (A), will continue in the same plane, neglecting infinitely small quantities of the second order;† all the particles situated on the parallel edges of (A), will [297']

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* (190b) The co-ordinates x, y, z, being functions of a, b, c, t, their complete variations relative to δ are as in [297], observing that the characteristic δ does not affect t, [294'].

† (191) This is analogous to the principles of the differential calculus. For if the extreme points of an infinitely small arch ds of a curve be given, the intermediate parts of this arch ds are supposed to fall on the straight line joining these two extreme points, neglecting quantities of the *second* order. In like manner on a curved surface the intermediate parts between the parallel edges of an infinitely small part of the surface may be

continue to be placed on equal and parallel right lines. Let us call this new
[297''] parallelopiped (*B*), and suppose through the extremities of the edge formed by the particles, which composed the edge dc of the parallelopiped (*A*), we draw two planes, parallel to the plane x, y. By prolonging the edges of (*B*) till they meet these two planes, we shall have another parallelopiped
[297'''] (*C*), contained between them, which is equal to (*B*); for it is evident that one of the two planes cuts off from the parallelopiped (*B*) as much as the other adds to it.* The parallelopiped (*C*) will have its two bases parallel to the plane of x, y; its height comprised between its bases, will evidently

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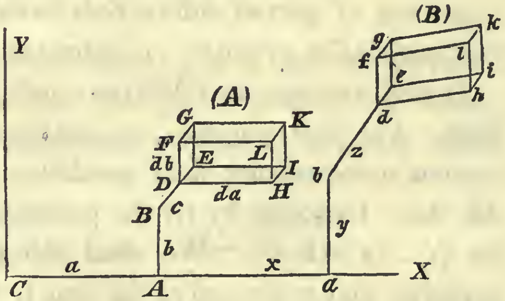
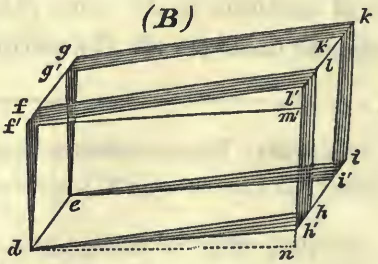
considered as being on the plane joining these parallel edges, neglecting quantities of the *second* order. And it is evident that the same must take place in the case under consideration. For the forces, acting on the different parts of any face of the parallelopiped, differ from each other only by infinitely small quantities, which vary gradually, from one point to another of the face, and the effect produced must be as above stated.





* (192) To illustrate what is here said, we have given the annexed figures, in which $CAaX$ is the axis of x, CY the axis of y, to which AB, ab are parallel, and BD, bd which are supposed to be perpendicular to the plane of the figure, are parallel to the axis of z; C being the origin of the co-ordinates; $DHIKG$ the rectangular parallelopiped (*A*), at the commencement of the motion; $dhikg$ the oblique parallelopiped (*B*) representing the situation and form of (*A*) at the end of the time t; the parallelopiped (*B*) is described in a separate figure, upon a larger scale, so as to make the letters and lines of reference more legible, and to this figure we must occasionally refer in the rest of this note. Then the co-ordinates of the point D are $CA = a$, $AB = b$, $BD = c$; those of the point d are $Ca = x$, $ab = y$, $bd = z$; the sides of the parallelopiped (*A*) are $DH = \mathrm{d}a$, $DF = \mathrm{d}b$, $DE = \mathrm{d}c$, and its solidity is the product of these three sides, and as its density is (ρ), its mass must be $(\rho) \, . \, \mathrm{d}a \, . \, \mathrm{d}b \, . \, \mathrm{d}c$, as in [297']. Suppose now a plane to be drawn through the point e, parallel to the plane xy, or Cab, it will cut the edges gf, kl, ih, continued if necessary, in the points g', k', i', respectively; and a similar parallel plane being drawn

be equal to the differential of z, taken on the supposition that c only varies; which gives $\left(\frac{dz}{dc}\right).\,\mathrm{d}c$ for this height.* [297$^{\mathrm{iv}}$]

We shall have its base, by observing that it is equal to the section of (B), by a plane parallel to that of x, y; let us call this section (ε). The value of z will be the same for all the particles of which it is composed, and we shall have† [297$^{\mathrm{v}}$]

$$0=\left(\frac{dz}{da}\right).\,\mathrm{d}a+\left(\frac{dz}{db}\right).\,\mathrm{d}b+\left(\frac{dz}{dc}\right).\,\mathrm{d}c. \qquad [298]$$

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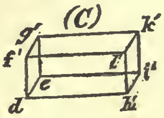
through d, will cut the same edges in the points f', l', h'; these parallel planes will thus form another parallelopiped $d\,h'\,i'\,k'\,g'\,f'\,d$, which for greater distinctness, is given separately on its proper scale; this is the solid called (C). Now it is evident that the part of the solid (B) included between the planes $e\,i\,k\,g$, $e\,i'\,k'\,g'$, must be equal to the part included by the parallel planes $d\,h\,l\,f$, $d\,h'\,l'\,f'$, and as the former is taken from (B), and the latter added, to make (C), it follows that the parallelopipeds (B) and (C) must be equal.



* (193) The height of the parallelopiped (C) comprised between its two faces, drawn parallel to the plane of x, y, must evidently be equal to the difference of the elevations of the points d, e, above that plane; that is, it must be equal to the value of z, corresponding to the point e, less that corresponding to the point d. Now z is evidently a function of a, b, c, t, representing the ordinate $b\,d$, corresponding to the point d; and by changing in it, c into $c+\mathrm{d}c$, we obtain the value of the ordinate corresponding to the point e, which is therefore $z+\left(\frac{dz}{dc}\right).\,\mathrm{d}c$; for by making this change in the ordinate c, corresponding to the point D, we obtain the ordinate corresponding to the point E, which last point, at the end of the time t, falls in e. The difference of these values $z+\left(\frac{dz}{dc}\right).\,\mathrm{d}c$, and z, namely, $\left(\frac{dz}{dc}\right).\,\mathrm{d}c$, is therefore, the required height of the parallelopiped, as in [297$^{\mathrm{iv}}$].

† (194) Though z is in general a function of a, b, c, t, whose complete differential is

$$dz=\left(\frac{dz}{da}\right).\,\mathrm{d}a+\left(\frac{dz}{db}\right).\,\mathrm{d}b+\left(\frac{dz}{dc}\right).\,\mathrm{d}c+\left(\frac{dz}{dt}\right).\,\mathrm{d}t, \qquad [298a]$$

yet in the present instance the term depending on $d\,t$ is to be neglected, because the object is to find the value of $d\,z$, *at the same instant of time* t, in different parts of the parallelopiped (B) or (C), so that we must put $d\,t=0$, and then the points of the parallelopiped (C), in which $d\,z=0$, will correspond to the equation [298].

Let δp and δq be two contiguous sides of the section (ε), of which the first is *formed by the particles of the face* $\mathrm{d}b\,.\,\mathrm{d}c$ of the parallelopiped (A), and the second is *formed by the particles of the face* $\mathrm{d}a\,.\,\mathrm{d}c$. If through the extremities of the side δp, we suppose two right lines parallel to the axis of x to be drawn, and prolong the side of the parallelogram (ε) which is parallel to δp, until it meets these lines; they will intercept between them another parallelogram (λ) equal to (ε)* and the base of which will be parallel to the axis of x. The side δp being formed by those particles of the face $\mathrm{d}b.\mathrm{d}c$ [298′], which have the same value of z [297′]; it is easy to perceive, that the height of the parallelogram (λ) is the differential of y, taken by supposing a, z, t, constant, which gives† [298′]

$$\mathrm{d}y=\left(\frac{dy}{db}\right).\,\mathrm{d}b+\left(\frac{dy}{dc}\right).\,\mathrm{d}'c;$$
$$0=\left(\frac{dz}{db}\right).\,\mathrm{d}b+\left(\frac{dz}{dc}\right).\,\mathrm{d}'c;$$ [299]

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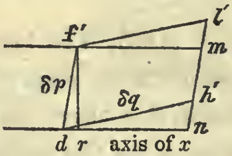
* (194a) The section (ε) is represented by the parallelogram $dh'l'f'd$ of the figure C, drawn separately in the annexed figure. The side $f'd=\delta p$, the side $h'd=\delta q$. Through the points d, f', draw the lines dn, $f'm$, parallel to the axis of x, meeting $l'h'$ in m, n, and forming the parallelogram $dnmf'$, which is here called (λ), whose base is equal to nd, and height is the perpendicular $f'r$, let fall from f' upon nd; and it is evident from the construction that the parallelograms (λ), (ε) are equal.



† (195) In these equations I have placed an accent on the letter d connected with c, in order to distinguish the term $\mathrm{d}'c$, found in this part of the calculation, from the side DE of the parallelogram (A), which is denoted by $\mathrm{d}c$. It being evident that DE may be increased or diminished at pleasure, without changing the value of the height $f'r$ of the parallelogram (λ), which is represented in [299], by $dy=\left(\frac{dy}{db}\right).\,\mathrm{d}b+\left(\frac{dy}{dc}\right).\,\mathrm{d}'c$; no term depending on the differential of a being introduced, because all the particles situated on the line $f'd$, appertain to the plane of the face $degf$ of the solid (B) or (C), on which the value of a is constant; neither is there any term depending on dt introduced, because at the same moment that the point (D) arrives at d, the particles of the face $DEGF$, corresponding to the line df', arrive at their proper places on that line. The relation of $\mathrm{d}b$ to $\mathrm{d}'c$, is determined by the condition that z is constant, or that $\mathrm{d}z$ is nothing, $[297^{\mathrm{v}}]$, for all particles situated on the line df'; this gives the second of the above equations [299], [299a]

whence we deduce

$$\mathrm{d}y=\frac{\left\{\left(\frac{dy}{db}\right)\cdot\left(\frac{dz}{dc}\right)-\left(\frac{dy}{dc}\right)\cdot\left(\frac{dz}{db}\right)\right\}}{\left(\frac{dz}{dc}\right)}\cdot\mathrm{d}b\,; \qquad [300]$$

this is the expression of the height of the parallelogram (λ). Its base is equal to the section of this parallelogram made by a plane parallel to the axis of x; this section is formed by the particles of the parallelopiped (A) which correspond to z and y constant; its length is therefore equal to the differential of x, taken on the supposition that z, y, and t are constant; which gives the three equations*

$$\begin{aligned}\mathrm{d}x&=\left(\frac{dx}{da}\right)\cdot\mathrm{d}a+\left(\frac{dx}{db}\right)\cdot\mathrm{d}_{,}b+\left(\frac{dx}{dc}\right)\cdot\mathrm{d}_{,}c\,;\\ 0&=\left(\frac{dy}{da}\right)\cdot\mathrm{d}a+\left(\frac{dy}{db}\right)\cdot\mathrm{d}_{,}b+\left(\frac{dy}{dc}\right)\cdot\mathrm{d}_{,}c\,;\\ 0&=\left(\frac{dz}{da}\right)\cdot\mathrm{d}a+\left(\frac{dz}{db}\right)\cdot\mathrm{d}_{,}b+\left(\frac{dz}{dc}\right)\cdot\mathrm{d}_{,}c\,;\end{aligned} \qquad [301]$$

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putting also, as in the first of the equations [299], t and a constant; or, in other words, rejecting the terms $\mathrm{d}a, dt$, from $dz=0$, [298a]. From this second equation [299], we get

$$\mathrm{d}'c=-\frac{\left(\frac{dz}{db}\right)}{\left(\frac{dz}{dc}\right)}\cdot\mathrm{d}b,$$

this being substituted in the first equation [299], gives the final value of dy, [300], which is proportional to $\mathrm{d}b$, or to the side DF of the parallelogram (A), and is independent of the sides $DH=\mathrm{d}a$, $DE=\mathrm{d}c$.

* (196) In these equations are put $\mathrm{d}_{,}b$, $\mathrm{d}_{,}c$, instead of $\mathrm{d}b$, $\mathrm{d}c$, to distinguish them from the sides $\mathrm{d}b$, $\mathrm{d}c$, or DF, DE, of the parallelogram (A); for the same reason that $\mathrm{d}'c$ was accented in the last note; it being evident that the length nd of the parallelogram (λ) is proportional to DH, or $\mathrm{d}a$, and that this length does not vary by increasing or decreasing DF, DE, or $\mathrm{d}b$, $\mathrm{d}c$. The first of these equations, [301], is the value of dx, the second that of $dy=0$, the third that of $dz=0$; dt being, as in the last notes, rejected. The values of $\mathrm{d}_{,}b$, $\mathrm{d}_{,}c$, being found from the two last, and substituted in the first, give the required value of dx. This may be found more simply by multiplying the first, second, and third equations, [301], respectively by the following factors, [299b]

$$\left(\frac{dy}{db}\right)\cdot\left(\frac{dz}{dc}\right)-\left(\frac{dy}{dc}\right)\cdot\left(\frac{dz}{db}\right); \qquad \left(\frac{dx}{dc}\right)\cdot\left(\frac{dz}{db}\right)-\left(\frac{dx}{db}\right)\cdot\left(\frac{dz}{dc}\right);$$

$$\left(\frac{dx}{db}\right)\cdot\left(\frac{dy}{dc}\right)-\left(\frac{dx}{dc}\right)\cdot\left(\frac{dy}{db}\right);$$

Put for abridgment

[302]
$$\beta=\left(\frac{dx}{da}\right).\left(\frac{dy}{db}\right).\left(\frac{dz}{dc}\right)-\left(\frac{dx}{da}\right).\left(\frac{dy}{dc}\right).\left(\frac{dz}{db}\right)+\left(\frac{dx}{db}\right).\left(\frac{dy}{dc}\right).\left(\frac{dz}{da}\right)$$
$$-\left(\frac{dx}{db}\right).\left(\frac{dy}{da}\right).\left(\frac{dz}{dc}\right)+\left(\frac{dx}{dc}\right).\left(\frac{dy}{da}\right).\left(\frac{dz}{db}\right)-\left(\frac{dx}{dc}\right).\left(\frac{dy}{db}\right).\left(\frac{dz}{da}\right);$$

we shall have

[303]
$$\mathrm{d}x=\frac{\beta.\mathrm{d}a}{\left(\frac{dy}{db}\right).\left(\frac{dz}{dc}\right)-\left(\frac{dy}{dc}\right).\left(\frac{dz}{db}\right)};$$

which is the expression of the base of the parallelogram (λ); the surface of this parallelogram will therefore be $\dfrac{\beta.\mathrm{d}a.\mathrm{d}b}{\left(\frac{dz}{dc}\right)}$. This quantity also expresses the surface of the parallelogram (ε); multiplying it by $\left(\frac{dz}{dc}\right).\mathrm{d}c$,

[303′] we shall have $\beta.\mathrm{d}a.\mathrm{d}b.\mathrm{d}c$, for the magnitude of the parallelopipeds (*C*) and (*B*). Let ρ be the density of the parallelopiped (*A*) after the time t; we shall have for its mass

[303″]
$$\rho.\beta.\mathrm{d}a.\mathrm{d}b.\mathrm{d}c;$$

putting this equal to its original mass $(\rho).\mathrm{d}a.\mathrm{d}b.\mathrm{d}c$ [297′], we shall have

[303‴]
$$\rho\beta=(\rho) \qquad (G)$$

Second important Equation of the continuity of the Fluid.

for the equation relative to the continuity of the fluid.

33. We may give to the equations (*F*) [296], and (*G*) [303‴], forms which will be more convenient in some circumstances. Let u, v and v be

[303^{iv}] the velocities of a fluid particle, parallel to the axes of x, y, z, respectively; we shall have*

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and adding them together; the coefficient of da in the second member, becomes equal to the quantity denoted by β, [302], while those of $\mathrm{d}_{,}b$, $\mathrm{d}_{,}c$, vanish; hence

$$\left\{\left(\frac{dy}{db}\right).\left(\frac{dz}{dc}\right)-\left(\frac{dy}{dc}\right).\left(\frac{dz}{db}\right)\right\}.dx=\beta.\mathrm{d}a,$$

which gives for dx the same value as in [303]. This multiplied by the height dy, [300], gives the area of the parallelogram (λ) or (ε), and this multiplied by the height of the parallelopiped (*C*), which by [297^{iv}] is $\left(\frac{dz}{dc}\right).\mathrm{d}c$, gives its solidity $\beta.\mathrm{d}a.\mathrm{d}b.\mathrm{d}c$, [303′].

* (197) The co-ordinates of any particle of the fluid, which were represented by a, b, c, [296″], at the commencement of the motion, when $t=0$, and by x, y, z, when $t=t$,

$$\left(\frac{dx}{dt}\right)=u\,; \qquad \left(\frac{dy}{dt}\right)=v\,; \qquad \left(\frac{dz}{dt}\right)=\mathrm{v}. \qquad [304]$$

Taking the differentials of these equations, supposing u, v, v, to be functions of the co-ordinates of the particle x, y, z, and of the time t; we shall have

$$\begin{aligned}\left(\frac{ddx}{dt^2}\right)&=\left(\frac{du}{dt}\right)+u\,.\left(\frac{du}{dx}\right)+v\,.\left(\frac{du}{dy}\right)+\mathrm{v}\,.\left(\frac{du}{dz}\right);\\ \left(\frac{ddy}{dt^2}\right)&=\left(\frac{dv}{dt}\right)+u\,.\left(\frac{dv}{dx}\right)+v\,.\left(\frac{dv}{dy}\right)+\mathrm{v}\,.\left(\frac{dv}{dz}\right);\\ \left(\frac{ddz}{dt^2}\right)&=\left(\frac{d\mathrm{v}}{dt}\right)+u\,.\left(\frac{d\mathrm{v}}{dx}\right)+v\,.\left(\frac{d\mathrm{v}}{dy}\right)+\mathrm{v}\,.\left(\frac{d\mathrm{v}}{dz}\right).\end{aligned} \qquad [305]$$

[296″], become respectively $x+u\,dt$, $y+v\,dt$, $z+\mathrm{v}\,dt$, [306′], when t is increased [305*a*]
to $t+dt$. In this notation the co-ordinates a, b, c, of any particular particle, do not vary
with the time, but differ for different particles, and they serve merely to denote the
primitive situation of the particular particle of the fluid, whose motion is to be considered.
Again, since the co-ordinates, x, y, z, of the particle, corresponding to the time t, are
increased during the following instant dt, by the quantities

Remarks on the co-ordinates x, y, z, a, b, c.

$$dx=u\,dt \qquad dy=v\,dt, \qquad dz=\mathrm{v}\,dt, \qquad [305b]$$

it follows that the velocity of the particle, resolved in directions parallel to these axes, will be represented by

$$\frac{dx}{dt}=u, \qquad \frac{dy}{dt}=v, \qquad \frac{dz}{dt}=\mathrm{v}\,; \qquad [305c]$$

but we must observe, that in taking the differentials of x, y, z, the quantity t only was considered variable, and since x, y, z, [296″], are denoted by functions of a, b, c, t, the preceding expressions $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, must be considered as the *partial* differentials of x, y, z, relative to t, and ought, in conformity to the usual notation of the author, to be [305*d*] included in parentheses, as in [304]. For the same reason the quantities $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$, [142], were included in parentheses, in [295], it being evident, from what has been said, that they are partial differentials relative to the time t. Moreover, the expressions of u, v, v, [304], must be considered as functions of x, y, z, t, since for the same value of t, these velocities will vary from one particle to another; and for the same co-ordinates x, y, z, the velocities will vary from one instant to another; so that in taking the differential of any one of the equations [304], as for example, $\left(\frac{dx}{dt}\right)=u$, we must consider u as a function of [305*e*]

Second form of the Equation of the Motion of a Fluid.

The equation (F) [296] of the preceding article will thus become

[306]
$$\begin{aligned}\delta V-\frac{\delta p}{\rho}=\delta x.&\left\{\left(\frac{du}{dt}\right)+u.\left(\frac{du}{dx}\right)+v.\left(\frac{du}{dy}\right)+\mathrm{v}.\left(\frac{du}{dz}\right)\right\}\\+\delta y.&\left\{\left(\frac{dv}{dt}\right)+u.\left(\frac{dv}{dx}\right)+v.\left(\frac{dv}{dy}\right)+\mathrm{v}.\left(\frac{dv}{dz}\right)\right\}; \qquad (H)\\+\delta z.&\left\{\left(\frac{d\mathrm{v}}{dt}\right)+u.\left(\frac{d\mathrm{v}}{dx}\right)+v.\left(\frac{d\mathrm{v}}{dy}\right)+\mathrm{v}.\left(\frac{d\mathrm{v}}{dz}\right)\right\}\end{aligned}$$

To obtain the equation relative to the continuity of the fluid; suppose in the value of β [302] of the preceding article, $a, b, c,$ to be equal to x, y, z; and $x, y, z,$ equal to $x+u\,dt,$ $y+v\,dt,$ $z+\mathrm{v}\,dt,$ respectively; which is [306′] equivalent to the supposition that the primitive co-ordinates $a, b, c,$ are infinitely near to $x, y, z,$ we shall have*

[307]
$$\beta=1+dt.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\};$$

and the equation (G) [303‴] will become

[308]
$$\rho.dt.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\}+\rho-(\rho)=0;$$

---

x, y, z, t; and $x, y, z,$ as functions of a, b, c, t. Therefore if we take the partial differential of $\left(\frac{dx}{dt}\right)=u,$ relative to t, it will become, according to the usual notation of partial differentials,

[305f]
$$\left(\frac{ddx}{dt^2}\right)=\left(\frac{du}{dt}\right)+\left(\frac{du}{dx}\right).\left(\frac{dx}{dt}\right)+\left(\frac{du}{dy}\right).\left(\frac{dy}{dt}\right)+\left(\frac{du}{dz}\right).\left(\frac{dz}{dt}\right).$$

Substitute in this the values of $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$, [304], and it will become as in the first of the equations [305]. The expressions $\left(\frac{ddy}{dt^2}\right)$, $\left(\frac{ddz}{dt^2}\right)$, [305], are found in the same manner, from $\left(\frac{dy}{dt}\right)=v,$ $\left(\frac{dz}{dt}\right)=\mathrm{v},$ [304]. These being substituted in [296], it becomes as in [306].

* (198) By changing a into x, and x into $x+u\,dt,$ the expression $\left(\frac{dx}{da}\right)$ becomes [306a] $\left(\frac{d.(x+u\,dt)}{dx}\right)$, and as $x, y, z, t,$ are considered as independent variable quantities, this becomes $\left(\frac{dx}{da}\right)=1+\left(\frac{du}{dx}\right).dt.$ In like manner

$$\left(\frac{dy}{db}\right)=1+\left(\frac{dv}{dy}\right).dt, \qquad \left(\frac{dz}{dc}\right)=1+\left(\frac{d\mathrm{v}}{dz}\right).dt,$$

If we consider ρ as a function of x, y, z, and t; we shall have*

$$(\rho)=\rho-dt.\left(\frac{d\rho}{dt}\right)-u\,dt.\left(\frac{d\rho}{dx}\right)-v\,dt.\left(\frac{d\rho}{dy}\right)-\mathrm{v}\,dt.\left(\frac{d\rho}{dz}\right); \qquad [309]$$

the preceding equation will therefore become

$$0=\left(\frac{d\rho}{dt}\right)+\left(\frac{d.\rho u}{dx}\right)+\left(\frac{d.\rho v}{dy}\right)+\left(\frac{d.\rho \mathrm{v}}{dz}\right); \qquad (K) \qquad [310]$$

Second form of the Equation of Continuity.

which is the equation relative to the continuity of the fluid, and it is easy to perceive that it is the differential of the equation (G) [303′′′] of the preceding article, taken relative to the time t.†

The remaining terms or factors of which β [302] is formed are of the order dt, thus

$$\left(\frac{dx}{db}\right)=\left(\frac{d.(x+u\,dt)}{dy}\right)=\left(\frac{du}{dy}\right).dt; \qquad \left(\frac{dx}{dc}\right)=\left(\frac{d.(x+u\,dt)}{dz}\right)=\left(\frac{du}{dz}\right).dt;$$

$$\left(\frac{dy}{da}\right)=\left(\frac{d.(y+v\,dt)}{dx}\right)=\left(\frac{dv}{dx}\right).dt, \text{ \&c.}$$

Therefore all the terms of β, except the first $\left(\frac{dx}{da}\right).\left(\frac{dy}{db}\right).\left(\frac{dz}{dc}\right)$, are of the order dt^2 or dt^3, and may be neglected; and this first term gives

$$\beta=\left\{1+\left(\frac{du}{dx}\right).dt\right\}.\left\{1+\left(\frac{dv}{dy}\right).dt\right\}.\left\{1+\left(\frac{d\mathrm{v}}{dz}\right).dt\right\},$$

which by developing and neglecting the terms multiplied by dt^2, becomes

$$\beta=1+dt.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\}$$

as in [307]. This being substituted in [303′′′], gives [308].

* (199) Supposing (ρ) to be a function of x, y, z, t, and ρ to be a similar function of $x+u\,dt$, $y+v\,dt$, $z+\mathrm{v}\,dt$, $t+dt$, we shall have by developing according to the powers of dt, by Taylor's theorem [607,—612],

$$\rho=(\rho)+dt.\left(\frac{d\rho}{dt}\right)+u\,dt.\left(\frac{d\rho}{dx}\right)+v\,dt.\left(\frac{d\rho}{dy}\right)+\mathrm{v}\,dt.\left(\frac{d\rho}{dz}\right),$$

neglecting the terms of the second and higher powers of dt, and by transposing all the terms of the second member, except the first, it becomes as in [309]. This value of (ρ) being substituted in [308] divided by dt, it becomes,

$$0=\rho.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\}+\left(\frac{d\rho}{dt}\right)+u.\left(\frac{d\rho}{dx}\right)+v.\left(\frac{d\rho}{dy}\right)+\mathrm{v}.\left(\frac{d\rho}{dz}\right), \qquad [310a]$$

which is the same as the equation [310], developed by writing

$$u.\left(\frac{d\rho}{dx}\right)+\rho.\left(\frac{du}{dx}\right) \text{ for } \left(\frac{d.\rho u}{dx}\right), \text{ \&c.}$$

† (200) The differential of the equation [303′′′], is $\rho.d\beta+\beta.d\rho=0$, because (ρ) [297′], is a constant quantity, not varying with the time, and its differential is nothing. [310b]

The equation (*H*) [306] is susceptible of integration, in a very extensive case, namely, when $u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z$ is an exact variation of x, y, z; ρ
[310′] *being any function whatever of the pressure p.* Let this variation be $\delta\varphi$, so that

[310″]
$$\delta\varphi=u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z\,;$$

the equation (*H*) [306] will give*

[311]
$$\delta V-\frac{\delta p}{\rho}=\delta.\left(\frac{d\varphi}{dt}\right)+\tfrac{1}{2}.\delta.\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\right\};$$

Now by [307], when $dt=0$, β becomes $=1$. Subtracting this value of β from that in [307], corresponding to the time $t+dt$, we get the change of the value of β during the time dt, or $d\beta=dt.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\}$. In like manner from [308], we get the value of ρ, corresponding to $dt=0$, which is (ρ); subtracting this from the general value of ρ, deduced from [309], we get the value of

$$d\rho=\rho-(\rho)=dt.\left(\frac{d\rho}{dt}\right)+udt.\left(\frac{d\rho}{dx}\right)+vdt.\left(\frac{d\rho}{dy}\right)+\mathrm{v}dt.\left(\frac{d\rho}{dz}\right);$$

substituting these in [310*b*], we get,

$$\rho.\left\{\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{d\mathrm{v}}{dz}\right)\right\}dt+\left\{dt.\left(\frac{d\rho}{dt}\right)+udt.\left(\frac{d\rho}{dx}\right)+v\,dt.\left(\frac{d\rho}{dy}\right)+\mathrm{v}dt.\left(\frac{d\rho}{dz}\right)\right\}.\beta,$$

in which β [307], may be put equal to 1, by neglecting terms of the order dt^2, and then the equation becomes precisely of the same form as [310*a*], which in the last note was shown to be equal to [310].

[311*a*] * (201) The expression of $\delta\varphi$ [310″], gives, $u=\left(\frac{\delta\varphi}{\delta x}\right)$, $v=\left(\frac{\delta\varphi}{\delta y}\right)$, $\mathrm{v}=\left(\frac{\delta\varphi}{\delta z}\right)$. The partial differential of the first of these expressions, relative to *t*, is

$$\left(\frac{du}{dt}\right)=\left(\frac{d\,\delta\varphi}{dt\,\delta x}\right)=\left(\frac{\delta\,d\varphi}{\delta x\,dt}\right),$$

and by putting $\left(\frac{d\varphi}{dt}\right)=\varphi'$, it becomes $\left(\frac{du}{dt}\right)=\left(\frac{\delta\varphi'}{\delta x}\right)$. In the same manner

$$\left(\frac{dv}{dt}\right)=\left(\frac{d\,\delta\varphi}{dt\,\delta y}\right)=\left(\frac{\delta\varphi'}{\delta y}\right),\qquad\text{and}\qquad\left(\frac{d\mathrm{v}}{dt}\right)=\left(\frac{\delta\varphi'}{\delta z}\right).\qquad\text{Hence}$$

$$\delta x.\left(\frac{du}{dt}\right)+\delta y.\left(\frac{dv}{dt}\right)+\delta z.\left(\frac{d\mathrm{v}}{dt}\right)=\delta x.\left(\frac{\delta\varphi'}{\delta x}\right)+\delta y.\left(\frac{\delta\varphi'}{\delta y}\right)+\delta z.\left(\frac{\delta\varphi'}{\delta z}\right),$$

the second member of which is evidently equal to $\delta\varphi'$ or $\delta.\left(\frac{d\varphi}{dt}\right)$, observing that the characteristic δ does not affect the time *t*, [294′]. Thus we shall have,

[312*a*]
$$\delta x.\left(\frac{du}{dt}\right)+\delta y.\left(\frac{dv}{dt}\right)+\delta z.\left(\frac{d\mathrm{v}}{dt}\right)=\delta.\left(\frac{d\varphi}{dt}\right),$$

whence by integration relative to δ,

$$V-\int\frac{\delta p}{\rho}=\left(\frac{d\varphi}{dt}\right)+\tfrac{1}{2}\cdot\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\right\}. \qquad [312]$$

We must add to this integral an arbitrary constant quantity, which is a function of t; but this may be considered as included in the function φ.* [312′] This function φ gives the velocities of the fluid particles, parallel to the axes of x, y, and z; for we have [312*b*]

$$u=\left(\frac{d\varphi}{dx}\right); \qquad v=\left(\frac{d\varphi}{dy}\right); \qquad \mathrm{v}=\left(\frac{d\varphi}{dz}\right). \qquad [313]$$

Again, since $\delta\varphi$ [310″], is an exact differential we have

$$u=\left(\frac{d\varphi}{dx}\right), \qquad v=\left(\frac{d\varphi}{dy}\right), \qquad \mathrm{v}=\left(\frac{d\varphi}{dz}\right), \qquad [312b]$$

φ being a function of the independent variable quantities x, y, z, t. Taking the partial differential of this value of v, relative to x, we get,

$$\left(\frac{dv}{dx}\right)=\left(\frac{dd\varphi}{dx\,dy}\right)=\left(\frac{d\cdot\left(\frac{d\varphi}{dx}\right)}{dy}\right)=\left(\frac{du}{dy}\right),$$

and in a similar manner $\left(\frac{d\mathrm{v}}{dx}\right)=\left(\frac{du}{dz}\right)$. These being substituted in

$$u\cdot\left\{\delta x\cdot\left(\frac{du}{dx}\right)+\delta y\cdot\left(\frac{dv}{dx}\right)+\delta z\cdot\left(\frac{d\mathrm{v}}{dx}\right)\right\},$$

it becomes $u\cdot\left\{\delta x\cdot\left(\frac{du}{dx}\right)+\delta y\cdot\left(\frac{du}{dy}\right)+\delta z\cdot\left(\frac{du}{dz}\right)\right\}$, which is evidently equal to $u\,\delta u=\tfrac{1}{2}\cdot\delta\cdot u^2$, and by substituting the value of u, [312*b*], we get,

$$u\cdot\left\{\delta x\cdot\left(\frac{du}{dx}\right)+\delta y\cdot\left(\frac{dv}{dx}\right)+\delta z\cdot\left(\frac{d\mathrm{v}}{dx}\right)\right\}=\tfrac{1}{2}\cdot\delta\cdot\left(\frac{d\varphi}{dx}\right)^2. \qquad [312e]$$

We may proceed, in the same manner, with the terms of [306], multiplied by v and by v, or we may obtain the same result, much more simply, by changing, in [312*c*], u into v, and x into y, and the contrary, hence we shall get,

$$v\cdot\left\{\delta x\cdot\left(\frac{du}{dy}\right)+\delta y\cdot\left(\frac{dv}{dy}\right)+\delta z\cdot\left(\frac{d\mathrm{v}}{dy}\right)\right\}=\tfrac{1}{2}\cdot\delta\cdot\left(\frac{d\varphi}{dy}\right)^2. \qquad [312d]$$

In like manner, changing in this v into v, and y into z, and the contrary, we get

$$\mathrm{v}\cdot\left\{\delta x\cdot\left(\frac{du}{dz}\right)+\delta y\cdot\left(\frac{dv}{dz}\right)+\delta z\cdot\left(\frac{d\mathrm{v}}{dz}\right)\right\}=\tfrac{1}{2}\cdot\delta\cdot\left(\frac{d\varphi}{dz}\right)^2. \qquad [312e]$$

The sum of the expressions [312*a*, *c*, *d*, *e*], constitutes the second member of [306], which is by this means, reduced to the form [311].

* (202) As the characteristic δ does not affect t, the integral of $u\delta x+v\delta y+\mathrm{v}\delta z=\delta\varphi$, [310″], taken relative to this characteristic, may be completed by adding an arbitrary

The equation (K) [310], relative to the continuity of the fluid, becomes*

[314] $$0=\left(\frac{d\rho}{dt}\right)+\left(\frac{d\rho}{dx}\right).\left(\frac{d\varphi}{dx}\right)+\left(\frac{d\rho}{dy}\right).\left(\frac{d\varphi}{dy}\right)+\left(\frac{d\rho}{dz}\right).\left(\frac{d\varphi}{dz}\right)$$
$$+\rho.\left\{\left(\frac{dd\varphi}{dx^2}\right)+\left(\frac{dd\varphi}{dy^2}\right)+\left(\frac{dd\varphi}{dz^2}\right)\right\};$$

thus, relative to homogeneous fluids, we have†

[315] $$0=\left(\frac{dd\varphi}{dx^2}\right)+\left(\frac{dd\varphi}{dy^2}\right)+\left(\frac{dd\varphi}{dz^2}\right).$$

We may observe that the function $u.\delta x+v.\delta y+\mathrm{v}.\delta z$ [310″] will always be an exact variation of x, y, z, if it be so during one instant. For if we
[315′] suppose, at any instant whatever, that it is equal to $\delta\varphi$; in the following instant, it will be‡

[316] $$\delta\varphi+dt.\left\{\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z\right\};$$

---

function of t; and as x, y, z, t, [306a], are supposed to be independent variable quantities, this function of t, will not affect the values of $\left(\frac{d\varphi}{dx}\right)$, $\left(\frac{d\varphi}{dy}\right)$, $\left(\frac{d\varphi}{dz}\right)$, in the expression [312]; and φ may therefore be supposed to contain the arbitrary function of t, [312′], which is required to complete the integral of [311].

* (203) This is easily deduced from [310], developed as in [310a], using the values [313], which give

[314a] $$\left(\frac{du}{dx}\right)=\left(\frac{dd\varphi}{dx^2}\right),\qquad \left(\frac{dv}{dy}\right)=\left(\frac{dd\varphi}{dy^2}\right),\qquad \left(\frac{d\mathrm{v}}{dz}\right)=\left(\frac{dd\varphi}{dz^2}\right).$$

† (204) The fluid being homogeneous, its density ρ is constant, therefore $d\rho=0$. Substitute this in [314], and divide by ρ, we get [315].

‡ (205) The ordinates x, y, z, being supposed to remain unaltered, but the time t, to increase by the differential dt, new particles of fluid taking the place of those which formerly corresponded to x, y, z, the partial velocities u, v, v, will become $u+\left(\frac{du}{dt}\right).dt$, $v+\left(\frac{dv}{dt}\right).dt$, $\mathrm{v}+\left(\frac{d\mathrm{v}}{dt}\right).dt$, respectively, for these new particles; consequently the expression $u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z$, will become,

$$u\,\delta x+v\,\delta y+z\,\delta y+dt.\left\{\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z\right\},$$

which, by substituting, for $u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z$, its assumed value $\delta\varphi$, [310″], becomes as in [316].

it will therefore be, in this last instant, an exact variation, if

$$\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z$$

be an exact variation at the first instant; now the equation (H) [306] will give at this instant*

$$\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z=\delta V-\tfrac{1}{2}.\delta.\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{\delta z}\right)^2\right\}-\frac{\delta p}{\rho}; \qquad [317]$$

consequently the first member of this equation will be an exact variation in x, y, z; therefore if the function $u.\delta x+v.\delta y+\mathrm{v}.\delta z$ be an exact variation at one instant, it will also be an exact variation in the next instant; and it will therefore be an exact variation at all times, if it be so at any instant.

When the motions are very small, we may neglect the squares and products of u, v, and v; the equation (H) [306] then becomes†

$$\delta V-\frac{\delta p}{\rho}=\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z; \qquad [318]$$

therefore, in this case, $u.\delta x+v.\delta y+\mathrm{v}.\delta z$ is an exact variation, if as

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* (206) The equation [306], by substituting the values computed in [312*c*, *d*, *e*] becomes
$$\delta V-\frac{\delta p}{p}=\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z+\tfrac{1}{2}.\delta.\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\varphi}{dy}\right)^2+\left(\frac{d\varphi}{dz}\right)^2\right\},$$
which by transposing the last term, becomes as in [317]; and as ρ is a function of p, by hypothesis [310'], the term $\frac{\delta p}{\rho}$ is an exact variation, and every term of the second member of the expression [317] is an exact variation, consequently the first member of [317] is also an exact variation.

† (207) In this hypothesis the terms $u.\left(\frac{du}{dx}\right)$, $u.\left(\frac{du}{dy}\right)$, &c. are to be neglected, being of the order of u^2, v^2, &c. Neglecting these terms, the expression [306] becomes like [318]. The first member of which being an exact variation, its second member must also be an exact variation, and by putting it equal to $\delta.\left(\frac{d\varphi}{dt}\right)$, we shall get $\delta V-\frac{\delta p}{\rho}=\delta.\left(\frac{d\varphi}{dt}\right)$, whose integral relative to the characteristic δ gives [319]. This assumed value of

$$\delta.\left(\frac{d\varphi}{dt}\right)=\left(\frac{du}{dt}\right).\delta x+\left(\frac{dv}{dt}\right).\delta y+\left(\frac{d\mathrm{v}}{dt}\right).\delta z,$$

multiplied by dt, and integrated relative to d gives $\delta\varphi=u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z$, since δx, δy, δz, are not affected by the time t, [294'].

we have supposed [310′], p is a function of ρ; still calling this variation $\delta\varphi$ [310″], we shall have

[319] $$V-\int\frac{\delta p}{\rho}=\left(\frac{d\varphi}{dt}\right);$$

and if the fluid is homogeneous, the equation of continuity will become [315]

[320] $$0=\left(\frac{dd\varphi}{dx^2}\right)+\left(\frac{dd\varphi}{dy^2}\right)+\left(\frac{dd\varphi}{dz^2}\right).$$

Equation of small Undulations of homogeneous Fluids. These two equations contain the whole theory of very small undulations in homogeneous fluids.

34. We shall now consider the motions of an homogeneous fluid mass, which has a uniform motion of rotation about the axis of x. *Let n be the angular velocity of rotation*, at a distance from the axis, which we shall take for the unity of distance; we shall have* $v=-nz$; $\mathrm{v}=ny$; the equation (H) [306] of the preceding article, will therefore become† [320′] [320″]

[321] $$\frac{\delta p}{\rho}=\delta V+n^2.\{y\,\delta y+z\,\delta z\};$$

which equation is possible, since its two members are exact differentials. The equation (K) [310] of the same article, will become‡

[322] $$0=dt.\left(\frac{d\rho}{dt}\right)+u.dt.\left(\frac{d\rho}{dx}\right)+v.dt.\left(\frac{d\rho}{dy}\right)+\mathrm{v}.dt.\left(\frac{d\rho}{dz}\right);$$

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* (208) The angular rotation about the axis of x, in the time dt, is $n\,dt$ [320′], and this is called $d\psi$ in the expressions of dy, dz, [230t], which, by this means, become $dy=-nz.dt$, $dz=ny.dt$. Substitute these in $dy=v\,dt$, $dz=\mathrm{v}\,dt$, [305b], and we shall get $v=-nz$, $\mathrm{v}=ny$, [320″].

[321a] † (209) The values $u=0$, $v=-nz$, $\mathrm{v}=ny$, [320″], give $\left(\frac{dv}{dz}\right)=-n$, $\left(\frac{d\mathrm{v}}{dy}\right)=n$, and all the other partial differentials, which occur in the second member of [306] vanish. This equation will therefore become

$$\delta V-\frac{\delta p}{\rho}=-\delta y.n\mathrm{v}+\delta z.nv=-n^2.\{y\,\delta y+z\,\delta z\},$$

as in [321].

‡ (210) In the equation [310], developed as in [310a], substitute $\left(\frac{du}{dx}\right)=0$, $\left(\frac{dv}{dy}\right)=0$, $\left(\frac{d\mathrm{v}}{dz}\right)=0$, [321$a$], and multiply by dt; in this manner we shall obtain the equation [322].

and it is evident that this equation will be satisfied, if the fluid mass be homogeneous. The equations of the motion of the fluid will then be satisfied, consequently this motion will be possible.

The centrifugal force at the distance $\sqrt{y^2+z^2}$ from the axis of rotation, is equal [54'] to the square of the velocity $n^2.(y^2+z^2)$ divided by this [322'] distance;* the function $n^2.(y\,\delta y+z\,\delta z)$ is therefore the product of the centrifugal force by the element of its direction; therefore by comparing the preceding equation of the motion of the fluid with the general equation of the equilibrium of fluids, given in § 17 [133]; we find that the conditions of the motion now treated of, are reduced to those of the equilibrium of a [322''] fluid mass, urged by the same forces, and by the centrifugal force arising from the rotation, which is evident from other considerations.

If the external surface of the fluid mass is free, we shall have $\delta p=0$, at this surface, consequently

$$0=\delta V+n^2.(y\,\delta y+z\,\delta z); \qquad [323]$$

whence it follows that the resultant of all the forces acting on each particle of the external surface, must be perpendicular to that surface;† it ought also to be directed towards the interior of the fluid mass. These conditions [323'] being satisfied, a mass of homogeneous fluid will be in equilibrium, even if we suppose it to cover a solid body of any figure whatever.

The case just examined is one of those in which the variation

$$u\,\delta x+v\,\delta y+\mathrm{v}\,\delta z$$

This equation will be satisfied if the fluid be homogeneous, or $\rho=$ constant, because all its partial differentials [322], would vanish. The equations [321, 322] being satisfied, the motion will be possible without any internal change in the situation of the particles.

[322a] * (211) Let r be the distance of a particle of the fluid from the axis of z, we shall have $r^2=y^2+z^2$, whose variation gives $r\,\delta r=y\,\delta y+z\,\delta z$. The centrifugal force of this particle is $n^2 r$, [138a], this being multiplied by the element of the direction δr, becomes $n^2.r\,\delta r$, or $n^2.(y\,\delta y+z\,\delta z)$, as in [322']. Multiplying this by the density ρ, and adding it to the second member of the equation of equilibrium [133], we get, by using δV, [295'], the same expression as in [321].

† (211a) This is proved by reasons similar to those in note 64. The equation of the surface of a fluid, [323], having a rotatory motion about the axis x, would agree with the [322b] result of a calculation made in note 65a, page 95, by a different method.

[323''] is not exact; for this variation becomes* $-n.(z\delta y - y\delta z)$; therefore in the theory of the tides we cannot suppose that variation to be exact; since it is not so in this very simple case, in which the sea has no other motion than that of rotation common both to the earth and sea.

35. Let us now determine the oscillations of a fluid mass, surrounding a spheroid, having a motion of rotation nt about the axis of x; *supposing it to be deranged but very little from the state of equilibrium, by the action of very*
[323'''] *small forces.* At the commencement of the motion, let r be the distance of a particle of the fluid from the centre of gravity of the spheroid which it
[323$^{\mathrm{iv}}$] covers. This centre of gravity we shall suppose to be at rest. Put θ for the angle which the radius r makes with the axis of x; and ϖ for the angle which the plane passing through the axis of x and this radius, makes with the plane of x, y.† Suppose that at the end of the time t the radius r

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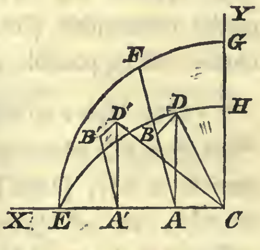
* (212) Substitute in $u\,\delta x + v\,\delta y + \mathrm{v}\,\delta z$, the values [321$a$,] $u=0$, $v=-nz$, $\mathrm{v}=ny$, and it becomes $-nz\delta y+nydz$. Now, upon the principles explained in note 61, it appears that if $P\delta y + R\,\delta z$, is an exact variation of a function of y, z, we ought to have

[322c] $$\left(\frac{dP}{dz}\right)=\left(\frac{dR}{dy}\right).$$

In the present example $P=-nz$, $R=ny$, and as n is constant, $\left(\frac{dP}{dz}\right)=-n$, $\left(\frac{dR}{dy}\right)=n$; and as these quantities are not equal, the expression $-nz\,\delta y + ny\,\delta z$, is not an exact variation.

† (213) To illustrate this we may refer to the annexed figure, in which C is the centre of gravity of the spheroid, CX the axis of x, CY the axis of y; the axis of z being drawn through C, perpendicular to the plane of the figure. Suppose a particle of the
[323a] fluid, whose motion is to be considered, to be at the point D', when $t=0$; and at the point D, when $t=t$. Draw $D'B'$, DB, perpendicular to the plane of the figure x, y, and $B'A'$, BA, perpendicular to the axis CX. Then the co-ordinates of the proposed particle, when $t=0$, will be, $CD'=r$, angle $D'CE'=\theta$, $D'A'B'=\varpi$; and when $t=t$, they will become $CD=r+\alpha s$, $DCE=\theta+\alpha u$, $DAB=nt+\varpi+\alpha v$, or $CA=x$, $AB=y$, $BD=z$. Now $CA=CD.\cos.DCE$; $AD=CD.\sin.DCE$; $AB=AD.\cos.DAB$; $BD=AD.\sin.DAB$; whence



becomes $r+\alpha s$, θ becomes $\theta+\alpha u$, and ϖ becomes $nt+\varpi+\alpha v$; αs, αu, and αv being very small quantities, whose squares and products we shall neglect; we shall have $[323^{v}]$

$$x=(r+\alpha s)\,.\,\cos.\,(\theta+\alpha u)\,;$$
$$y=(r+\alpha s)\,.\,\sin.\,(\theta+\alpha u)\,.\,\cos.\,(nt+\varpi+\alpha v)\,;$$
$$z=(r+\alpha s)\,.\,\sin.\,(\theta+\alpha u)\,.\,\sin.\,(nt+\varpi+\alpha v)\,;$$
[324]

If we substitute these values in the equation (F) § 32 [296], we shall have, by neglecting the square of α,*

$$AB=CD\,.\,\sin.\,DCE\,.\,\cos.\,DAB\,;\qquad BD=CD\,.\,\sin.\,DCE\,.\,\sin.\,DAB\,;$$ [323b]
which, by substituting the values of CD, DCE, DAB, give x, y, z, as in [324].

It may be observed that αu represents nearly the motion of the particle in latitude, and αv its motion in longitude [347'''], from a meridian of the earth, which has the angular motion nt in the time t.

* (214) In the notation here used, the quantities α and n are constant; r, θ, ϖ, take the place of a, b, c, [305a], and are constant in the differentials relative to the characteristic d; but s, u, v, are variable, and take the place of the quantities x, y, z, [305a]. The characteristic δ, as usual, does not affect t, [294']. Put now for brevity, [323c]

$$\rho=(r+\alpha s)\,.\,\sin.\,(\theta+\alpha u)\qquad \tau=nt+\varpi+\alpha v,$$ [324a]

and the expression of y, z, [324] will become,

$$y=\rho\,.\,\cos.\,\tau,\qquad z=\rho\,.\,\sin.\,\tau.$$ [324b]

The variation of z, and its second differential being taken, we shall find,

$$\delta z=\delta\rho\,.\,\sin.\,\tau+\rho\,\delta\tau\,.\,\cos.\,\tau,$$
$$ddz=(dd\rho-\rho\,d\tau^2)\,.\,\sin.\,\tau+(2\,d\rho\,.\,d\tau+\rho\,.\,dd\tau)\,.\,\cos.\,\tau.$$ [324c]

Multiply these two expressions together, and put A for the coefficient of $\sin.\,\tau\,.\,\cos.\,\tau$, in the product, we shall get,

$$\delta z\,.\,ddz=(dd\rho-\rho\,.\,d\tau^2)\,.\,\delta\rho\,.\,\sin.^2\tau+(2\,.\,d\rho\,.\,d\tau+\rho\,.\,dd\tau)\,.\,\rho\,.\,\delta\tau\,.\,\cos.^2\tau+A\,.\,\sin.\,\tau\,.\,\cos.\,\tau.$$ [324d]

From this we can easily obtain $\delta y\,.\,ddy$, by putting $\tfrac{1}{2}\pi+\tau$ for τ, $\tfrac{1}{2}\pi$ being a right angle. This changes $\sin.\,\tau$ into $\cos.\,\tau$, and $\cos.\,\tau$ into $-\sin.\,\tau$. By this means z changes into y, [324b], and the preceding expression [324d], becomes

$$\delta y\,.\,ddy=(dd\rho-\rho\,.\,d\tau^2)\,.\,\delta\rho\,.\,\cos.^2\tau+(2\,.\,d\rho\,.\,d\tau+\rho\,.\,dd\tau)\,.\,\rho\,.\,\delta\tau\,.\,\sin.^2\tau-A\,.\,\cos.\,\tau\,.\,\sin.\,\tau.$$ [324e]

Add together [324d, e], and reduce by putting $\sin.^2\tau+\cos.^2\tau=1$, we get

$$\delta y\,.\,ddy+\delta z\,.\,ddz=(dd\rho-\rho\,.\,d\tau^2)\,.\,\delta\rho+(2\,.\,d\rho\,.\,d\tau+\rho\,.\,dd\tau)\,.\,\rho\,\delta\tau.$$ [324f]

If we now put

$$\rho'=r+\alpha s,\qquad \tau'=\theta+\alpha u,$$ [324g]

the values of x, ρ, [324, 324a], will become

$$x=\rho'\,.\,\cos.\,\tau',\qquad \rho=\rho'\,.\,\sin.\,\tau',$$ [324h]

[325]

General Equation for all parts of the Fluid, in Motion.

$$\begin{aligned}&\alpha r^2.\delta\theta.\left\{\left(\frac{d\,d\,u}{d\,t^2}\right)-2\,n\,.\,\sin.\,\theta\,.\,\cos.\,\theta\,.\left(\frac{d\,v}{dt}\right)\right\}\\&+\alpha\,.\,r^2.\,\delta\varpi.\left\{\sin.^2\theta.\left(\frac{d\,d\,v}{d\,t^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{d\,u}{dt}\right)+\frac{2\,n\,.\,\sin.^2\theta}{r}.\left(\frac{d\,s}{dt}\right)\right\};\quad(L)\\&+\alpha\,\delta\,r\,.\left\{\left(\frac{d\,d\,s}{d\,t^2}\right)-2\,n\,r\,.\,\sin.^2\theta\,.\left(\frac{d\,v}{dt}\right)\right\}\\&=\frac{n^2}{2}\,.\,\delta\,.\,\{(r+\alpha\,s)\,.\,\sin.\,(\theta+\alpha\,u)\}^2+\delta\,V-\frac{\delta\,p}{\rho}\end{aligned}$$

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which are similar to y, z, [324b], and may be derived from them by changing y into x, z into ρ, and accenting the letters ρ, ϖ. Making these changes in [324f], it will become, by this principle of derivation,

[324i] $$\delta x\,.\,d\,d\,x+\delta\rho\,.\,d\,d\,\rho=(d\,d\,\rho'-\rho'\,.\,d\,\varpi'^2)\,.\,\delta\,\rho'+(2\,.\,d\,\rho'\,.\,d\,\varpi'+\rho'\,.\,d\,d\,\varpi')\,.\,\rho'\,.\,\delta\,\varpi',$$

and since $d\,\varpi'=\alpha\,d\,u$, $d\,\rho'=\alpha\,d\,s$, we may neglect $d\,\varpi'^2$ and $d\,\rho'\,.\,d\,\varpi'$, which are of the order α^2; then adding [324i], thus corrected, to [324f], and rejecting the term $\delta\,\rho\,.\,d\,d\,\rho$, which occurs in both members, we shall get

[324k] $$\begin{aligned}\delta x\,.\,d\,d\,x+\delta y\,.\,d\,d\,y+\delta z\,.\,d\,d\,z=&-\rho\,d\,\varpi^2.\,\delta\,\rho+(2\,.\,d\,\rho\,.\,d\,\varpi+\rho\,.\,d\,d\,\varpi)\,.\,\rho\,.\,\delta\,\varpi\\&+d\,d\,\rho'\,.\,\delta\,\rho'+\rho'^2\,.\,d\,d\,\varpi'\,.\,\delta\,\varpi'.\end{aligned}$$

If we now suppose the differentials to refer to the time t only, the first member of this expression, being divided by $d\,t^2$, will be equal to the second member of [296], and it will therefore be equal $\delta\,V-\frac{\delta\,p}{\rho}$, and if we add to both members the variation of ρ^2, or $2\,\rho\,.\,\delta\,\rho$, multiplied by $\frac{1}{2}\,n^2$, the first member of the sum will be $\frac{n^2}{2}\,.\,\delta\,.\,(\rho^2)+\delta\,V-\frac{\delta\,p}{p}$, which is the same as the second member of [325], and the second member of this sum will be

[324l] $$\left\{n^2-\left(\frac{d\,\varpi}{d\,t}\right)^2\right\}.\,\rho\delta\rho+2\,.\,\rho\delta\varpi\,.\left(\frac{d\,\rho}{d\,t}\right).\left(\frac{d\,\varpi}{d\,t}\right)+\rho^2\,.\,\delta\varpi\,.\left(\frac{d\,d\,\varpi}{dt^2}\right)+\left(\frac{d\,d\,\rho'}{d\,t^2}\right).\,\delta\rho'+\rho'^2.\left(\frac{d\,d\,\varpi'}{d\,t^2}\right).\,\delta\varpi',$$

and it now remains to be proved that this is equal to the first member of [325].

If we neglect terms of the order α^2, we shall get from [324a, g], noticing the remarks in [323c],

[324m] $$\begin{aligned}&\left(\frac{d\,\rho}{d\,t}\right)=\alpha\,.\left(\frac{d\,s}{d\,t}\right).\,\sin.\,\theta+\alpha\,r\,.\,\cos.\,\theta\,.\left(\frac{d\,u}{d\,t}\right);\\&\left(\frac{d\,\varpi}{d\,t}\right)=n+\alpha\,.\left(\frac{d\,v}{d\,t}\right);\qquad\left(\frac{d\,d\,\varpi}{d\,t^2}\right)=\alpha\,.\left(\frac{d\,d\,v}{d\,t^2}\right);\\&\left(\frac{d\,d\,\rho'}{d\,t^2}\right)=\alpha\,.\left(\frac{d\,d\,s}{d\,t^2}\right);\qquad\left(\frac{d\,d\,\varpi'}{d\,t^2}\right)=\alpha\,.\left(\frac{d\,d\,u}{d\,t^2}\right);\end{aligned}$$

Substituting these in [324l], it becomes,

[324n] $$\begin{aligned}&-2\,\alpha\,n\,.\left(\frac{d\,v}{d\,t}\right).\,\rho\,\delta\,\rho+2\,\rho\,\delta\,\varpi\,.\,\alpha\,n\,.\left\{\left(\frac{d\,s}{d\,t}\right).\,\sin.\,\theta+r\,.\,\cos.\,\theta\,.\left(\frac{d\,u}{d\,t}\right)\right\}+\alpha\rho^2.\,\delta\varpi.\left(\frac{d\,d\,v}{d\,t^2}\right)\\&+\alpha\,.\,\delta\,\rho'\,.\left(\frac{d\,d\,s}{d\,t^2}\right)+\alpha\,\rho'^2\,.\,\delta\,\varpi'\,.\left(\frac{d\,d\,u}{d\,t^2}\right);\end{aligned}$$

at the external surface of the fluid, we have $\delta p = 0$; moreover, in the state of equilibrium, we have* [325′]

Equation of Equilibrium at the Surface.

$$0 = \frac{n^2}{2} \,.\, \delta \,.\, \{(r + \alpha s) \,.\, \sin.(\theta + \alpha u)\}^2 + (\delta V)\,; \qquad [326]$$

(δV) being the value of δV corresponding to this state. Suppose the fluid in question to be the sea; the variation (δV) will be the product of gravity multiplied by the element of its direction. Let g be the force of gravity, αy† the elevation of a particle of the fluid above its surface of equilibrium, which we shall consider as the true level of the sea. The variation (δV) will increase by this elevation in the state of motion, by the quantity‡ [326′] [326″] [326‴]

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and as every term of this expression is of the order α, we may neglect the terms of the order α in $\delta\rho$, $\delta\tau$, $\delta\rho'$, $\delta\tau'$, and we shall get from [324a, g], and [323c]

$$\begin{array}{ll} \rho = r \,.\, \sin.\,\theta, & \delta\rho = \delta r \,.\, \sin.\,\theta + r\,\delta\theta \,.\, \cos.\,\theta, \\ \rho' = r, & \delta\rho' = \delta r, \\ \delta\tau = \delta\varpi, & \delta\tau' = \delta\theta. \end{array} \qquad [324p]$$

Substituting these in [324n], it becomes,

$$\begin{array}{l} -2\alpha \,.\, n \,.\, \left(\frac{dv}{dt}\right) \,.\, \{r\,\delta r \,.\, \sin.^2\theta + r^2 \,.\, \delta\theta \,.\, \sin.\,\theta \,.\, \cos.\,\theta\} \\ + 2r \,.\, \sin.\,\theta \,.\, \delta\varpi \,.\, \alpha n \,.\, \left\{ \left(\frac{ds}{dt}\right) \,.\, \sin.\,\theta + r \,.\, \cos.\,\theta \,.\, \left(\frac{du}{dt}\right) \right\} \\ + \alpha \,.\, r^2 \,.\, \sin.^2\theta \,.\, \delta\varpi \,.\, \left(\frac{ddv}{dt^2}\right) + \alpha \,.\, \delta r \,.\, \left(\frac{dds}{dt^2}\right) + \alpha r^2 \,.\, \delta\theta \,.\, \left(\frac{ddu}{dt^2}\right), \end{array}$$

this, by connecting together the terms depending on $\delta\theta$, $\delta\varpi$, δr, becomes as in the first member of [325], agreeing with the above.

* (215) In the state of equilibrium, u, v, s, are constant, and their differentials relative to t are nothing, which makes the first member of the equation [325] vanish, and at the external surface $\delta p = 0$, and δV becomes (δV), [326′]: these being substituted in [325], it becomes as in [326].

† (215a) It may be observed that the quantity y is here wholly different from the rectangular co-ordinate y, [324], but as this is not used in the rest of the chapter, it cannot produce any ambiguity or mistake.

‡ (216) The function δV, [295′], represents the sum of all the forces acting upon a particle of the fluid, multiplied each by the element of its direction. These forces may be composed into one single force g', [16], acting in the direction of a line r'', which we may suppose to be drawn towards the origin of that force. This origin is very near to the origin

$-\alpha g \,.\, \delta y$; because gravity acts nearly in the direction of αy, towards the origin of that line. Then denoting by $\alpha\delta V'$, the part of δV depending on the new forces which in the state of motion, act on the particle, and which depend either on the changes in the attractions of the spheroid arising from that state, or on any external attractions ; we shall have at the surface [326iv]

$$\delta V = (\delta V) - \alpha g \,.\, \delta y + \alpha \,.\, \delta V'. \qquad [327]$$

The variation $\frac{n^2}{2} \,.\, \delta \,.\, \{(r + \alpha s) \,.\, \sin.\, (\theta + \alpha u)\}^2$ is increased by the quantity

[327′] $\alpha n^2 \,.\, \delta y \,.\, r : \sin.^2 \theta$, by means of the elevation of the particle of water above the level of the sea ; but this quantity may be neglected in comparison with

[327″] $-\alpha g \,.\, \delta y$, because the ratio $\frac{n^2 r}{g}$, of the centrifugal force at the equator, to

[327‴] gravity, is a very small fraction, equal to $\frac{1}{289}$.* Lastly, the radius r is very

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of the co-ordinates, or the centre of the earth ; so that the direction of the line r'', and that of the radius $r + \alpha s$, or r' [334], differ but very little, and the quantity g' is nearly equal to the gravity g at the earth's surface. Now from the formulas [295′, 16] we obtain

[327a] $\delta V = -g' \,.\, \delta r''$, the negative sign being prefixed, because the force g' tends to decrease r'', instead of increasing it, as is supposed in [295a].

The co-ordinates of the particle, upon the momentary surface of the sea, are at the end of the time t, represented by $r + \alpha s$, $\theta + \alpha u$, $nt + \varpi + \alpha v$, [323v], which may, for

[327b] brevity, be denoted by r', θ', ϖ', respectively, as in [334]. The corresponding co-ordinates at the point of the surface of equilibrium, treated of in [326″], will be $r' - \alpha y$, θ', ϖ', and if the same force g', acted at this point, and in the same direction, the formula [327a], would become for this point, $(\delta V) = -g' \,.\, \delta \,.\, (r' - \alpha y) = -g' \,.\, \delta r + \alpha g' \,.\, \delta y$. This, by substituting the value of δV, [327a], and in the very small terms multiplied by δy, putting g for g', becomes $(\delta V) = \delta V + \alpha g \,.\, \delta y$, or $\delta V = (\delta V) - \alpha g \,.\, \delta y$. To which must be added the quantity $\alpha\, \delta V'$, [326iv], depending on the difference in the direction and in the value of the force g', at the two points, arising from the change of situation of the attracting mass in the state of motion, and from the attraction of other bodies, as the sun and moon. By this means we finally get $\delta V = (\delta V) - \alpha g \,.\, \delta y + \alpha\, \delta V'$, as in [327].

* (217) For the sake of brevity, let the function,

$$\frac{n^2}{2} \,.\, \delta \,.\, \{(r + \alpha s) \,.\, \sin.\, (\theta + \alpha u)\}^2 \qquad \text{or} \qquad \frac{n^2}{2} \,.\, \delta \,.\, \{(r' \,.\, \sin.\, \theta')\}^2, \qquad [327c]$$

corresponding to the point of the surface of equilibrium, treated of in [327b], be represented by (δN) ; and the same function, at the corresponding point of the momentary surface, by $(\delta N) + \delta N$. Put M for the second member of the equation [325], in the state of nearly constant at the surface of the sea, because it differs but very little from a spherical surface; we may therefore suppose δr nothing. The equation (L) thus becomes, at the *surface of the sea*,* [327iv]

Equation at the momentary Surface of the Sea.

$$r^2.\delta\theta.\left\{\left(\frac{ddu}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{dv}{dt}\right)\right\}$$
$$+r^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{ddv}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du}{dt}\right)+\frac{2n.\sin.^2\theta}{r}.\left(\frac{ds}{dt}\right)\right\} \qquad [328]$$
$$=-g.\delta y+\delta V';$$

the variations δy and $\delta V'$ correspond to the two variable quantities θ and ϖ. [328′]

Let us now consider the equation relative to the continuity of the fluid. For this purpose, suppose at the origin of the motion, a rectangular parallelopiped

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motion, and we shall have, $\mathcal{M}=(\delta\mathcal{N})+\delta\mathcal{N}+\delta V-\frac{\delta p}{p}$. The same notation being used in [326], it becomes $0=(\delta\mathcal{N})+(\delta V)$, whence $(\delta V)=-(\delta\mathcal{N})$. Substitute this in [327], and we shall get $\delta V=-(\delta\mathcal{N})-ag.\delta y+\alpha\delta V'$; therefore the preceding value of $\mathcal{M}$ will become $\mathcal{M}=\delta\mathcal{N}-\frac{\delta p}{p}-ag.\delta y+\alpha\delta V'$. Now $\delta\mathcal{N}$ is the increment of the function $\frac{n^2}{2}.\delta.\{r'.\sin.\theta'\}^2=n^2.r'\delta r'.\sin.^2\theta'$, arising from the change of $r'-\alpha y$ into r', by which means, the variation $\delta r'$ is increased by the quantity $\alpha\delta y$; so that we shall have $\delta\mathcal{N}=\alpha n^2 r'.dy.\sin.^2\theta'$, [327′]. This, being compared with the term $-\alpha g.\delta y$, [327e], is of the order $\frac{n^2 r}{g}$; being of the same order as the centrifugal force [138a], is to gravity, or $\frac{1}{289}$ [1594a]. Therefore we may neglect $\delta\mathcal{N}$; and if we also put $\delta p=0$, as in [325′], the value of $\mathcal{M}$, [327e], will become, $\mathcal{M}=-\alpha g.\delta y+\alpha\delta V'$. [327d] [327e] [327f] [327g]

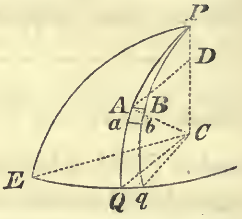
It may be observed, that the quantity $\frac{n^2}{2}.\delta.(r'.\sin.\theta')^2$, [327c], depends on the centrifugal force, [322′], and this force might have been included among the forces on which δV, [327a], depends, and it would then correspond to the whole force of gravity, g, acting in the direction r'', perpendicular to the surface of equilibrium; in which case the variation $\delta r''$, of the line of direction of that force, along the surface of equilibrium, would be nothing, [19a]. In this view of the subject we also perceive the propriety of neglecting the term [327′], depending on n^2.

* (218) The second member of the equation [325], represented by $\mathcal{M}$ in the preceding note, and reduced to the form [327g], is to be substituted in [325], neglecting δr in both members of the equations, on account of its smallness. Then dividing by the common factor α, the equation [325] will become of the form [328]. This last equation corresponds to the momentary surface of the sea.

[328''] whose height is $\mathrm{d}r$, width $r\mathrm{d}\varpi.\sin.\theta$, and length $r\mathrm{d}\theta$.* Let r', θ', ϖ', be the values of r, θ, ϖ, corresponding to the time t. Pursuing the same method of investigation as in § 32, we shall find, that at the end of that time, the magnitude of the fluid particle will be equal to a rectangular

[328'''] parallelopiped, whose height is $\left(\frac{dr'}{dr}\right).\mathrm{d}r$;† its width

[329] $$r'.\sin.\theta'.\left\{\left(\frac{d\varpi'}{d\varpi}\right).\mathrm{d}\varpi+\left(\frac{d\varpi'}{dr}\right).\mathrm{d}'r\right\},$$



* (219) The dimensions of this parallelopiped are found as in [275a], using the same figure, and changing R, s, p', into r, ϖ, θ, so that $CA=r$, $ECQ=\varpi$, $ACP=\theta$; from which we get $AB=r\,\mathrm{d}\varpi.\sin.\theta$, $Aa=r\,\mathrm{d}\theta$, and the height of the parallelopiped formed on the base $ABba$, is the other dimension $\mathrm{d}r$. These correspond to the commencement of the motion. At the end of the time t, the terms r, θ, ϖ, become r', θ', ϖ', [328'']. In the equations [329—331], the letter d is accented for the same reason as it was done in [299a, b]. [328a]

† (220) These dimensions are easily deduced from those of the parallelopiped (C), § 32, in the following manner. The dimensions of the parallelopiped (A) [297'], at the commencement of the motion, are $\mathrm{d}a$, $\mathrm{d}b$, $\mathrm{d}c$; these correspond to the rectangular elements $r\,\mathrm{d}\theta$, $r\,\mathrm{d}\varpi.\sin.\theta$, $\mathrm{d}r$, [328''], respectively. At the end of the time t, a, b, c, become x, y, z, [305a], and θ, ϖ, r, become θ', ϖ', r', [328'']. Now if, for the sake of brevity, we put $\rho=r.\sin.\theta$, $\rho'=r'.\sin.\theta'$, and follow, in every respect, the method of calculation detailed in [297'—303'''], it will evidently appear that we may change in all these equations,

[329a]
$$\begin{array}{lcccccc} & \mathrm{d}a, & \mathrm{d}b, & \mathrm{d}c, & \mathrm{d}x, & \mathrm{d}y, & \mathrm{d}z; \\ \text{into} & r\,\mathrm{d}\theta, & \rho\,\mathrm{d}\varpi, & \mathrm{d}r, & r'\,\mathrm{d}\theta', & \rho'\,\mathrm{d}\varpi', & \mathrm{d}r', \text{ respectively;} \end{array}$$

and by this means, we shall obtain the dimensions of the parallelopiped (C), in conformity with the present notation. *First*, The height corresponding to $\left(\frac{dz}{dc}\right).\mathrm{d}c$, [297$^{\mathrm{iv}}$], will become $\left(\frac{dr'}{dr}\right).\mathrm{d}r$, as in [328''']. *Second*, The width $\mathrm{d}y$, [300], deduced from the two equations [299], will, in the present case, be the value of $\rho'\,\mathrm{d}\varpi'$, deduced from the two following equations, which were obtained from [299], by changing the symbols, as in [329a],

[329b] $$\rho'\,\mathrm{d}\varpi'=\left(\frac{\rho'\,d\varpi'}{\rho\,d\varpi}\right).\rho\,\mathrm{d}\varpi+\left(\frac{\rho'\,d\varpi'}{dr}\right).\mathrm{d}'r;\qquad 0=\left(\frac{dr'}{\rho\,d\varpi}\right).\rho\,\mathrm{d}\varpi+\left(\frac{dr'}{dr}\right).\mathrm{d}'r,$$

and if we bring the quantities ρ, ρ', without the parentheses, they will become as in [329, 330].

exterminating $\mathrm{d}'r$ by means of the equation

$$0=\left(\frac{dr'}{d\varpi}\right).\mathrm{d}\varpi+\left(\frac{dr'}{dr}\right).\mathrm{d}'r; \qquad [330]$$

and whose length is $r'.\left\{\left(\frac{d\theta'}{dr}\right).\mathrm{d}_{,}r+\left(\frac{d\theta'}{d\theta}\right).\mathrm{d}\theta+\left(\frac{d\theta'}{d\varpi}\right).\mathrm{d}_{,}\varpi\right\}$, [330′]

exterminating $\mathrm{d}_{,}r$ and $\mathrm{d}_{,}\varpi$ by means of the equations

$$\begin{aligned} 0&=\left(\frac{dr'}{dr}\right).\mathrm{d}_{,}r+\left(\frac{dr'}{d\theta}\right).\mathrm{d}\theta+\left(\frac{dr'}{d\varpi}\right).\mathrm{d}_{,}\varpi; \\ 0&=\left(\frac{d\varpi'}{dr}\right).\mathrm{d}_{,}r+\left(\frac{d\varpi'}{d\theta}\right).\mathrm{d}\theta+\left(\frac{d\varpi'}{d\varpi}\right).\mathrm{d}_{,}\varpi. \end{aligned} \qquad [331]$$

Supposing therefore

$$\begin{aligned} \beta'=&\left(\frac{dr'}{dr}\right).\left(\frac{d\theta'}{d\theta}\right).\left(\frac{d\varpi'}{d\varpi}\right)-\left(\frac{dr'}{dr}\right).\left(\frac{d\theta'}{d\varpi}\right).\left(\frac{d\varpi'}{d\theta}\right)+\left(\frac{dr'}{d\theta}\right).\left(\frac{d\theta'}{d\varpi}\right).\left(\frac{d\varpi'}{dr}\right) \\ &-\left(\frac{dr'}{d\theta}\right).\left(\frac{d\theta'}{dr}\right).\left(\frac{d\varpi'}{d\varpi}\right)+\left(\frac{dr'}{d\varpi}\right).\left(\frac{d\theta'}{dr}\right).\left(\frac{d\varpi'}{d\theta}\right)-\left(\frac{dr'}{d\varpi}\right).\left(\frac{d\theta'}{d\theta}\right).\left(\frac{d\varpi'}{dr}\right); \end{aligned} \qquad [332]$$

the magnitude of the particle, at the end of the time t, will be*

$$\beta'.r'^{2}.\sin.\theta'.\mathrm{d}r.\mathrm{d}\theta.\mathrm{d}\varpi; \qquad [332']$$

therefore supposing the density of the fluid at the commencement to be (ρ), and at the end of the time t to be ρ; we shall have, by putting the expressions of the mass at these times equal to each other,†

General Equation of the continuity of a Fluid, first form.

$$\rho.\beta'r'^{2}.\sin.\theta'=(\rho).r^{2}.\sin.\theta; \qquad [333]$$

The value of $\mathrm{d}'r$, being found from [330], and substituted in [329], gives the required width. *Third*, The length $\mathrm{d}x$, deduced from the three equations [301], will become, in the present notation, equal to the value of $r'd\theta'$, deduced from the three following equations, by the elimination of $\mathrm{d}_{,}\varpi$, $\mathrm{d}_{,}r$,

$$\begin{aligned} r'\mathrm{d}\theta'&=\left(\frac{r'd\theta'}{rd\theta}\right).r\mathrm{d}\theta+\left(\frac{r'd\theta'}{\rho d\varpi}\right).\rho\mathrm{d}_{,}\varpi+\left(\frac{r'd\theta'}{dr}\right).\mathrm{d}_{,}r; \\ 0&=\left(\frac{\rho'd\varpi'}{rd\theta}\right).r\mathrm{d}\theta+\left(\frac{\rho'd\varpi'}{\rho d\varpi}\right).\rho\mathrm{d}_{,}\varpi+\left(\frac{\rho'd\varpi'}{dr}\right).\mathrm{d}_{,}r; \\ 0&=\left(\frac{dr'}{rd\theta}\right).r\mathrm{d}\theta+\left(\frac{dr'}{\rho d\varpi}\right).\rho\mathrm{d}_{,}\varpi+\left(\frac{dr'}{dr}\right).\mathrm{d}_{,}r. \end{aligned} \qquad [329c]$$

These, by reduction, become as in [330′, 331], changing the order of the two last equations.

* (220*a*) If in the value of β, [302], we make the same changes as in [329*a*], it will become $\beta=\frac{r'^{2}.\sin.\theta'}{r^{2}.\sin.\theta}.\beta'$, using β', [332]. The same changes being made in the magnitude $\beta.\mathrm{d}a.\mathrm{d}b.\mathrm{d}c$, [303′], it becomes, by reduction, as in [332′]. [329*d*]

† (220*b*) This is the same as [303‴], multiplied by $r^{2}.\sin.\theta$, substituting β, [329*d*].

which is the equation of the continuity of the fluid. In the present case, [323^{v}, 328$''$],

[334] $$r'=r+\alpha s\,; \qquad \theta'=\theta+\alpha u\,; \qquad \varpi'=nt+\varpi+\alpha v\,;$$

we shall therefore have,* by neglecting quantities of the order α^2

[335] $$\beta'=1+\alpha.\left(\frac{ds}{dr}\right)+\alpha.\left(\frac{du}{d\theta}\right)+\alpha.\left(\frac{dv}{d\varpi}\right).$$

Suppose that at the end of the time t, the original density (ρ) of the fluid [335$'$] becomes $(\rho)+\alpha\rho'$, the preceding equation of the continuity of the fluid will give†

Second form of the same general Equation.

[336] $$0=r^2.\left\{\rho'+(\rho).\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u\,.\cos.\theta}{\sin.\theta}\right\}\right\}+(\rho).\left(\frac{d\,.r^2s}{dr}\right).$$

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* (221) The values of r', θ', ϖ', [334], give

$$\left(\frac{dr'}{dr}\right)=1+\alpha.\left(\frac{ds}{dr}\right); \qquad \left(\frac{d\theta'}{d\theta}\right)=1+\alpha.\left(\frac{du}{d\theta}\right); \qquad \left(\frac{d\varpi'}{d\varpi}\right)=1+\alpha.\left(\frac{dv}{d\varpi}\right);$$

all the other terms of β' are of the order α, $\left(\frac{dr'}{d\theta}\right)=\alpha.\left(\frac{ds}{d\theta}\right)$; $\left(\frac{dr'}{d\varpi}\right)=\alpha.\left(\frac{ds}{d\varpi}\right)$, &c. Therefore by neglecting terms of the order α^2, the value of β', [332], will be reduced to its first term,

$$\left(\frac{dr'}{dr}\right).\left(\frac{d\theta'}{d\theta}\right).\left(\frac{d\varpi'}{d\varpi}\right)=\left\{1+\alpha.\left(\frac{ds}{dr}\right)\right\}.\left\{1+\alpha.\left(\frac{du}{d\theta}\right)\right\}.\left\{1+\alpha.\left(\frac{dv}{d\varpi}\right)\right\}$$
$$=1+\alpha.\left\{\left(\frac{ds}{dr}\right)+\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)\right\},$$

as in [335].

† (222) From r', θ', [334], we get $r'^2=r^2.\left(1+2\alpha.\frac{s}{r}\right)$,

$$\sin.\theta'=\sin.(\theta+\alpha u)=\sin.\theta+\alpha u\,.\cos.\theta,$$

(60 Int.), or $\frac{\sin.\theta'}{\sin.\theta}=\left(1+\alpha u\,.\frac{\cos.\theta}{\sin.\theta}\right)$. These values, and that of β', [335], being substituted in the equation of continuity, [333], put under the following form

$$\rho\,.\beta'\,.r'^2\,.\frac{\sin.\theta'}{\sin.\theta}-r^2\,.(\rho)=0, \qquad \text{it becomes,}$$

$$\left\{(\rho)+\alpha\rho'\right\}.\left\{1+\alpha.\left(\frac{ds}{dr}\right)+\alpha.\left(\frac{du}{d\theta}\right)+\alpha.\left(\frac{dv}{d\varpi}\right)\right\}.r^2.\left\{1+2\alpha.\frac{s}{r}\right\}.\left\{1+\alpha u\,.\frac{\cos.\theta}{\sin.\theta}\right\}-r^2(\rho)=0,$$

reducing and dividing by α, it becomes,

$$0=r^2\rho'+r^2.(\rho).\left\{\left(\frac{ds}{dr}\right)+\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{2s}{r}+u.\frac{\cos.\theta}{\sin.\theta}\right\},$$

which, by a slight reduction, is easily reduced to the form of the equation [336]. If the fluid be homogeneous, and $\rho=(\rho)$, we shall have $\rho'=0$, and the equation [336], divided by (ρ), will become as in [337].

36. Let us apply these results to the oscillations of the sea. Its mass being homogeneous, we shall have $\rho' = 0$; consequently [336]. [336′]

$$0 = \left(\frac{d \,.\, r^2 s}{d r}\right) + r^2 . \left\{ \left(\frac{d u}{d \theta}\right) + \left(\frac{d v}{d \varpi}\right) + \frac{u \,.\, \cos.\, \theta}{\sin.\, \theta} \right\}. \qquad [337]$$

Third general form for homogeneous Fluids.

Suppose, conformably to what appears to be the case, that the depth of the sea is very small in comparison with the radius r of the terrestrial spheroid; let us represent it by γ, γ being a very small function of θ and ϖ, depending on the law of the depth. [337′] If we integrate the preceding equation with respect to r, from the surface of the solid which the sea covers to the surface of the sea, ;* the value of s will be a function of θ, ϖ, and t, independent of r, increased by a small function, which will be, with respect to u or v, of the same order as the function $\frac{\gamma}{r}$; [337″] now at the surface of that solid, when the angles θ and ϖ become $\theta + \alpha u$, and $n t + \varpi + \alpha v$, it is evident that the distance from a particle of water contiguous to that surface, to the centre of gravity of the earth, varies but a very small quantity in comparison with αu, or αv, and that variation is of the same order as the product of those [337‴] quantities αu or αv by the eccentricity of the spheroid covered by the sea: the function independent of r, which occurs in the expression of s, is therefore a very small quantity of the same order ;† therefore we may in

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* (223) This method of integration will be more easily understood after reading the part included between [343—346]. In speaking of the order of the terms depending on r, γ, s, in [337″], and in other parts of this chapter, it will be convenient to refer all the linear measures to the mean radius of the earth, considered as unity, so that we may say indifferently [337a] either that a term is of the order $\frac{\gamma}{r}$, or of the order γ.

† (224) A particle of the fluid at the bottom of the sea, being supposed in its motion always to touch the solid spheroid, which is very nearly spherical; the value of αs for a particle so situated must be very small; being to αu or αv, of the order of the eccentricity of the spheroid, to its mean radius taken as unity. Now this function of θ, ϖ, t, added in [337″], to complete the integral s, being independent of r, must be the same, on all parts of the radius r, as it is at the bottom of the sea; and as we have just shown, that s varies but very little at the bottom of the sea, by changing θ into $\theta + \alpha u$, and ϖ into $\varpi + \alpha v$, it follows that the function of θ, ϖ, t, here treated of, must be very small, and of the order mentioned in [337″].

[337iv] general neglect s in comparison with u and v. The equation of the motion of the sea *at its surface*, given in § 35 [328], becomes by this means*

Equation of the Motion at the Surface of the Sea.

$$r^2.\delta\theta.\left\{\left(\frac{ddu}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{dv}{dt}\right)\right\} \qquad (M)$$

[338]
$$+r^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{ddv}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du}{dt}\right)\right\}=-g.\delta y+\delta V';$$

The equation (L) [325] of the same article relative to any point whatever of the *interior* of the fluid mass gives in the state of equilibrium†

[339]
$$0=\frac{n^2}{2}.\delta.\{(r+\alpha s).\sin.(\theta+\alpha u)\}^2+(\delta V)-\frac{(\delta p)}{\rho};$$

(δV) and (δp) being the values of δV and δp, which, in the state of equilibrium, correspond to the quantities $r+\alpha s$, $\theta+\alpha u$, and $\varpi+\alpha v$. [339′] Suppose that in the state of motion we have

[340]
$$\delta V=(\delta V)+\alpha\delta V'; \qquad \delta p=(\delta p)+\alpha\delta p';$$

the equation (L) [325] will give

[341]
$$\left\{\frac{d.\left(V'-\frac{p'}{\rho}\right)}{dr}\right\}=\left(\frac{dds}{dt^2}\right)-2nr.\sin.^2\theta.\left(\frac{dv}{dt}\right).$$

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* (225) This is the same as the equation [328], neglecting s, as in [337iv].

† (226) In the state of equilibrium, αs, αu, αv, must be constant, and their differentials relative to t are nothing, therefore the first member of [325] will vanish; and if we put, as in [339′], (δV), $\frac{(\delta p)}{\rho}$; for δV, $\frac{\delta p}{\rho}$; the second member will become as in [339]. This equation gives $\frac{n^2}{2}.\delta.\{(r+\alpha s).(\sin.\theta+\alpha u)\}^2=-(\delta V)+\frac{(dp)}{\rho}$, and by neglecting, as in [327′], the variation arising from $\alpha n^2\delta y$, on account of its smallness, we may substitute this in the second member of [325], which will make it

[339a]
$$-(\delta V)+\frac{(\delta p)}{p}+\delta V-\frac{\delta p}{\rho}.$$

This, by substituting the values of δV, δp, [340], becomes $+\alpha\delta V'-\alpha.\frac{\delta p'}{\rho}$, and the part of this expression relative to the *independent* variation δr, is

$$\alpha.\left\{\frac{d.\left(V'-\frac{p'}{\rho}\right)}{dr}\right\}.\delta r,$$

ρ being constant as in [336′]. This, being put equal to the term depending on $\alpha\delta r$, in the first member of [325], produces an equation, which, being divided by $\alpha\delta r$, becomes as in [341].

The equation (M) [338] shows that $n.\left(\frac{dv}{dt}\right)$ is of the same order as y or s,* consequently of the order $\frac{\gamma u}{r}$ [337″]; the value of the first member of [341′] this equation [341] is therefore of the same order; so that if we multiply this value by dr, and then integrate, from the surface of the spheroid which the sea covers, to the surface of the sea, we shall have $V'-\frac{p'}{\rho}$ equal to a very small function of the order $\frac{\gamma s}{r}$, increased by a function of θ, ϖ, t, independent of r, which we shall denote by λ;† considering therefore in the [341″]

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* (227) Making the coefficients of the independent variations $\delta\theta$, $\delta\varpi$, in the equation [338], separately equal to nothing, we shall get,

$$r^2.\left(\frac{ddu}{dt^2}\right)-2nr^2.\sin.\theta.\cos.\theta.\left(\frac{dv}{dt}\right)=-g.\left(\frac{dy}{d\theta}\right)+\left(\frac{dV'}{d\theta}\right);$$

$$r^2.\sin.^2\theta.\left(\frac{ddv}{dt^2}\right)+2nr^2.\sin.\theta.\cos.\theta.\left(\frac{du}{dt}\right)=-g.\left(\frac{dy}{d\varpi}\right)+\left(\frac{dV'}{d\varpi}\right).$$

Add the differential of this last relative to t, to the first equation, multiplied by $-2.n.\sin.\theta.\cos.\theta$, and let the second member of this sum be represented by $y'.r^2.\sin.\theta$. Then divide by $r^2.\sin.^2\theta$, and put for brevity $2n.\cos.\theta=a$, we shall get

$$\left(\frac{d^3v}{dt^3}\right)+a^2.\left(\frac{dv}{dt}\right)=y'.$$

This becomes of the form of the equation [865], by changing y into $\left(\frac{dv}{dt}\right)$, and αQ into $-y'$; and the value of y, [870], being multiplied by a or $2n.\cos.\theta$, will give,

$$2n.\cos.\theta.\left(\frac{dv}{dt}\right)=\sin.at.\int y'.dt.\cos.at-\cos.at.\int y'.dt.\sin.at;$$

the constant quantities, produced by the integration, being supposed to be included under the signs $\int$ of integration. The second member of this equation being of the order y', the first member, or $2n.\left(\frac{dv}{dt}\right)$, will be of the same order. But y' depends on y, V', and by note 231, we shall see that V' is of the order y, therefore y' is also of the order y; hence we finally perceive that $n.\left(\frac{dv}{dt}\right)$ is of the order y, as in [341′].

† (228) This integration is made as in [337′], and if we put $\varphi(r)$, for the small function of the order $\frac{\gamma s}{r}$, mentioned in [341″], we shall have $V'-\frac{p'}{\rho}=\lambda+\varphi(r)$. [341a]

equation (*L*) § 35 [325], only the two variable quantities θ and ϖ, it will be
[341'''] changed into the equation (*M*) [338], with this difference only, that the
second member will become $\delta\lambda$.* But λ being [341''] independent of the
depth at which the particle under consideration is found; if we suppose
this particle to be very near the surface, the equation (*L*) [325] ought
$[341^{iv}]$ evidently to coincide with the equation (*M*) [338]; therefore we shall have
$\delta\lambda=\delta V'-g.\delta y$; consequently†

[342]
$$\delta.\left\{V'-\frac{p'}{\rho}\right\}=\delta V'-g.\delta y\,;$$

the value of $\delta V'$ in the second member of this equation corresponds to the
[342'] surface of the sea. We shall see, in the theory of the tides, that this value
is nearly the same for all the particles situated on the same radius of
the earth, from the surface of the solid covered by the sea, to the surface

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* (229) The second member of [325] is in [339*a*], reduced to the form $\alpha\,\delta V'-\alpha.\frac{\delta p'}{\rho}$,
[341*b*] or $\alpha.\delta.\left(V'-\frac{p'}{\rho}\right)$, and this by means of [341*a*], becomes $\alpha.\delta\lambda+\alpha.\delta.\varphi(r)$. Therefore
if we consider only the parts of the equation [325], depending on the two variable quantities
θ, ϖ, as is directed in [341''], we shall get, by dividing by α,

General Equation for all parts of the Fluid.

[341c]
$$r^2.\delta\theta.\left\{\left(\frac{d\,d\,u}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{d\,v}{d\,t}\right)\right\}$$
$$+r^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{d\,d\,v}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du}{d\,t}\right)\right\}=\delta\lambda,$$

which is the same as the equation [338], changing its second member, $-g.\delta y+\delta V$, into
$\delta\lambda$; observing that the equation [341*c*] corresponds to any point in the *interior* of the fluid,
whereas, [338] refers only to its external surface. Now λ being a function of θ, ϖ, t,
independent of r, [341''], it must be the same upon any part of r, either at the surface of
the fluid, or below it, and it must therefore, in all cases, be equal to the value [338], so that
[341*d*] we shall have $\delta\lambda=-g.\delta y+\delta V'$, in which the values δy and $\delta V'$, correspond to
particles at the external surface of the fluid. This agrees with $[341^{iv}]$.

† (230) Noticing only the variations of θ, ϖ, we get, from [341*b*], $\delta\lambda=\delta.\left\{V'-\frac{p'}{\rho}\right\}$.
Substitute this in [341*d*], and it will become as in [342]. In the last member of the
equation [342], the quantities V', y, correspond to the *surface* of the sea; in the first
member, to any point of the *interior*, on the same radius.

of the sea;* therefore we shall have, relative to all these particles,
$\frac{\delta p'}{\rho} = g\,.\,\delta y$; which gives $p' = \rho g y$, increased by an arbitrary function, [342″]
independent of θ, ϖ and r;† now at the level surface of the sea, the value
of $\alpha p'$ is equal to the pressure of the small column of water αy, which rises [342‴]
above this surface, and this pressure is equal to $\alpha \rho\,.\,g y$; hence we shall have, in all the interior of the fluid mass, from the surface of the solid which the sea covers, to the level surface of the sea,

$$p' = \rho g y ;$$ [342iv]

therefore any point whatever of the surface of that solid is more pressed

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* (231) It is shown in Book IV, § 1, [2130′—2135″], that the value of $\alpha \delta V'$, consists of two parts, the one arising from the attraction of the sun and moon, &c. on any particle of the fluid; the other from the attraction on the same particle by an aqueous stratum, whose interior radius is r', and exterior $r' + \alpha y$; r' being the radius of the earth, corresponding to the state of equilibrium, and $r' + \alpha y$, that in the state of motion. The part $\alpha V'$, depending on the first of these forces, computed in [2134], is equal to

$$\frac{\alpha Z^{(0)}}{r} + \frac{\alpha Z^{(2)}}{r^3} + \frac{\alpha Z^{(3)}}{r^4} + \&c.,$$

in which r is taken for the distance of the sun or moon from the centre of the earth, the polar radius of the earth being unity; if the earth's radius had been put equal to r', corresponding to a stratum below the surface, the expression would have been

$$\frac{\alpha Z^{(0)}}{r} + \frac{\alpha r'^2\, Z^{(2)}}{r^3} + \frac{\alpha r'^3\, Z^{(3)}}{r^4} + \&c.,$$

and the part of V', depending on this, would be,

$$\frac{Z^{(0)}}{r} + \frac{r'^2}{r^3}\,.\,Z^{(2)} + \frac{r'^3}{r^4}\,.\,Z^{(3)} + \&c.,$$

which does not sensibly vary for all the particles situated on the same radius, from the surface to the bottom of the sea, because the variation of r' is only γ, which is very small in comparison with r', and $Z^{(0)}$, $Z^{(2)}$, &c. are independent of r'. The second of these forces, [2135″], arising from the attraction of the shell whose thickness is αy, is computed in [1501], which varies but very little, for all particles situated on the part γ of the radius, from the top to the bottom of the sea; the greatest variation being of the order $\frac{\gamma}{r'}\,.\,\alpha y$, in comparison with the whole attraction of the earth, and it may therefore be neglected. The author has given a short note on this subject at the end of Book XIII.

† (231*a*) This arbitrary function is added to complete the integral, and as usual, it must be independent of the variable quantities θ, ϖ, r.

than in the state of equilibrium, by all the weight of the small column of water, comprised between the surface of the sea and the level surface. This excess of pressure becomes negative, in those parts where the surface of the sea falls below the level surface.

[342$^{\text{v}}$] It follows from what we have said, that if we notice only the variations of θ and ϖ, the equation (L) [325], will change into the equation (M) [338], for all the particles in the interior of the fluid mass. The values of u and v, relative to all the particles of the sea, situated on the same radius of the [342$^{\text{vi}}$] earth, are therefore determined by the same differential equations: hence if we suppose, as we shall do in the theory of the tides, that at the commencement of the motion, the values of u, $\left(\frac{du}{dt}\right)$, v, $\left(\frac{dv}{dt}\right)$, were the same for all the particles situated in the same radius; these particles will [342$^{\text{vii}}$] remain in the same radius during the oscillations of the fluid.* The values of r, u, v, may therefore be supposed nearly the same on the small part of the radius of the earth, comprised between the solid covered by the sea, and the surface of the sea; therefore by integrating, with respect to r, the equation [337],

$$0=\left(\frac{d\,.r^2 s}{dr}\right)+r^2.\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u\,.\cos.\theta}{\sin.\theta}\right\}; \qquad [343]$$

we shall have†

$$0=r^2 s-(r^2 s)+r^2\gamma\,.\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u\,.\cos.\theta}{\sin.\theta}\right\}; \qquad [344]$$

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* (232) The quantities u, v, at the commencement of the motion, being supposed to change into u', v', at the end of the time dt; we should have, by the usual rules of the differential calculus, $u'=u+\left(\frac{du}{dt}\right).dt$, $v'=v+\left(\frac{dv}{dt}\right).dt$. Now u, v, [323$^{\text{v}}$], are the same for all particles situated upon the same radius; and if we suppose, as in [342$^{\text{vii}}$], that at the commencement of the motion $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, were the same for all these particles, we should have also u' and v' the same for all the particles, and these values of u', v', would therefore, [323$^{\text{v}}$], correspond to the same radius, at the end of the time dt; and for the like reasons, the same would happen at any successive instant.

† (233) Multiply [343] by dr, and integrate it, supposing

$$r^2.\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u\,.\cos.\theta}{\sin.\theta}\right\}$$

to be constant throughout the length γ, of the radius, we shall get [344]. The term $-(r^2 s)$,

$(r^2 s)$ being the value of $r^2 s$, at the surface of the spheroid covered by the sea. The function $r^2 s-(r^2 s)$ is nearly equal to $r^2.\{s-(s)\}+2r\gamma.(s)$; (s) being the value of s at the surface of the spheroid; we may neglect [344'] the term $2r\gamma.(s)$, on account of the smallness of γ and (s): we shall thus have

$$r^2 s-(r^2 s)=r^2.[s-(s)].$$ [345]

Now the depth of the sea corresponding to the angles $\theta+\alpha u$, and $nt+\varpi+\alpha v$, is* $\gamma+\alpha.\{s-(s)\}$; if we take the origin of the angles [345'] θ, and $nt+\varpi$, at a fixed point and a fixed meridian upon the surface of the earth, which may be done, as we shall soon show; this depth will be† $\gamma+\alpha u.\left(\frac{d\gamma}{d\theta}\right)+\alpha\varpi.\left(\frac{d\gamma}{d\varpi}\right)$, increased by the elevation αy of the fluid particle [345''] at the surface of the sea above its level; we shall therefore have

$$s-(s)=y+u.\left(\frac{d\gamma}{d\theta}\right)+v.\left(\frac{d\gamma}{d\varpi}\right).$$ [346]

The equation relative to the continuity of the fluid will therefore become‡

$$y=-\left(\frac{d.\gamma u}{d\theta}\right)-\left(\frac{d.\gamma v}{d\varpi}\right)-\frac{\gamma u.\cos.\theta}{\sin.\theta}. \qquad (N)$$ [347]

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by putting $r-\gamma$ for the value of r, at the bottom of the sea, becomes $-r^2(s)+2r\gamma.(s)$, neglecting γ^2, as in [344'.]

* (234) The depth γ corresponds to the angles θ and ϖ or $nt+\varpi$, and when these increase by αu, αv, the elevation at the upper surface increases by αs, [323$^{\text{v}}$], and at the bottom by $\alpha(s)$, [344'], hence the whole depth becomes $\gamma+\alpha s-\alpha(s)$.

† (235) γ is a function of θ, ϖ, corresponding to the surface of equilibrium, and these quantities, in the state of motion, become $\theta+\alpha u$, $\varpi+\alpha v$. Developing this function according to the powers of αu, αv, neglecting their squares and products, on account of their smallness, it becomes as in [345'']; this depth is to be augmented by the elevation αy, of the particle above its natural level. The expression thus found, being put equal to the former, $\gamma+\alpha.\{s-(s)\}$, [345'], it becomes, by neglecting γ, common to both sides of the equation, and dividing by α, the same as in [346].

‡ (235a) Substitute [345] in [344], divide by r^2, and add the equation [346], we shall get, by rejecting $s-(s)$, from both members of the equation, the following expression,

$$0=y+u.\left(\frac{d\gamma}{d\theta}\right)+v.\left(\frac{d\gamma}{d\varpi}\right)+\gamma.\left\{\left(\frac{du}{d\theta}\right)+\left(\frac{dv}{d\varpi}\right)+\frac{u.\cos.\theta}{\sin.\theta}\right\};$$

We may observe, that in this equation, the angles θ and $nt+\varpi$ are reckoned
[347'] from a fixed point and from a fixed meridian upon the earth, and that in the
equation (M) [338] these angles are reckoned relative to the axis of x, and
to a plane which, passing through that axis, will have about it a rotatory
motion equal to n; now this axis and this plane are not fixed on the surface
of the earth, because the attraction and the pressure of the incumbent fluid,
ought to alter a little their position upon this surface, and vary a little the
[347''] rotatory motion of the spheroid.* But it is easy to perceive, that these
alterations are to the values of αu, αv, in the ratio of the mass of the sea to
that of the terrestrial spheroid; therefore, in order to refer the angles θ and
$nt+\varpi$, to a fixed point and meridian on the surface of the spheroid, in the
two equations (M) [338], and (N) [347]; it is only necessary to vary u
and v by quantities of the order $\frac{\gamma u}{r}$ and $\frac{\gamma v}{r}$, which may be neglected;
[347'''] we may therefore suppose, in these equations, that *αu and αv are the motions
of the fluid in latitude and longitude.*

We may also observe, that the centre of gravity of the spheroid being
supposed immoveable [323$^{\text{iv}}$], we must transfer to the particles of the fluid,
[347$^{\text{iv}}$] in an opposite direction, the forces with which that centre is urged by the

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but $u.\left(\frac{d\gamma}{d\theta}\right)+\gamma.\left(\frac{du}{d\theta}\right)=\left(\frac{d.\gamma u}{d\theta}\right)$; and $v.\left(\frac{d\gamma}{d\varpi}\right)+\gamma.\left(\frac{dv}{d\varpi}\right)=\left(\frac{d.\gamma v}{d\varpi}\right)$, hence by substitution and reduction, we get, $y=-\left(\frac{d.\gamma u}{d\theta}\right)-\left(\frac{d.\gamma v}{d\varpi}\right)-\frac{\gamma u.\cos.\theta}{\sin.\theta}$, as in [347].

* (236) The co-ordinates θ, ϖ, of a particle, at the beginning of the motion, become $\theta+\alpha u$, and $nt+\varpi+\alpha v$, at the end of the time t, [323$^{\text{v}}$]; and the rotatory velocity of the particle instead of being n, will be $n+\alpha.\left(\frac{dv}{dt}\right)$, as evidently appears, by taking the differential of the angle $nt+\varpi+\alpha v$, relative to t. This change of velocity $\alpha.\left(\frac{dv}{dt}\right)$, would produce in the whole fluid mass quantities of the order $\alpha.\frac{\gamma}{r}.\left(\frac{dv}{dt}\right)$, when compared with the motion of the whole spheroid; and we may neglect such quantities as in [347'']. A similar variation arises from the change of θ into $\theta+\alpha u$. It may be remarked that instead of saying, as above, [347'], that the angle $nt+\varpi$, is reckoned from a fixed meridian on the earth, it would be more correct to neglect nt, and call the angle simply ϖ, but this does not affect the reasoning.

reaction of the sea; but the common centre of gravity of the spheroid and the sea does not change its situation by this reaction; it is therefore evident that the ratio of these forces to those with which the particles are urged by the action of the spheroid, is of the same order as the ratio of the mass of fluid to that of the spheroid; consequently of the order $\frac{\gamma}{r}$; we [347ᵛ]
may therefore neglect them in the calculation of $\delta V'$.

37. Let us now consider in the same manner the motions of the atmosphere. [347ᵛⁱ] In this research, we shall neglect the consideration of the variation of heat, in different latitudes and at different heights, as well as all the irregular causes of agitation, and we shall only notice the regular causes which act upon it as upon the ocean. We shall therefore suppose the sea to be surrounded by an elastic fluid of uniform temperature; and we shall also suppose, conformably to observation, that its density is proportional [347ᵛⁱⁱ]
to the pressure. This supposition makes the height of the atmosphere infinite; but it is easy to prove that at a very moderate elevation, its density is so small that it may be regarded as nothing.*

This being premised, we shall put s', u', v', to denote, for the particles of [347ᵛⁱⁱⁱ]
the atmosphere, the quantities which were named s, u, v, for the particles of the ocean [323ᵛ]; the equation (L) § 35 [325], will become

General Equation for all parts of the Atmosphere.

$$\begin{aligned}
&\alpha r^2.\delta\theta.\left\{\left(\frac{ddu'}{dt^2}\right)-2n.\sin.\theta.\cos.\theta.\left(\frac{dv'}{dt}\right)\right\}\\
&+\alpha r^2.\delta\varpi.\left\{\sin.^2\theta.\left(\frac{ddv'}{dt^2}\right)+2n.\sin.\theta.\cos.\theta.\left(\frac{du'}{dt}\right)+\frac{2n.\sin.^2\theta}{r}.\left(\frac{ds'}{dt}\right)\right\}\\
&+\alpha.\delta r.\left\{\left(\frac{dds'}{dt^2}\right)-2nr.\sin.^2\theta.\left(\frac{dv'}{dt}\right)\right\}\\
&\qquad=\frac{n^2}{2}.\delta.\{(r+\alpha s').\sin.(\theta+\alpha u')\}^2+\delta V-\frac{\delta p}{\rho}.
\end{aligned}\qquad [348]$$

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* (237) The formula for computing the density ρ, [355], rejecting the term $\frac{r'}{R'}$, on account of its smallness, and putting $g=g'$, [349′, 353′], becomes $\rho=\Pi.c^{-\frac{r'}{l}}$. Now if, for an example, we suppose r' to be equal to $10l$, which, as will be seen in the next note, is about 55 English miles, it will become $\rho=\Pi.c^{-10}$; and as $c=2.71828$, we shall find $\rho=\frac{\Pi}{22026}$, which is extremely small, as is observed above. [348a]

[348′] Let us at first consider the atmosphere in a state of equilibrium, in which s', u', and v' are nothing. The preceding equation, in that case, will give by integration

General Equation of Equilibrium.

[349] $$\frac{n^2}{2}\cdot r^2\cdot\sin.^2\theta+V-\int\frac{\delta p}{\rho}=\text{constant}.$$

The pressure p being supposed proportional to the density, we shall put

[349′] $$p=l\cdot g\cdot\rho\,;$$

g being the force of gravity at a given place, which we shall suppose to be the equator, and l a constant quantity, denoting the height of a homogeneous atmosphere of the same density as at the surface of the sea; this height is very small in comparison with the radius of the earth, being less than

[349″] $\frac{1}{720}$th part.*

[349‴] The integral† $\int\frac{\delta p}{\rho}$ is equal to $lg\,.\log.\rho$; therefore the preceding equation of the equilibrium of the atmosphere will become

Second form of the General Equation of Equilibrium.

[350] $$lg\,.\log.\rho=\text{constant}+V+\frac{n^2}{2}\cdot r^2\cdot\sin.^2\theta.$$

[350′] At the surface of the sea, the value of V is the same for a particle of air, as for the particle of water contiguous to it, because the forces acting on both are the same; but the conditions of the equilibrium of the sea, require that we should have,‡

[351] $$V+\frac{n^2}{2}\cdot r^2\cdot\sin.^2\theta=\text{constant};$$

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* (238) By Book X, § 9, [8500], the general expression of l, corresponding to the temperature of x degrees of the centigrade thermometer, is $7974^{met.}\cdot\{1+0.00375.x\}$, and if, for example, we take $x=29$, it will become equal to 8841 metres, or 5½ English miles, nearly, which is about $\frac{1}{720}$ part of the earth's radius.

† (239) From $\delta p=lg\,.\,\delta\rho$, [349′], we get $\int\frac{\delta p}{\rho}=lg\,.\int\frac{\delta\rho}{\rho}=lg\,.\log.\rho$. This substituted in [349], gives [350].

‡ (240) The equation [326] corresponds to the surface of the sea. In the case of equilibrium, and when $\alpha u=0$, $\alpha v=0$, it becomes $0=\frac{n^2}{2}\cdot\delta\cdot(r^2\cdot\sin.^2\theta)+\delta V$; its integral relative to δ is as in [351]. This, being substituted in [350], we obtain $lg\,.\log.\rho=\text{constant}$, whence $\rho=\text{constant}$.

we have therefore, at this surface, ρ constant; consequently the density of the stratum of air contiguous to the sea, is constant in the state of equilibrium. [351′]

[351″] If we put R for the part of the radius r, comprised between the centre of the spheroid and the surface of the sea, and r' the part included between this surface and a particle of air elevated above it; r' will be the height of this particle above the surface of the sea, neglecting quantities of the order* $\frac{\left(\frac{n^2}{g}\cdot r'\right)^2}{R}$, and quantities of this order we shall neglect. The equation between ρ and r will give† [351‴]

Third general form of the Equation of Equilibrium.

$$lg\,.\log.\rho=\text{constant}+V+r'.\left(\frac{dV}{dr}\right)+\frac{r'^2}{2}.\left(\frac{ddV}{dr^2}\right)+\frac{n^2}{2}R^2.\sin.^2\theta+n^2Rr'.\sin.^2\theta; \qquad [352]$$

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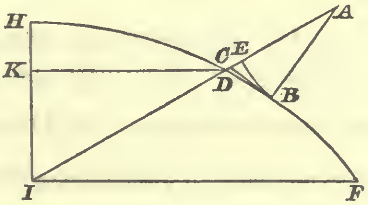
* (241) Let $FBDH$ be a meridian of the earth considered as an ellipsoid of revolution, whose semi-axis is HI, equal to unity, centre I, equatorial semi-axis IF. A the place of a particle of air; $ID=R$, $DA=r'$, $AI=r$; AB the perpendicular let fall from A upon the surface at B, BC a tangent to the meridian touching it in B, BE an arch of a circle described about the centre A, meeting IA in E; then by § 25 of Book III, [1648], the ellipticity of the meridian, is proportional to the centrifugal force, consequently the angle BAD is of the same order as the ratio of the centrifugal force to gravity, or of the order $\frac{n^2r}{g}$, [327″], or $\frac{n^2}{g}$, [337*a*], therefore BD or BC is of the order $\frac{n^2}{g}\cdot r'$. But $EC=\frac{BC^2}{2.AB}$, nearly, and CD is nearly equal to the square of BC divided by twice the radius of curvature of the meridian at B, and is therefore of the same order as the square of BC^2, divided by $2R$. The sum of CE, CD, expresses the difference between AB and r'. The first of these quantities is of the order $\frac{\left(\frac{n^2}{g}\cdot r'\right)^2}{2r'}$, the other is of the order $\frac{\left(\frac{n^2}{g}\cdot r'\right)^2}{2R}$. This last term is that mentioned by the author; the other is larger, but even this is very small, being in comparison with r' of the order $\frac{n^4}{2g^2}$, or $\frac{1}{2.289^2}$, [327″].



† (242) Supposing the values of V, $\left(\frac{dV}{dr}\right)$, $\left(\frac{ddV}{dr^2}\right)$, to correspond to the surface of the sea, or to the distance R from the centre, the general value of V, at the distance

the values of V, $\left(\frac{d\,V}{d\,r}\right)$ and $\left(\frac{d\,d\,V}{d\,r^2}\right)$ *being taken to correspond to the surface of the sea*, where we have*

[353] $$\text{constant} = V + \frac{n^2}{2}\,.\,R^2.\sin.^2\theta\,;$$

[353′] the quantity $-\left(\frac{d\,V}{d\,r}\right) - n^2 R\,.\sin.^2\theta$ is the gravity at the same surface;†

$R + r'$, θ and ϖ remaining unaltered, will become by development, by the formulas [607, 608], $V + r'\,.\left(\frac{d\,V}{d\,r}\right) + \frac{1}{2}\,.\,r'^2.\left(\frac{d\,d\,V}{d\,r^2}\right) + \&c.$ This is to be substituted for V, and $R + r'$, for r, in [350], neglecting r'^2, on account of the smallness of the term $n^2.r'^2$; we shall thus obtain [352].

* (243) This is the same as [351], substituting R for r, as in [351″], to make it correspond to the surface of the sea.

† (244) The function V represents the integral of the sum of the products, formed by multiplying each force acting on the particle, by the element of its direction, [295′]. It is [352a] similar to the function φ, [16, 17], and in the same manner as $\left(\frac{\delta\varphi}{\delta x}\right)$ was proved in [17], to be the resultant of all the forces resolved in a direction parallel to x, we shall find that $\left(\frac{d\,V}{d\,r}\right)$ is the resultant of the forces P, Q, R, acting on the particle, resolved in the direction r. Again, by drawing, in the preceding figure, DK perpendicular to the axis, we shall have nearly, $DK = R\,.\sin.\theta$, and the centrifugal force in the direction KD is, by [138a], $n^2\,.\,DK = n^2.R\,.\sin.\theta$. This resolved, in the direction ID, is nearly

[352b] $$n^2.R\,.\sin.\theta \times \frac{DK}{ID}, \quad\text{or}\quad n^2.R\,.\sin.^2\theta\,;$$

therefore the whole force acting at D, in the direction ID, is $\left(\frac{d\,V}{d\,r}\right) + n^2.R\,..\sin.^2\theta$, this force being supposed to tend to increase ID as in note 190*a*; now this ought to be equal to gravity $-g'$, [353″], the sign $-$ being prefixed, because gravity acts in the opposite [353a] direction DI nearly; hence $g' = -\left(\frac{d\,V}{d\,r}\right) - n^2.R\,.\sin.^2\theta$, as in [353′]. The last term of the second member being much smaller than the other, we have nearly $\left(\frac{d\,V}{d\,r}\right) = -g'$, whence $\left(\frac{d\,d\,V}{d\,r^2}\right) = -\left(\frac{d\,g'}{d\,r}\right)$.

In [470], it will be shown that for a sphere $g' = \frac{m}{r^2}$, r being the radius of the sphere, [353b] and m its mass; therefore $\left(\frac{d\,g'}{d\,r}\right) = -\frac{2m}{r^3}$, nearly; hence $\left(\frac{d\,d\,V}{d\,r^2}\right) = \frac{2m}{r^3} = \frac{2g'}{r}$, and this

we shall denote it by g'. The function $\left(\frac{ddV}{dr^2}\right)$ being multiplied by the [353″]
very small quantity r'^2, we may determine it on the supposition that the earth is spherical, and neglect the density of the atmosphere relative to that of the earth; we shall therefore have very nearly

$$-\left(\frac{dV}{dr}\right)=g=\frac{m}{R^2}; \qquad [354]$$

m being the mass of the earth; therefore $\left(\frac{ddV}{dr^2}\right)=\frac{2m}{R^3}=\frac{2g'}{R}$; we shall
therefore have* $lg.\log.\rho=\text{constant}-r'g'+\frac{r'^2}{R}.g'$; whence we deduce [354′]

$$\rho=\Pi.c^{-\frac{r'g'}{lg}.\left(1-\frac{r'}{R}\right)}, \qquad [355]$$

c being the number whose hyperbolic logarithm is unity, and Π a constant [355′]
quantity, which is evidently equal to the density of the air at the surface
of the sea.† Put h and h' for the lengths of pendulums vibrating in a second
at the surface of the sea, at the equator, and in the latitude of the particle [355″]

Equation of the Density of the Atmosphere.

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at the surface, where $r=R$, becomes $\frac{2g}{R}$, agreeing with the author, in the original work, except in the sign, which is changed in the present translation, and likewise the signs of the term $\frac{r}{R}$, in the equations [355, 356], which required the same correction.

[*] (245) Substituting in [352], the value of $V+\frac{n^2}{2}.R^2.\sin.^2\theta$, [353]; also for $+r'\left(\frac{dV}{dr}\right)+n^2.Rr'.\sin.^2\theta$, its value $-g'r'$, [353*a*], and for $\left(\frac{ddV}{dr^2}\right)$, its value $\frac{2g'}{R}$, [353*b*], it will become $lg.\log.\rho=\text{constant}-r'g'+\frac{r'^2}{R}.g'$. Divide this by lg, and put the constant quantity equal to $lg.\log.\Pi$, we shall get $\log.\rho=\log.\Pi-\frac{r'g'}{lg}.\left(1-\frac{r'}{R}\right)$, which is easily reduced to the form [355].

[†] (246) Because when $r'=0$, we shall have from [355], $\rho=\Pi$.

[355‴] of air under consideration; we shall have $\frac{g'}{g}=\frac{h'}{h}$, consequently*

[356]
$$\rho=\Pi\,.\,c^{-\frac{r'h'}{lh}\,.\,\left(1-\frac{r'}{R}\right)}.$$

This expression of the density of the air, shows that a stratum of the same density is in all parts equally elevated above the sea, neglecting the quantity† [356′] $\frac{r'\,.\,(h'-h)}{h}$; but in the exact calculation of the heights of mountains, by observations with a barometer, this quantity ought not to be neglected.

We shall now consider the atmosphere in a state of motion; and shall determine the oscillations of a level surface, or surface of the same density, [356″] in the state of equilibrium. Let $\alpha\varphi$ *be the elevation of a particle of air above the level surface to which it appertains in the state of equilibrium*; it is evident that by means of this elevation, the value of δV will be increased by [356‴] the variation $-\alpha g\,.\,\delta\varphi$; therefore we shall have $\delta V=(\delta V)-\alpha g\,.\,\delta\varphi+\alpha\delta V'$;‡ (δV) being the value of δV, which in the state of equilibrium corresponds to the level surface, and to the angles $\theta+\alpha u'$ and $nt+\varpi+\alpha v'$; and $\delta V'$ being the part of δV, arising from the new forces which in a state of motion agitate the atmosphere.

Let $\rho=(\rho)+\alpha\rho'$, (ρ) being the density of the level surface, in the state of [356$^{\text{iv}}$] equilibrium. If we make $\frac{l\rho'}{(\rho)}=y'$, we shall have§

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* (247) Deduced from [355], by putting $\frac{h'}{h}$ for $\frac{g'}{g}$, observing that when the time T of the vibration of a pendulum is given, the length of the pendulum will be proportional to the force of gravity g, or g', [86].

† (248) If in the exponent of c, [356], we change h' into $h+(h'-h)$, and neglect the very small quantity $\frac{r'\,.\,(h'-h)}{lh}$, the expression of ρ, [356], will become very nearly $\rho=\Pi\,.\,c^{-\frac{r'}{l}\,.\,\left(1-\frac{r'}{R}\right)}$, which is nearly constant at the same elevation r'.

‡ (249) This equation is precisely similar to [327], changing y into φ, the demonstration is made as in note 216.

§ (250) From [349′] we get $\delta p=lg\,.\,\delta\rho$. Substituting ρ, [356$^{\text{iv}}$], we find by neglecting α^2,

$$\frac{\delta p}{\rho} = \frac{lg . \delta . (\rho)}{(\rho)} + \alpha g . \delta y' ;$$ [357]

now we have in the state of equilibrium*

$$0 = \frac{n^2}{2} . \delta . \{(r + \alpha s') . \sin. (\theta + \alpha u')\}^2 + (\delta V) - \frac{lg . \delta . (\rho)}{(\rho)} ;$$ [358]

the general equation of the motion of the atmosphere [348], corresponding to these level surfaces, in which δr is nearly evanescent, will therefore become†

$$\begin{aligned} & r^2 . \delta \theta . \left\{ \left(\frac{ddu'}{dt^2}\right) - 2n . \sin. \theta . \cos. \theta . \left(\frac{dv'}{dt}\right) \right\} \\ & + r^2 . \delta \varpi . \left\{ \sin.^2 \theta . \left(\frac{ddv'}{dt^2}\right) + 2n . \sin. \theta . \cos. \theta . \left(\frac{du'}{dt}\right) + \frac{2n . \sin.^2 \theta}{r} . \left(\frac{ds'}{dt}\right) \right\} \\ & \qquad = \delta V' - g . \delta \varphi - g . \delta y' + n^2 r . \sin.^2 \theta . \delta . \{s' - (s')\}, \end{aligned}$$ [359]

$$\frac{\delta p}{\rho} = \frac{lg . \{\delta . (\rho) + \alpha . \delta \rho'\}}{(\rho) + \alpha \rho'} = \frac{lg . \delta (\rho)}{(\rho)} + \alpha g l . \left\{ \frac{(\rho) . \delta \rho' - \rho' \delta . (\rho)}{(\rho)^2} \right\} = \frac{lg . \delta (\rho)}{(\rho)} + \alpha g . \delta . \left\{ \frac{l \rho'}{(\rho)} \right\} .$$

Putting as in [356$^{\text{iv}}$], $\frac{l \rho'}{(\rho)} = y'$, it becomes as in [357],

* (251) In the case of equilibrium u', v', s', are constant, and their differentials relative to t vanish, which makes the first member of [348] vanish, and the second member becomes as in [358]; observing that in this case δV becomes (δV), and ρ becomes (ρ), therefore, $\rho' = 0$, [356$^{\text{iv}}$]; also $y' = 0$; hence [357] changes into $\frac{\delta p}{\rho} = \frac{lg . \delta (\rho)}{(\rho)}$.

† (252) These level surfaces being nearly spherical, we shall have, in like manner as in [327$^{\text{iv}}$], $\delta r = 0$, and in the same way that [328] was obtained from [325], we may deduce [359], from [348]. For, by neglecting the term δr in the first member of [348], and dividing by α, it becomes identical with the first member of [359]. The second member of [348], divided by α, becomes $\frac{1}{\alpha} . \left\{ \frac{n^2}{2} . \delta \{(r + \alpha s') . \sin. (\theta + \alpha u')\}^2 + \delta V - \frac{\delta p}{\rho} \right\}$; [358a] in which $\frac{n^2}{2} . \delta . \left\{ (r + \alpha s') . \sin. (\theta + \alpha u') \right\}^2$, corresponding to the state of motion, may be divided into two parts, the one, being its value in the state of equilibrium, is equal to $-(\delta V) + \frac{lg . \delta . (\rho)}{(\rho)}$, [358]; the other depending on the change in the value of this quantity, arising from the motion of the particles. Now in the state of motion the distance of the particle from the centre of the spheroid, has increased from r to $r + \alpha s'$, the increment being $\alpha s'$; but in the same time the motions of the particle in the directions $\alpha u'$, $\alpha v'$, along

[359′] $\alpha . (s')$ being the variation of r, corresponding in the state of equilibrium,
to the variations $\alpha u'$, and $\alpha v'$ of the angles θ and ϖ.

Suppose that all the particles of air, which are situated at the beginning of
the motion on the same radius of the earth, remain constantly on that radius
during the motion which, as we have seen, takes place in the oscillations of
the sea [342′]; and let us see whether this hypothesis will satisfy the
equations of the motion and the continuity of the atmospherical fluid. For
this purpose it is necessary that the values of u' and v' should be the same
[359″] for all these particles; now the value of $\delta V'$ is nearly the same for all these
particles, as will be seen when we shall hereafter compute the forces from
which this variation results;* it is therefore necessary that the variations
$\delta\varphi$ and $\delta y'$ should be the same for all these particles, and that the quantities
[359‴] $2nr.\delta\varpi.\sin.^2\theta.\left(\frac{ds'}{dt}\right)$, and $n^2r.\sin.^2\theta.\delta.\{s'-(s')\}$, should be neglected
in the preceding equation.

[359ⁱᵛ] At the surface of the sea we have [356″, 326″], $\varphi=y$, αy being the
elevation of that surface above its level. Let us now see whether the
[359ᵛ] supposition of $\varphi=y$, and y constant for all the particles of air, situated upon
the same radius, can subsist with the equation of continuity of the fluid.

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the surface of equilibrium increase the height of that point of this surface by $\alpha(s')$, [359′], consequently the elevation of the particle above the surface of equilibrium is increased by $\alpha.\{s'-(s')\}$, hence the variation of the term $\frac{n^2}{2}.\delta.\{(r+\alpha s').\sin.(\theta+\alpha u')\}^2$, will be nearly $\alpha.n^2.r.\sin.^2\theta.\delta.\{s'-(s')\}$; and by reasoning as in note 217, it will be evident, that this expresses the whole variation of $\frac{n^2}{2}.\delta.\{(r+\alpha s').\sin.(\theta+\alpha u')\}$, arising from the motion of the particles. Its complete value therefore, in the state of motion will be

$$-(\delta V)+\frac{lg.\delta.(\rho)}{(\rho)}+\alpha.n^2r.\sin.^2\theta.\delta.\{s'-(s')\}.$$

Substituting this and $\delta V=(\delta V)-\alpha g.\delta\varphi+\alpha\delta V'$, [356‴]; $\frac{\delta p}{\rho}=\frac{lg.\delta.(\rho)}{(\rho)}+ag.\delta y'$, [357], in [358a], it becomes, by reduction, $\delta V'-g\delta\varphi-g\delta y'+n^2r.\sin.^2\theta.\delta.\{s'-(s')\}$, as in the second member of [359].

* (254) This may be shown as in note 231.

This equation by § 35 [336] is*

$$0=r^2.\left\{\rho'+(\rho).\left\{\left(\frac{du'}{d\theta}\right)+\left(\frac{dv'}{d\varpi}\right)+\frac{u'.\cos.\theta}{\sin.\theta}\right\}\right\}+(\rho).\left(\frac{d.r^2s'}{dr}\right);$$ [360]

whence we deduce

$$y'=-l.\left\{\left(\frac{d.r^2s'}{r^2dr}\right)+\left(\frac{du'}{d\theta}\right)+\left(\frac{dv'}{d\varpi}\right)+\frac{u'.\cos.\theta}{\sin.\theta}\right\}.$$ [361]

$r+\alpha s'$ is equal to the value of r of the level surface, which corresponds to the angles $\theta+\alpha u'$, and $\varpi+\alpha v'$, increased by the elevation of the particle of air above this surface; the part of $\alpha s'$ which depends on the variation of [361'] the angles θ and ϖ, being of the order† $\frac{\alpha n^2.u'}{g}$, may be neglected in the preceding expression of y', therefore we may suppose, in this expression, $s'=\varphi$; if we then make $\varphi=y$, we shall have $\left(\frac{d\varphi}{dr}\right)=0$, since the value [361''] of φ is then the same relative to all the particles situated on the same radius. Moreover y is, by what precedes, of the order l, or $\frac{n^2}{g}$;‡ the expression of y' will thus become

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* (255) This is the same as [336], changing s,u,v, into s',u',v', as in [347^{viii}]. Multiplying this by $\frac{l}{r^2(\rho)}$, we get $0=\frac{l\rho'}{(\rho)}+l.\left\{\left(\frac{du'}{d\theta}\right)+\left(\frac{dv'}{d\varpi}\right)+\frac{u'.\cos.\theta}{\sin.\theta}+\left(\frac{d.r^2s'}{r^2.dr}\right)\right\}$, and by substituting for $\frac{l\rho'}{(\rho)}$ its assumed value y', [356^{iv}], we obtain [361].

† (256) By note 241, page 229, the angle *BAD*, which the perpendicular to the surface of the ellipsoid makes with the radius *ID*, is of the order $\frac{n^2}{g}$. This multiplied by αu, will give the order of the increment of the radius, arising from the motion of any particle of air along its surface of equilibrium, through the angular space $\alpha u'$. The increment will therefore be of the order $\frac{\alpha n^2 u'}{g}$, and this, on account of its smallness, may be neglected, as in note 217.

‡ (257) Developing the term $\frac{d.(r^2s')}{r^2dr}$, [361], it becomes $2.\frac{s'}{r}+\left(\frac{ds'}{dr}\right)$, and since by [361''], $s'=\varphi=y$, and $\left(\frac{ds'}{dr}\right)=\left(\frac{d\varphi}{dr}\right)=0$, it becomes simply $\frac{d.(r^2s')}{r^2dr}=2.\frac{y}{r}$. Now by [347], y is of the order γu, or γv, so that this term $2.\frac{y}{r}$, must be of the order

[362] $$y' = -l . \left\{ \left(\frac{du'}{d\theta}\right) + \left(\frac{dv'}{d\varpi}\right) + \frac{u' . \cos. \theta}{\sin. \theta} \right\}.$$

therefore u' and v' being the same for all the particles which were at the beginning on the same radius, the value of y' will be the same for all these particles. Again, it is evident from what we have said, that the quantities

[362'] $2nr . \delta\varpi . \sin.^2\theta . \left(\frac{ds'}{dt}\right)$, and $n^2 r . \sin.^2\theta . \delta . \{s' - (s')\}$, may be neglected in the preceding equation of the motion of the atmosphere,* which may then be satisfied, by supposing u' and v' to be the same for all the particles of air situated originally on the same radius; the supposition that all these particles remain constantly on the same radius during the oscillations of the fluid, is therefore admissible with the equations of the motion and of the continuity of the atmospherical fluid. In this case, the oscillations of the different level strata are the same, and are determined by these equations,† [362'']

Equation of the Oscillations of the level strata of the Atmosphere.

[363] $$r^2 . \delta\theta . \left\{ \left(\frac{ddu'}{dt^2}\right) - 2n . \sin. \theta . \cos. \theta . \left(\frac{dv'}{dt}\right) \right\}$$
$$+ r^2 . \delta\varpi . \left\{ \sin.^2\theta . \left(\frac{ddv'}{dt^2}\right) + 2n . \sin. \theta . \cos. \theta . \left(\frac{du'}{dt}\right) \right\} = \delta V' - g . \delta y' - g\delta y;$$
$$y' = -l . \left\{ \left(\frac{du'}{d\theta}\right) + \left(\frac{dv'}{d\varpi}\right) + \frac{u' . \cos. \theta}{\sin. \theta} \right\}.$$

These oscillations of the atmosphere ought to produce corresponding oscillations in the heights of the barometer. To determine these, by means of the former, let us consider a barometer fixed at any height above the surface of the sea. The height of the mercury is proportional to the pressure

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$2 . \frac{\gamma}{r} . u$, which is to the other terms of the formula, [361], $\left(\frac{du'}{d\theta}\right)$, $\left(\frac{dv'}{d\varpi}\right)$, u', of the order $\frac{\gamma}{r}$, and by neglecting it on account of its smallness, the expression [361] will become as in [362].

* (258) Because s', (s), are small in comparison with u and v, and these terms are also multiplied by n or n^2 in [359]; they are therefore so small that they may be neglected.

† (259) These are the equations [359, 362], neglecting the terms depending on s', (s), on account of their smallness, and putting $\varphi = y$, [361''].

its surface experiences when exposed to the action of the air; it may [363']
therefore be represented by $lg.\rho$ [349']; but this surface is successively
exposed to the action of different level strata, which rise and fall like the
surface of the sea; thus the value of ρ, at the surface of the mercury, varies;
First, Because it appertains to a level stratum, which, in the state of
equilibrium, was less elevated by the quantity αy. *Second*, Because the
density of the stratum increases in the state of motion, by* $\alpha\rho'$, or $\frac{\alpha(\rho).y'}{l}$.
By means of the first cause, the variation of ρ is† $-\alpha y.\left(\frac{d\rho}{dr}\right)$, or $\frac{\alpha(\rho)y}{l}$; [363'']
the whole variation of the density ρ, at the surface of the mercury, is therefore
$\alpha(\rho).\frac{(y+y')}{l}$. Hence it follows, that if we put k for the height of the mercury [363''']
in the barometer, corresponding to the state of equilibrium, its oscillations
in the state of motion, will be expressed by the function‡ $\frac{\alpha k.(y+y')}{l}$; they [363^{iv}]

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* (261) In [356^{iv}], $\rho=(\rho)+\alpha\rho'$, (ρ) being the value of ρ, corresponding to the level surface; therefore $\alpha\rho'$ is the increment arising from the state of motion. Using the value of y', [356^{iv}], it becomes $\alpha\rho'=\frac{\alpha(\rho).y'}{l}$.

† (262) The density ρ is a function of r which decreases when r increases, therefore when the increment of r is αy, the decrement of ρ will be $-\alpha y.\left(\frac{d\rho}{dr}\right)$. Now the equation [355], neglecting $\frac{r'}{R}$ on account of its smallness, and putting $\frac{g'}{g}=1$, becomes $\rho=\Pi.c^{-\frac{r'}{l}}$, nearly, and as $r'=r-R$, [351''], we shall get $\rho=\Pi c^{\frac{R-r}{l}}$, hence $\left(\frac{d\rho}{dr}\right)=-\frac{\Pi}{l}.c^{\frac{R-r}{l}}$, or, by substituting the value of $c^{\frac{R-r}{l}}=\frac{\rho}{\Pi}$; $-\left(\frac{d\rho}{dr}\right)=\frac{\rho}{l}$, or $\frac{(\rho)}{l}$, nearly, therefore

$$-\alpha y.\left(\frac{d\rho}{dr}\right)=\frac{\alpha.(\rho).y}{l}.$$

‡ (263) For if the density (ρ) give the height k, the increment of density $\alpha.(\rho).\frac{(y+y')}{l}$, must, by proportion, give a corresponding increase of height of the barometer denoted by $\frac{\alpha k.(y+y')}{l}$.

are therefore similar, at all elevations above the sea, and proportional to the heights of the barometer.

To determine the oscillations of the sea and the atmosphere, it is now only necessary to know the forces which act upon these two fluid masses, and to integrate the preceding differential equations; which will be done in the course of this work.

SECOND BOOK.

ON THE LAW OF UNIVERSAL GRAVITATION, AND THE MOTIONS OF THE CENTRES OF GRAVITY OF THE HEAVENLY BODIES.

CHAPTER I.

ON THE LAW OF UNIVERSAL GRAVITATION, DEDUCED FROM OBSERVATION.

1. Having explained the laws of motion; we shall now proceed to deduce from these laws, and from the phenomena of the motions of the heavenly bodies, given in detail in the work entitled, "Exposition du Systême du Monde," the general law regulating the motions of those bodies. Of all these phenomena, the elliptical motion of the planets and comets about the sun, seems the best adapted to this investigation; we shall therefore use [363$^{\rm v}$]
it for this object, and shall suppose x and y to be the rectangular co-ordinates of a planet in the plane of its orbit; the origin of these co-ordinates being in the centre of the sun. Let P and Q be the forces acting on the planet, parallel to the axes of x and y, in its relative motion about the sun, *these* [363$^{\rm vi}$]
forces being supposed to tend towards the origin of the co-ordinates. Lastly, let dt be the element of the time, which we shall suppose to be constant. We shall have, by Chapter II of the first book,*

$$0=\frac{ddx}{dt^2}+P; \qquad (1)$$

[364]

$$0=\frac{ddy}{dt^2}+Q; \qquad (2)$$

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* (264) From [38] we have $\frac{ddx}{dt^2}=P$, $\frac{ddy}{dt^2}=Q$; but in the computation [34$^{\rm vii}$], the forces P, Q, were supposed to tend to *increase* the co-ordinates; whereas in the present case, [363$^{\rm vi}$], these forces tend to *decrease* the co-ordinates; we must therefore put P and Q

If we add the first of these equations, multiplied by $-y$, to the second multiplied by x, we shall have

[365] $$0=\frac{d.(x\,dy-y\,dx)}{dt^2}+x.Q-y.P.$$

[365′] It is evident that $xdy-ydx$ is double the area which the radius vector of the planet describes about the sun in the instant dt [167*a*]; this area is proportional to the element of the time, according to the first law of Kepler; so that we shall have

[366] $$x\,dy-y\,dx=c\,dt,$$

c being a constant quantity; the differential of the first member of this equation must therefore be nothing; hence we shall find*

[367] $$x.Q-y.P=0.$$

It follows from this equation, that the forces P, Q, are to each other in the ratio of x to y; consequently the resultant of these forces must pass through the origin of the co-ordinates, or in other words, through the sun's centre.†

[367′] Moreover, the curve described by the planet being concave towards the sun; it is evident, that the force which causes it to describe this curve tends towards the sun.

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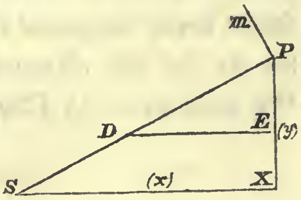
negative, and then, by transposition, we shall get [364]. Multiplying the first by $-y$, the second by x, and in their sum putting $d.(x\,dy-y\,dx)$ for $x\,ddy-y\,ddx$, we shall obtain the equation [365].

* (265) The differential of [366] being nothing, it reduces [365] to [367].

† (266) Let S be the origin of the co-ordinates, or the centre of the sun; P the centre of the planet, SX the axis of x, and PX the perpendicular let fall on it from P, making $SX=x$, $PX=y$. On PX, take PE equal to Q, and draw ED parallel to SX, to meet PS in D; then from the similar triangles PXS, PED, we get



$$PX(=y):SX(=x)::PE(=Q):DE=\frac{x.Q}{y};$$

but from [367] we have $\frac{x.Q}{y}=P$, consequently, $DE=P$. The two forces, $DE=P$, $PE=Q$, being composed [11 &c.], form the single force $PD=\varphi=\sqrt{P^2+Q^2}$, in

[367*a*] the direction PS, towards the origin of the co-ordinates S; the curve described being concave towards the sun.

The law of the areas proportional to the times of description, leads therefore to this first remarkable result; namely, that the force which acts [367″] on the planets and comets, is directed towards the centre of the sun.

2. We shall now investigate the law according to which this force acts at different distances from the sun. It is evident, since the planets and comets alternately approach to, and recede from, the sun, at each revolution, that the nature of the elliptical motion ought to conduct us to this law. For this purpose, we shall resume the differential equations (1), (2), [364] of the preceding article. If we add the first multiplied by dx, to the second multiplied by dy, we shall obtain

$$0=\frac{dx\,.\,ddx+dy\,.\,ddy}{dt^2}+P\,dx+Q\,dy\,;$$ [368]

and by integration,

$$0=\frac{dx^2+dy^2}{dt^2}+2\,.\int(P\,dx+Q\,dy),$$ [369]

the arbitrary constant quantity being indicated by the sign of integration. Substituting, instead of dt, its value [366] $\frac{x\,dy-y\,dx}{c}$, given by the law of [369′] the proportionality of the areas to the times, we shall find

$$0=\frac{c^2\,.\,(dx^2+dy^2)}{(x\,dy-y\,dx)^2}+2\,.\int(P\,dx+Q\,dy).$$ [370]

We shall transform, for greater simplicity, the co-ordinates x, y, into a radius vector, and polar angle, in conformity to the usage of astronomers. Let r be the line drawn from the centre of the sun to the centre of the planet, or its radius vector; v the angle which this radius forms with the axis of x; we shall have* [370′]

Polar Co-ordinates.

$$x=r\,.\,\cos.\,v\,;\qquad y=r\,.\,\sin.\,v\,;\qquad r=\sqrt{x^2+y^2}\,;$$ [371]

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* (267) This is evident from the preceding figure, where $SP=r$, $PSX=v$, whence $SX=SP\,.\,\cos.\,PSX$, $PX=SP\,.\,\sin.\,PSX$, which in symbols are the same as the above values of x, y, [371]. These agree with the values of x, y, in the note page 109, changing ρ into r, and ϖ into v. The same changes being made in [167c], it becomes as in the first of the equations [372], and from [167b], we find, that double the area described by the radius vector r, in the time dt, is represented by $r^2\,.\,dv=x\,dy-y\,dx$, as [372a] in [372].

whence we deduce

[372] $$dx^2+dy^2=r^2dv^2+dr^2; \qquad x\,dy-y\,dx=r^2dv.$$

If we then denote by φ the principal force which acts on the planet, we shall have, by the preceding article,*

[373] $$P=\varphi.\cos.v; \qquad Q=\varphi.\sin.v; \qquad \varphi=\sqrt{P^2+Q^2};$$

which give†

[374] $$P\,dx+Q\,dy=\varphi\,dr;$$

we shall therefore have

[375] $$0=\frac{c^2.\{r^2dv^2+dr^2\}}{r^4dv^2}+2\int\varphi\,dr;$$

whence we deduce‡

[376] $$dv=\frac{c\,dr}{r.\sqrt{-c^2-2r^2\int\varphi\,dr}}. \qquad (3)$$

This equation will give, by means of the quadrature of curves, the value of v in r, when the force φ is given in a function of r. If this force is unknown, but the nature of the curve it causes the body to describe is given, then by taking the differential of the preceding expression of $2\int\varphi dr$, we shall have the following equation§ to determine φ;

[377] $$\varphi=\frac{c^2}{r^3}-\frac{c^2}{2}.\frac{d.\left\{\frac{dr^2}{r^4dv^2}\right\}}{dr}. \qquad (4)$$

---

* (268) By [367*a*], we have $PD=\varphi=\sqrt{P^2+Q^2}$, and $P=DE=PD\cos.PDE=\varphi.\cos.v$, also $Q=PD.\sin.PDE=\varphi.\sin.v$, as in [373].

† (269) Substitute in $P\,dx+Q\,dy$, the values of P, Q, [373], also those of dx, dy, deduced from [371], it becomes

$$\varphi.\cos.v.\{dr.\cos.v-r\,dv.\sin.v\}+\varphi.\sin.v.\{dr.\sin.v+r\,dv.\cos.v\},$$

which by reduction is $\varphi\,dr.\{\cos.^2v+\sin.^2v\}$, or $\varphi.dr$, as in [374]. This equation is the same as [16], putting $V=\varphi$, $P=S$, $Q=S'$, $u=r$, $s=x$, $s'=y$; the forces P, Q, being equivalent to φ. Substituting the values [372, 374] in [370] we get [375].

‡ (270) Multiply [375] by $r^4.dv^2$, transpose the terms $c^2r^2dv^2$, $2r^4.dv^2.\int\varphi\,dr$, divide by $-c^2r^2-2r^4.\int\varphi\,dr$, and extract the square root, we shall get [376].

§ (271) The equation [375], divided by 2 is $\int\varphi\,dr=-\frac{c^2}{2r^2}-\frac{c^2}{2}.\left(\frac{dr^2}{r^4dv^2}\right)$; its differential divided by dr gives [377].

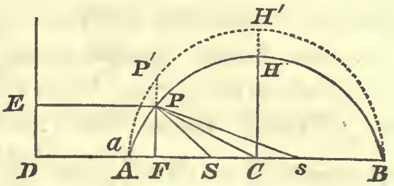
The orbits of the planets are ellipses, in one of whose foci the centre of the sun is placed : now if in the ellipsis we put* [377′]

$\varpi =$ the angle included between the axis of x and the transverse axis;

$a =$ the semi-transverse axis; [377″]

$e =$ the ratio of the excentricity to the semi-transverse axis;

Properties of the Conic Sections.



* (271*a*) Let $APHB$ be an ellipsis whose transverse axis is AB, conjugate semi-axis CH, centre C, foci S, s, vertices A, B, and *directrix* DE, perpendicular to BA. Then if from any point P of the curve, we let fall upon DE the perpendicular PE, and join PS, we shall have $SP : PE :: e : 1$, e being [378a] a constant quantity. This property appertains to all the conic sections, and it may serve to define them.

In the ellipsis $e < 1$; in the parabola $e = 1$; in the hyperbola $e > 1$. We shall, in the [378b] first place, demonstrate the formula [378], by means of this property of the directrix, and shall afterwards give another demonstration, depending upon the rectangular co-ordinates of the curve. Put $CA = CB = a$, $SA = D$, $SB = 2a - D$, $SP = r$, angle [378c] $ASP = v - \varpi$. Then from [378*a*] we get $SA = e.AD$, $SB = e.BD$. Therefore [378d] $SB - SA = e.(BD - AD)$, or $2.CS = 2e.CA$, and in symbols, $CS = ae$; [378e] also $SA = CA - CS$ becomes $SA = D = a - ae = a.(1 - e)$, and $AD = \frac{SA}{e} = \frac{a.(1-e)}{e}$; the sum of these two last expressions is [378f]

$$SD = a.(1-e) + \frac{a.(1-e)}{e} = \frac{a.(1-e^2)}{e};$$

subtracting from this $PE = \frac{SP}{e} = \frac{r}{e}$, [378*a*], we shall get $SF = \frac{a.(1-e^2) - r}{e}$, and as this [378g] is evidently $= SP.\cos. ASP$, or $r.\cos.(v - \varpi)$, we shall get, by multiplying by e, $re.\cos.(v - \varpi) = a.(1 - e^2) - r$, whence we easily deduce the value of r, [378].

We may also demonstrate the formula [378], by showing that the usual equation of the ellipsis, referred to the rectangular co-ordinates $CF = x$, $FP = y$, may be derived from it. For in the rectangular triangle SFP, we have $PF = SP.\sin. PSF$, $SF = CF - CS = SP.\cos. PSF$, or in symbols,

$$y = r.\sin.(v - \varpi), \qquad x - ae = r.\cos.(v - \varpi). \qquad [378h]$$

If we eliminate r and $v - \varpi$ from these equations, by means of the assumed relation between r and $v - \varpi$, [378], we shall obtain the equation of the curve, corresponding to [378], expressed in terms of x and y. Now from [378] we get

$$r.\cos.(v - \varpi) = \frac{a.(1-e^2) - r}{e} = \frac{a - r}{e} - ae. \qquad [378i]$$

Polar Equation of an Ellipsis.

the origin of the co-ordinates being fixed at the focus, we shall have

[378] $$r=\frac{a\,.\,(1-e^2)}{1+e\,.\,\cos.\,(v-\varpi)}.$$

Put this equal to the preceding value of $r\,.\,\cos.\,(v-\varpi)$, [378*h*], reject $-a\,e$ from both
[378*k*] members, and multiply the result by e, we shall get $e\,x=a-r$, or $r=a-e\,x$.
The sum of the squares of the two equations [378*h*], is evidently equal to r^2, and by
substituting the preceding value of r, [378*k*], we shall get $y^2+(x-a\,e)^2=(a-e\,x)^2$;
[378*l*] which, by development and reduction, becomes $y^2=(1-e^2)\,.\,(a^2-x^2)$, and as this
is the well known equation of an ellipsis, it proves that the curve defined by [378]
corresponds to that curve. When $x=0$, the ordinate y will correspond to the semi-conjugate
[378*m*] axis $C\,H=b$, hence $b^2=(1-e^2)\,.\,a^2$. Dividing the preceding value of y^2 by that of
[378*n*] b^2, we get $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$, or, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is one of the usual methods of
expressing the equation of an ellipsis.

If the ellipsis differs but very little from a circle, whose radius is 1, and we put $b=1$,
[378*o*] $a=1+\alpha$, $C\,P=\rho$, angle $H\,C\,P=\theta$, neglecting α^2, the preceding equation [378*n*]
will become $x^2\,.\,(1-2\,\alpha)+y^2=1$, hence $\sqrt{x^2+y^2}=\sqrt{1+2\,\alpha\,x^2}=1+\alpha\,x^2$, and
[378*p*] since $\sqrt{x^2+y^2}=\rho$, and $x=\rho\,.\,\sin.\,\theta=\sin.\,\theta$, nearly, we shall have $\rho=1+\alpha\,.\,\sin.^2\theta$, nearly.

In using the focus S, we have found r or $S\,P=a-e\,x$, [378*k*]; if the focus s had
been used, e would have been negative, and $s\,P=a+e\,x$; the sum of these two expressions
[378*q*] is $S\,P+s\,P=2\,a=A\,B$, a noted property of the ellipsis. When $x=0$, the preceding
[378*r*] value of $S\,P=a-e\,x$, will correspond to the point H, and will become $S\,H=a=C\,A$.

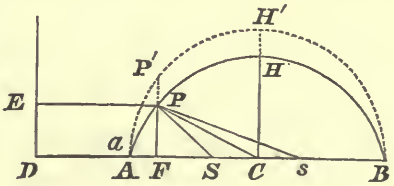
The whole ordinate $2\,y$, corresponding to the focus S, is called the *parameter* of the curve.
At this point x becomes $a\,e$, [378*e*], and the value of $2\,y=2\,.\sqrt{1-e^2}\,.\sqrt{a^2-x^2}$, [378*l*]
[378*s*] becomes $2\,a\,.\,(1-e^2)$. If we represent this by $2\,p$, we shall have, $p=a\,.\,(1-e^2)=\frac{b^2}{a}$,
[378*m*].

Upon the diameter $A\,B$ describe the semi-circle $A\,P'\,H'\,B$, to meet the ordinates $F\,P$,
$C\,H$, continued in P' and H'. Put $F\,P'=y'=\sqrt{a^2-x^2}$, and since $y=\frac{b}{a}\,.\sqrt{a^2-x^2}$,
[378*t*] [378*n*], we shall find $\frac{y}{y'}=\frac{b}{a}$, or $\frac{F\,P}{F\,P'}=\frac{C\,H}{C\,H'}$, which is a property sometimes used to
define the ellipsis. This value of $y=\frac{b}{a}\,.\,y'$, gives $\int y\,d\,x=\frac{b}{a}\,.\int y'\,d\,x$, whence it follows
that the area of the elliptical segment $A\,P\,F$, is to the area of the corresponding circular
segment $A\,P'\,F$, as b to a, also the area of the semi-ellipsis $A\,H\,B$ is equal to the area of
[378*u*] the semi-circle $A\,H'\,B$ multiplied by $\frac{b}{a}$, that is $\tfrac{1}{2}\,\pi\,.\,a^2\times\frac{b}{a}=\tfrac{1}{2}\,\pi\,.\,a\,b$, π being the
semi-circumference of a circle, whose radius is unity. Therefore the area of the whole
[378*v*] ellipsis is $\pi\,.\,a\,b=\pi\,.\,a^2\,.\sqrt{1-e^2}$, [378*m*].

This equation becomes that of a parabola, when $e=1$ and a is infinite ;* [378] and it corresponds to an hyperbola, when e exceeds unity, and a is negative. This equation gives†

$$\frac{dr^2}{r^4\,dv^2}=\frac{2}{ar.(1-e^2)}-\frac{1}{r^2}-\frac{1}{a^2.(1-e^2)}; \qquad [379]$$

The solidity of an ellipsoid of revolution about the axis AB is represented by $\pi.\int y^2\,dx$, because the area of the circle described by the radius FP, during this revolution is πy^2. In like manner $\pi.\int y'^2\,dx$, represents the solidity of the sphere, $\frac{4\pi}{3}.a^3$, [275*b*], described by the revolution of the semi-circle $AH'B$, about the same diameter. Now since $y=\frac{b}{a}.y'$, we shall have $\pi.\int y^2.dx=\frac{b^2}{a^2}.\pi.\int y'^2.dx=\frac{b^2}{a^2}.\frac{4\pi.a^3}{3}=\frac{4\pi}{3}.ab^2$. Therefore the solidity of the ellipsoid, formed by the revolution of the ellipsis, about the transverse axis $2a$, is represented by $\frac{4\pi}{3}.ab^2$; and if the revolution be about the conjugate axis $2b$, the solidity would be represented by $\frac{4\pi}{3}.a^2b$. [378*w*]



* (272) Substitute $D=a.(1-e)$, [378*f*], in the numerator of [378], it will become for the ellipsis $r=\frac{D.(1+e)}{1+e.\cos.(v-\varpi)}$. Put $e=1$, and we shall get the equation of a parabola $r=\frac{2D}{1+\cos.(v-\varpi)}$, in which $a=\frac{D}{1-e}$, is infinite. If e exceed unity, we shall have as before, $r=\frac{D.(1+e)}{1+e.\cos.(v-\varpi)}$, corresponding to an hyperbola, in which case $a=\frac{D}{1-e}$, [378*f*], becomes negative. [379*a*] [379*b*]

Put $AF=x'$, or $x=a-x'$, in the equation of the ellipsis, [378*n*], and it will become $\frac{(a-x')^2}{a^2}+\frac{y^2}{b^2}=1$, hence $y^2=\frac{b^2}{a}.\left(2x'-\frac{x'^2}{a}\right)$; and if we substitute for $\frac{b^2}{a}$ its value p, [378*s*], also $\frac{1}{a}=\frac{1-e}{D}$, [378*f*] we shall get for the ellipsis, [379*c*]

$$y^2=p.\left(2x'-\frac{(1-e)}{D}.x'^2\right). \qquad [379d]$$

In the parabola, where $e=1$, it becomes $y^2=2px'$. [379*e*]

In the hyperbola, where e exceeds unity, it becomes $y^2=p.\left(2x'+\frac{(e-1)}{D}.x'^2\right)$.

† (273) From [378] we get, $\frac{1}{r}=\frac{1+e.\cos.(v-\varpi)}{a.(1-e^2)}$; whose differential divided by $-dv$, is $\frac{dr}{r^2dv}=\frac{e.\sin.(v-\varpi)}{a.(1-e^2)}$. This being squared, putting $\sin.^2(v-\varpi)=1-\cos.^2(v-\varpi)$,

consequently*

[380] $$\varphi = \frac{c^2}{a.(1-e^2)} \cdot \frac{1}{r^2};$$

therefore, the orbits of the planets and comets being conic sections, the force
[380′] φ will be inversely proportional to the square of the distance of the centre of the planet from the centre of the sun.

We also perceive, that if the force φ be inversely proportional to the square of the distance, or expressed by $\frac{h}{r^2}$, h being a constant coefficient, the preceding equation of the conic sections, will satisfy the differential equation (4) [377] between r and v,† which gives the expression of the force, when we change φ into $\frac{h}{r^2}$. We shall then have

[380″] $$h = \frac{c^2}{a.(1-e^2)},$$

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we get $\frac{d\,r^2}{r^4\,d\,v^2} = \frac{e^2 - e^2.\cos.^2(v-\varpi)}{a^2.(1-e^2)^2}$. Substituting in the numerator of the second member, $e.\cos.(v-\varpi) = -1 + \frac{a.(1-e^2)}{r}$, it becomes $e^2 - 1 + \frac{2a.(1-e^2)}{r} - \frac{a^2.(1-e^2)^2}{r^2}$, or $(1-e^2).\left\{-1+\frac{2a}{r}-\frac{a^2.(1-e^2)}{r^2}\right\}$. Dividing the numerator and denominator by $1-e^2$, we find, $\frac{d\,r^2}{r^4\,d\,v^2} = \frac{-1+\frac{2a}{r}-\frac{a^2.(1-e^2)}{r^2}}{a^2.(1-e^2)}$, which is easily reduced to the form [379].

* (274) The differential of $\frac{d\,r^2}{r^4\,d\,v^2}$, [379], divided by $d\,r$ is $-\frac{2}{a\,r^2.(1-e^2)} + \frac{2}{r^3}$. Substitute this in [377], it becomes as in [380].

† (275) If we substitute $\varphi = \frac{h}{r^2}$ in [377], multiplied by $-d\,r$, we shall get by
[380a] integration $\frac{h}{r} = \frac{c^2}{2\,r^2} + \frac{c^2}{2}.\left\{\frac{d\,r^2}{r^4\,d\,v^2}\right\}$ + constant. Put this constant equal to $\frac{c^2}{2.a^2.(1-e^2)}$, and $h = \frac{c^2}{a.(1-e^2)}$; divide by $\frac{c^2}{2}$, and it will become by transposition,

$$\frac{d\,r^2}{r^4\,d\,v^2} = \frac{2}{a\,r.(1-e^2)} - \frac{1}{r^2} - \frac{1}{a^2.(1-e^2)},$$

as in [379], which was found above to be the differential of the equation [378], corresponding to the conic sections.

which forms an equation of condition between the two arbitrary constant quantities a and e of the equation of conic sections; the three arbitrary constant quantities a, e, ϖ, of this equation, will thus be reduced to two distinct arbitrary constant quantities; and as the differential equation between r and v is only of the second order, the finite equation of the conic sections [380'''] will be the complete integral.

Hence it follows, that if the described curve be a conic section, the force will be in the inverse ratio of the square of the distance; and conversely, [380iv] if the force be in the inverse ratio of the square of the distance, the described curve will be a conic section.

3. The intensity of the force φ, relative to each planet and comet, depends on the coefficient $\frac{c^2}{a\,.\,(1-e^2)}$ [380'']; the laws of Kepler furnish the means of determining it. For if we put T for the time of revolution of a planet; [380v] the area which its radius vector would describe during that time, would be equal to the surface of the planetary ellipsis, represented by $\pi\,.\,a^2\,.\,\sqrt{1-e^2}$ [380vi] [378v], π being the ratio of the semi-circumference of a circle to its radius; but by what precedes [365', 366], the area described during the instant dt is $\frac{1}{2}c\,dt$; the law of the proportionality of the areas to the times, will therefore give this proportion,

$$\tfrac{1}{2}\,.\,c\,dt : \pi a^2\,.\,\sqrt{1-e^2} :: dt : T\,; \qquad [381]$$

hence we deduce

$$c=\frac{2\,\pi\,.\,a^2\,.\sqrt{1-e^2}}{T}. \qquad [382]$$

With respect to the planets, the law of Kepler, according to which the squares of the times of their revolutions, are as the cubes of the transverse axes of their ellipses, gives $T^2=k^2.a^3$, k being the same for all the planets; [382'] therefore we shall have

$$c=\frac{2\pi\,.\sqrt{a\,.\,(1-e^2)}}{k}\,; \qquad [383]$$

$2\,a\,.\,(1-e^2)$ is the parameter of the orbit [378s], and in different orbits, [383'] the values of c are as the areas described by the radius vector in equal

times ;* these areas are therefore as the square roots of the parameters of
[383''] the orbits.

This proportion takes place also in comparing the orbits of the comets,
either with each other, or with those of the planets; this is one of the
fundamental points of their theory, and it agrees exactly with all their
observed motions. The transverse axes of their orbits, and the times of
their revolutions being unknown, their motions are calculated in a parabolic
[383'''] orbit, denoting the perihelion distance by D, and putting† $c=\frac{2\pi.\sqrt{2D}}{k}$,
which is equivalent to making e equal to unity, and a infinite, in the preceding
expression of c; we shall therefore have, with respect to the comets,
[383iv] $T^2=k^2a^3$; whence we may find the transverse axes of their orbits, when the
times of their revolution are known. Now, the expression of c [383] gives

[384] $$\frac{c^2}{a.(1-e^2)}=\frac{4\pi^2}{k^2};$$

therefore we shall have‡

[385] $$\varphi=\frac{4\pi^2}{k^2}.\frac{1}{r^2}.$$

The coefficient $\frac{4\pi^2}{k^2}$ being the same for all the planets and comets, it
follows that for each of these bodies, the force φ is inversely proportional to
the square of the distance from the centre of the sun, and that it varies
[385'] from one body to another, only by reason of these distances; whence it

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* (277) Putting A for double the area described in the time t, we shall have, [365′, 366],
$dA=c\,dt$, whose integral is $A=c\,t$, A being supposed to commence with t. Let A', c',
[383a] be the values of A, c, corresponding to another planet; then $A'=c'\,t$, consequently,
$A:A'::ct:c't::c:c'$.

† (278) Substitute $a.(1-e)=D$, [378f], in [383], and we shall find,

$$c=\frac{2\pi.\sqrt{a.(1-e).(1+e)}}{k}=\frac{2\pi.\sqrt{D.(1+e)}}{k},$$

and in a parabola, where $e=1$, [378′], it becomes $c=\frac{2\pi.\sqrt{2D}}{k}$.

‡ (279) By substituting the value of $\frac{c^2}{a.(1-e^2)}$, [384] in [380].

follows that it would be the same for all those bodies, supposing them at equal distances from the sun.

We are thus induced, by the beautiful laws of Kepler, to consider the centre of the sun as the focus of an attractive force, which extends infinitely in every direction, decreasing in the ratio of the square of the distance. The law of the proportionality of the areas described by the radius vector
to the times of description, shows that the principal force acting on the [$385''$]
planets and comets, is always directed towards the centre of the sun; the ellipticity of the planetary orbits, and the almost parabolic orbits of the comets, prove that, for each planet and comet, this force is inversely proportional to the square of the distance of the body from the sun; lastly, from the law of the proportionality of the square of the times of revolutions,
to the cubes of the great axes of the orbits, or from that of the proportionality [$385'''$]
of the areas described in equal times by the radius vector, in different orbits, to the square roots of the parameters of the orbits, which law comprises the preceding, and extends to comets; it follows that this force is the same for all the planets and comets, placed at equal distances from the
sun, so that in this case, these bodies fall towards it with the same velocity. [385^{iv}]

4. If from the planets we pass to the satellites, we shall find that as the laws of Kepler are very nearly observed in the motions of the satellites about their primary planets, they ought to gravitate towards the centres of these planets, in the inverse ratio of the square of their distances from those
centres; the satellites ought likewise to gravitate towards the sun in nearly [385^{v}]
the same manner as their planets, in order that the relative motions about their primary planets may be very nearly the same as if these planets were at rest. The satellites are therefore attracted towards the planets and towards the sun, by forces inversely proportional to the squares of the
distances. The ellipticity of the orbits of the three first satellites of Jupiter [385^{vi}]
is small, but that of the fourth is very sensible. The great distance of Saturn has hitherto prevented the discovery of the ellipticity in the orbits of any of its satellites except the sixth, which is sensibly elliptical. But the law of gravitation of the satellites of Jupiter, Saturn, and Uranus, is most apparent in the ratio of their mean motions, to their mean distances from the
centres of their planets. Which ratio for each system of satellites is, that [385^{vii}]
the squares of the times of their revolutions are as the cubes of their mean

distances from the centre of the planet. Suppose therefore that a satellite describes a circular orbit, with a radius equal to that of its mean distance from the centre of its primary planet; let this distance be a, and T the number of seconds contained in its sidereal revolution, π being the ratio of the semi-

[385viii] circumference of a circle to its radius; $\frac{2a\pi}{T}$ will be the small arch which the satellite describes in a second. If it was not retained in its orbit by the attractive force of the planet, it would fly off, in the direction of the tangent, increasing its distance from the centre by a quantity equal to the versed sine

[385ix] of the arch $\frac{2a\pi}{T}$, which is,* $\frac{2a\pi^2}{T^2}$; the attractive force causes it therefore to fall towards the planet by the same quantity. Relative to another satellite, whose mean distance from the centre of the planet is a', and T' the time of revolution in seconds, the fall in one second would be

[385x] $\frac{2a'\pi^2}{T'^2}$; now if we put φ and φ' for the attractive forces of the planet at the distances a and a', it is evident that they are as the spaces fallen through in a second; therefore we shall have

[386] $$\varphi : \varphi' :: \frac{2a\pi^2}{T^2} : \frac{2a'\pi^2}{T'^2}.$$

The law of the squares of the times of revolution, proportional to the cubes of the mean distances of the satellites from the centre of their planet, gives

[387] $$T^2 : T'^2 :: a^3 : a'^3;$$

from these two proportions it is easy to deduce

[388] $$\varphi : \varphi' :: \frac{1}{a^2} : \frac{1}{a'^2};$$

therefore the forces φ and φ' are inversely proportional to the squares of the distances a and a'.

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* (280) The versed sine of an arch of a circle, is equal to the square of the corresponding chord divided by the diameter, and the chord of a very small arch $\frac{2a\pi}{T}$, is nearly equal to this arch. The square of this arch, divided by the diameter $2a$, gives the versed sine $\frac{2a\pi^2}{T^2}$, as above.

5. The earth having but one satellite, the ellipticity of the lunar orbit is the only celestial phenomenon, which would lead to the discovery of the law of the attractive force; but the elliptical motion of the moon is very sensibly affected by the disturbing forces, which would leave some doubt about the law of the diminution of the attractive force of the earth, in the ratio of [388′]
the square of the distance from its centre. However, the analogy which exists between this force, and the attractive forces of the Sun, Jupiter, Saturn, and Uranus, leads us to believe, that it follows the same law of diminution; but the experiments on gravity upon the surface of the earth, afford a direct method to verify this law.

For this purpose we shall investigate the parallax of the moon, from the experiments of the length of a pendulum vibrating in a second, and shall compare it with astronomical observations. On the parallel on which the square of the sine of the latitude is $\frac{1}{3}$, the space through which gravity causes a heavy body to descend in a second of time, is, according to the observations of the [388″]
length of the pendulum, equal to $3^m, 65548$, as we shall see in the third book;* we have chosen this parallel, because the attraction of the earth on the corresponding points of its surface, when compared with that at the distance of the moon, is very nearly as the mass divided by the square of the [388‴]
distance from the centre of gravity of the earth.† On this parallel, the force

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* (281) The formula given in Book III, [2054], for the length of this pendulum, is $0^m,739502 + 0^m,004208 \,.\, (\text{sin. lat.})^2$, and if $(\text{sin. lat.})^2 = \frac{1}{3}$, it becomes $0^m,740905$. Putting this $= r$, and $T = 1''$, in the theorem $T = \pi \,.\, \sqrt{\frac{r}{g}}$, [86], we obtain $g = \pi^2 \,.\, r = 7^m,31244$. The space z, fallen through in one second of time, by the force of gravity is [67], $z = \frac{1}{2} g\, t^2$, and by putting $t = 1''$, it becomes equal to $\frac{1}{2} g$, or $3^m,65622$, which differs a little from the above, but it will be unnecessary to revise the calculation, as the whole is to be considered merely as an approximation.

† (282) That the attraction of the earth is nearly as its mass, divided by the square of the distance of the moon from its centre of gravity, is proved in [470^{vii}]. Suppose now that the earth is a homogeneous ellipsoid of revolution, whose polar semi-axis is denoted by $b = 1$, its equatorial semi-axis $a = 1 + \alpha$; its solidity will be $\frac{4\pi}{3} \,.\, (1 + \alpha)^2$, [378$w$], and if we put this equal to the area of a sphere, whose radius is ρ, which is $\frac{4\pi}{3} \,.\, \rho^3$, [275$b$]; we shall

of gravity is less than that depending on the attraction of the earth, by two
thirds of the centrifugal force, corresponding to the rotatory motion at the
equator;* this force is $\frac{1}{288}$ of gravity; we must therefore increase the
[388iv] preceding space by its $\frac{1}{432}$ part, to obtain the whole space arising from the
attraction of the earth, which on this parallel is equal to the mass divided by
the square of the radius of the earth: we shall therefore have $3^m, 66394$ for
[388v] this space. At the distance of the moon, it ought to be diminished in the
ratio of the square of the radius of the terrestrial spheroid, to the square of
the distance of the moon from the earth; and it is evident that this is effected
by multiplying it by the square of the sine of the moon's parallax;† putting
[388vi] therefore x for this sine, corresponding to the parallel under consideration, we
shall have $x^2 . 3^m, 66394$, for the space the moon ought to fall through, by the
attraction of the earth, in one second of time. But we shall see, in the

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get $\rho^3 = (1+\alpha)^2$, hence $\rho = 1 + \frac{2}{3}\alpha$, nearly. Putting this equal to the expression of
[388a] $\rho = 1 + \alpha . \sin.^2 \theta$, [378p], we get $\sin.^2 \theta = \frac{2}{3}$, hence $\cos.^2 \theta = \frac{1}{3}$, θ being very
nearly equal to the complement of the latitude of the place whose radius is ρ. Therefore the
mass of the ellipsoid is equal to the mass of a sphere described with a radius equal to that of
the ellipsoid, in a latitude whose sine is equal to $\sqrt{\frac{1}{3}}$.

* (283) Referring to the figure in page 229, we find that the centrifugal force, resolved in the direction of the radius ID, is $n^2 . R . \sin.^2 \theta$, [352b]. At the equator, where $\sin. \theta = 1$, it becomes $n^2 . R$. If we suppose this to be $\frac{1}{289}$ of the attractive force A of the earth, at the equator, [1594a], the actual force of gravity g, at the equator, will be $g = \frac{288}{289} . A$, hence $A = \frac{289}{288} . g$, and the centrifugal force at the equator $\frac{A}{289}$ becomes $\frac{g}{288} = n^2 . R$; therefore the preceding expression of the centrifugal force, in the direction of the radius ID, will be $\frac{g}{288} . \sin.^2 \theta$; and since, in the case now under consideration, [388a], we have $\sin.^2 \theta = \frac{2}{3}$, this will become $\frac{g}{288} . \frac{2}{3} = \frac{g}{432}$, as in [388iv]. Adding to 3,65548, [388″], its $\frac{1}{432}$ part, or 0,00846, the sum becomes 3.66394, as in [388v]. We may observe that in all the calculations, relative to the figure of the earth, in this work, terms of the order α^2, are generally neglected, and for this reason the centrifugal force might be
[388b] taken indifferently for $\frac{1}{289} . A$, or $\frac{1}{289} . g$, without departing from the usual limits of accuracy.

† (285) This corresponds with the usual rule for finding the horizontal parallax of any body, seen from the earth's surface, by saying, as the distance of the observed body from the centre of the earth, is to the earth's semi-diameter, so is radius to the sine of the horizontal parallax, nearly.

theory of the moon, that the action of the sun diminishes its gravity towards the earth, by a quantity, whose constant part is the $\frac{1}{358}$ part of gravity;* [388$^{\text{vii}}$] moreover, the moon, in its relative motion about the earth, is acted upon by a force equal to the sum of the masses of the earth and moon, divided by the square of their distance from each other; we must therefore diminish the preceding space, by $\frac{1}{358}$, and increase it in the ratio of the sum of the masses of the earth and moon to that of the earth; now we shall see in the fourth book, that the phenomena of the tides give the mass of the moon equal to

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* (286) A student in astronomy, who has not examined the calculations of the lunar theory, had better pass over this, and assume, with the author, that the decrement of gravity, arising from the sun's disturbing force is $\frac{1}{358}$ part. This may be safely done, as the present calculation is not used for any other purpose in the rest of the work. After reading the theory of the moon's motion in Book VII, the subject may be again resumed, and this decrement of gravity may be investigated in the following manner.

If we represent the masses of the earth, moon, and sun, by M, m, m', respectively, the quantity $\left(\frac{dQ}{dr}\right)$, [499$a$], will represent the force acting on the moon m, in the direction of the radius vector r of her relative about the earth. From the general value of Q, [4806], we may obtain the mean value, required in the present calculation, by neglecting the terms depending on the angle $v-v'$, and its multiples, which nearly destroy each other in every revolution; we may also neglect the terms depending on the tangent of the moon's latitude s, on account of their smallness, by which means [4806] will become, $Q=u+m'u'+\frac{m'.u'^3}{4u^2}$, and since by neglecting s^2 we have $u=\frac{1}{r}$, $u'=\frac{1}{r'}$, [4776, 4779'], we shall get $Q=\frac{1}{r}+\frac{m'}{r'}+\frac{m'.r^2}{4r'^3}$; hence $\left(\frac{dQ}{dr}\right)=-\frac{1}{r^2}.\left\{1-\frac{m'.r^3}{2r'^3}\right\}$. If the sun did not disturb [388c] the motion, or $m'=0$, this would become $-\frac{1}{r^2}$. The ratio of the former expression to the latter is represented by $1-\frac{m'.r^3}{2r'^3}$; therefore the gravity g of the moon towards the earth, is decreased by the sun's disturbing force, a quantity equal to $\frac{m'.r^3}{2r'^3}.g$, nearly; and if we use the mean values $r=a$, $r'=a'$, [4791], it is $\frac{m'.a^3}{a'^3}.g$. Substitute $\frac{1}{a^3}=1$, [4795], and $\frac{m'}{a'^3}=m^2$, [4794], it becomes $\frac{m^2}{2}.g$; and since $m=0{,}0748013$, [5117], [388d] it is nearly $\frac{m^2}{2}.g=\frac{1}{358}.g$, as in [388$^{\text{vii}}$].

[388viii] $\frac{1}{58{,}7}$ of that of the earth;* therefore we shall have $\frac{357}{358}\cdot\frac{59{,}7}{58{,}7}\cdot x^2\cdot 3^m{,}66394$, for the space through which the moon falls towards the earth in one second of time.

Now, if we put a for the mean radius of the moon's orbit, and T for the
[388ix] number of seconds in the time of its sidereal revolution, $\frac{2a\pi^2}{T^2}$ will be, as we have shown [385ix], the versed sine of the arch described in one second
[388x] of time. This expresses the space through which the moon falls towards the earth in that time. The value of a is equal to the radius of the earth under
[388xi] the parallel of latitude just mentioned, divided by x; this radius is equal to† 6369514^m; therefore we have

$$a=\frac{6369514^m}{x}; \qquad [389]$$

but to obtain the value of a, independent of the inequalities of the motion of the moon, we must take for its mean parallax, whose sine is x, the part of the parallax which is independent of those inequalities, and which, for
[389′] that reason, is usually called *the constant term of the parallax.* Hence, by
[389″] taking for π the ratio of 355 to 113, and for T its value $2732166''$,‡ the

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[*] (289) In [2706], the disturbing forces of the moon and sun on the tides, are found to be nearly as 3 to 1; and from this, in [4321], the mass of the moon was found to be $\frac{1}{58{,}6}$, of that of the earth, which nearly agrees with the above. Further observations induced the
[389a] author, in [4631], to change this into $\frac{1}{68{,}5}$; and afterwards in Book XIII, § 9 to $\frac{1}{75}$, nearly; making the force of the moon on the tides to that of the sun as 2,35 to 1, nearly.

[†] (290) Using the ellipticity $\frac{1}{334}$, given by the author, [2034], we get in [2035b], the polar semi-diameter $=6356676^m$, and the equatorial semi-diameter $=6375709^m$, their
[389b] difference being 19033^m. Now the decrement of the radius, in proceeding from the equator to the pole, being nearly as the square of the sine of the latitude, [378p], the decrement corresponding to the latitude whose sine is $\sqrt{\tfrac{1}{3}}$, will be $19033^m\times\frac{1}{3}=6344^m$, which subtracted from the equatorial semi-diameter, leaves the radius of that latitude 6369365^m, which is rather less than that above given; the difference may have arisen from using another ratio of ellipticity; this however has but a very little effect on the result of the calculations.

[‡] (291) This is the time of a sidereal revolution of the moon in seconds, corresponding to $27^d\ 7^h\ 43^m\ 11^s{,}4$.

mean space through which the moon falls towards the earth, will be

$$\frac{2\,.\,(355)^2\,.\,6369514^m}{(113)^2\,.\,x\,.\,(2732166)^2}. \qquad [390]$$

Putting the two expressions [388$^{\rm viii}$, 390] of this space equal to each other, we shall have

$$x^3 = \frac{2\,.\,(355)^2\,.\,358.58{,}7.6369514}{(113)^2.357.59{,}7\,.\,3{,}66394\,.\,(2732166)^2}\,; \qquad [391]$$

whence we deduce 10536″,2, for the constant term of the moon's parallax, [391′]
under the parallel of latitude before mentioned. This value differs but very little from* 10540″,7, computed by Triesnecker, from a great number of observations of eclipses and occultations of stars by the moon; it is therefore certain that the principal force which retains the moon in her orbit, is the attraction of the earth, decreased in the duplicate ratio of the distance; thus [391″]
the law of the diminution of gravity, which, for the planets accompanied by several satellites, is proved by the comparison of the times of their revolutions, and of their distances, is demonstrated for the moon by the comparison of her motion with that of projectiles on the surface of the earth. Hence it follows, that we must fix the origin of the distances, at the centre of gravity [391‴]
of any heavenly body, in computing its attraction upon bodies placed upon its surface, or without it; since this has been proved to be the case with respect to the earth, whose attractive force is of the same nature as that of the other heavenly bodies, as we have shown.

6. Hence it follows that the sun, and the planets which have satellites, are endowed with an attractive force, extending infinitely, decreasing inversely as the square of the distance, and including all bodies in the sphere of their activity. Analogy leads us to infer that a similar force exists [391$^{\rm iv}$]
generally in all the planets and comets; and it may be proved in the following

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* (293) The constant term corresponding to the equator in Burg's tables, is by Book VII, § 26, [5603], equal to 10558″,64. The decrement for any other latitude, according to Burg, [5604], is found by multiplying this by $\frac{(\text{sin. lat.})^2}{330}$. In the present case, $(\text{sin. lat.})^2 = \frac{1}{3}$, hence the decrement is 10″,66, and the constant term becomes 10547″,98, differing a few seconds from the above. The calculation of the parallax in [5331], differs a few seconds from this.

manner. It is an invariable law of nature, that a body cannot act on another, without experiencing an equal and contrary reaction; therefore, since the planets and comets are attracted towards the sun, they must in like manner attract that body. For the same reason the satellites attract their planets; this attractive property is therefore common to the planets, comets and [391^{v}] satellites; consequently we may consider the gravitation of the heavenly bodies, towards each other, as a general law of the universe.

We have shown that this law follows the inverse ratio of the square of distances. It is true, that this ratio was deduced from the supposition ~~the~~ [391^{vi}] of aperfect elliptical motion, which does not rigorously accord with the observed motions of the heavenly bodies. But we ought to consider that the most simple laws should always be preferred, until we are compelled by observation to abandon them. It is natural at first to suppose that the law of gravitation is inversely as a power of the distance; and we find by calculation that the slightest difference between this power and the square, [391^{vii}] would become extremely sensible in the position of the perihelia of the planetary orbits,* in which, however, no motions have been discovered by observation, except such as are very small, the cause of which will be explained hereafter.† In general we shall see, in the course of this work, that the law of gravitation, in the inverse ratio of the square of the distances, represents with the greatest precision, all the known inequalities of the motions of the heavenly bodies; and this accordance, taken in connexion with the simplicity of the law, authorizes the belief that it is rigorously the law of nature.

Gravitation is proportional to the masses; for it follows from § 3 [385^{iv}], [391^{viii}] that if the planets and comets are supposed to be at equal distances from the sun, they would fall freely towards it through equal spaces in equal times;

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* (294) This is very sensible in the motion of the moon, which would move in a fixed ellipsis, if the moon was affected only by the mutual attraction of the moon and earth. But the disturbing force of the sun, which is about $\frac{1}{358}$ of that of their gravity towards each other, [388^{vii}], produces a motion of the perigee of nearly 40° in a year, as is easily proved from the value of c, Book VII, § 16, [5117].

† (295) As in Book VI, § 25, where $d\varpi$ is determined for the planets, and in Book VII, § 16, for the moon, &c.

consequently their gravities would be proportional to their masses. The motions of the satellites about their primary planets, in nearly circular orbits, prove that the satellites gravitate, like the planets, towards the sun, in the ratio of their masses; the slightest difference, in this respect, would be sensible in the motions of the satellites; but no inequality depending on this [391^{ix}] cause has been discovered by observation.* Hence we see that the comets,

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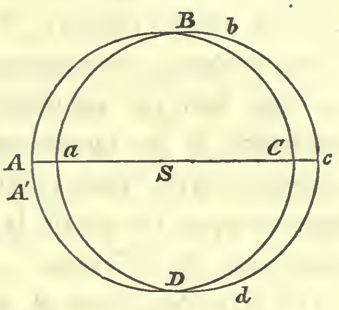

* (296) To point out the effect of this difference in the attraction, let us suppose that a very small body, or particle of matter revolves about the sun S, in an elliptical orbit $a\,b\,c\,d$, whose transverse axis is $a\,c$, and one of its foci S; and that another similar body or particle revolves about S as a centre, in the circular orbit $A\,B\,C\,D$, whose diameter $A\;C$ is equal to $a\,c$. Then as the mean distances from the sun are equal, their times of revolution, by Kepler's law, will be equal, [382']; neglecting the mutual attractions of the two revolving bodies, and the sun will attract them equally at equal distances, [385^{iv}]. Now if we suppose both the bodies to revolve in the same direction, the one of them being at A, when the other is at a, the distances of the two bodies will always be of the same order as the quantity $A\,a$; and if $A\,a$ be small in comparison with $S\,A$, as for example $\frac{1}{400}$ part, the two bodies will be somewhat similarly situated to that of a primary planet and its satellite; the primary being at A, B, C, D, when the secondary is at a, b, c, d, respectively; the distances $A\,a$, $B\,b$, $C\,c$, $D\,d$, being of the same order as $\frac{1}{400}$ part of the distance $S\,A$. The satellite will be in conjunction with the sun at a, in opposition at c, and in the quadratures at b and d; and the same will happen in the successive revolutions of the bodies. And it may not be amiss to notice, particularly, that in this case [391a] one of the bodies would appear to revolve about the other, as a satellite, without being in the least attracted by it; the motion being maintained wholly by the sun's attraction. Suppose now that the action of the sun on the planet is less than on the satellite, when at the same distance, by $\frac{1}{1000}$ part; so that instead of falling through the space, or versed sine g, in a second of time, when at the point A, it should only fall through the space $g-\frac{g}{1000}$. In this case, to make the planet continue in its orbit it is necessary to decrease its velocity, in the proportion of $\sqrt{g}$ to $\sqrt{g-\frac{g}{1000}}$, or nearly, as $1:1-\frac{1}{2000}$; because the versed sines of small arcs are nearly as the squares of the arcs. The time of describing the circle $A\,B\,C\,D$, [391b] must in this case be varied in the inverse ratio of the velocities, and it must therefore be increased about $\frac{1}{2000}$ part; consequently, at the end of one revolution, when the satellite has arrived at a, the planet will be at A', a little short of A; and this distance will increase in the

planets and satellites, placed at the same distance from the sun, would
gravitate towards it, in the ratio of their masses; and as action and reaction
are equal and contrary, it follows that they attract the sun in the same ratio;
[391^{x}] consequently their actions on the sun are proportional to their masses divided
by the square of their distances from its centre.

The same law is observed upon the surface of the earth; for it has been
found by very exact experiments, made with a pendulum, that if we neglect
[391^{xi}] the resistance of the air, all bodies would fall towards the centre of the earth
with an equal velocity. Such bodies gravitate therefore towards the earth,
in proportion to their masses, in like manner as the planets gravitate towards
the sun, and the satellites towards their primary planets. This perfect
conformity in the operations of nature, upon the surface of the earth and in
[391^{xii}] the immensity of space, proves, in the most striking manner, that the gravity
observed upon the earth, is only a particular case of a general law extending
throughout the universe.

The attractive force of any one of the heavenly bodies does not appertain
[391^{xiii}] exclusively to its *aggregated* mass, for the property is common to each
component particle. If the sun acted only on the centre of the earth,
without attracting each of its particles, there would result, in the ocean,
incomparably greater and extremely different oscillations, from those now
observed; the gravity of the earth towards the sun results therefore from the
gravitations of all the particles of the earth; consequently these particles
must also attract the sun, in the ratio of their respective masses. Moreover,
each body upon the surface of the earth gravitates towards the centre of the
earth, in proportion to the mass of the body. It therefore reacts on the
earth, and attracts it in the same proportion. If this were not the case, and
[391^{xiv}] any part of the earth, however small it might be, did not attract the rest of

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successive revolutions. At the end of about 1000 revolutions, when the satellite is at a, the planet will be at C. Thus we see that by only varying the gravity $\frac{1}{1000}$ part, it would have the effect to increase the distance of the bodies so much, that they could no longer be considered as a planet and satellite. If we had not supposed the velocity of the planet to be decreased, the circular orbit $ABCD$, would have become elliptical, and its greater axis would have exceeded AC or ac, consequently the periodical time of revolution would have been increased, and a similar effect, in the separation of the two particles, would have been produced.

the earth, in the same manner as it is attracted; the centre of gravity of the earth would be put in motion, by gravity, which is impossible.*

Observations of the heavenly bodies, compared with the laws of motion,
lead therefore to this great principle of nature, namely, that all the particles of
matter attract each other in the direct ratio of their masses, and the inverse [391^{xv}]
ratio of the square of their distances. And in this universal gravitation we
perceive the cause of the perturbations of the motions of the heavenly
bodies. For the planets and comets, in obeying their mutual attractions,
must vary a little from the elliptical motion, which they would exactly follow,
if they were attracted only by the sun. The satellites, disturbed in their [391^{xvi}]
motions about their planets, by their mutual attractions, and by that of
the sun, vary also from these laws. We find also, that the particles of
each heavenly body, united by their attraction, ought to form nearly a
spherical mass; and the resultant of their attractions on the surface of the
body, ought to produce all the phenomena of gravity. We also perceive that [391^{xvii}]
the rotatory motion of the heavenly bodies must produce a small change in
their spherical form, by compressing the poles, and then the resultant of the
mutual attraction of the particles, will not pass exactly through their centres [391^{xviii}]
of gravity; in consequence of which there will arise, in their axes of rotation,
motions similar to those discovered by observation. Lastly, we see that the
particles of the ocean, being unequally attracted by the sun and moon, ought
to have an oscillatory motion, similar to the flux and reflux of the tide. But [391^{xix}]
the development of these effects of the general gravitation of matter requires
a profound analysis. To embrace this subject in the most general manner,

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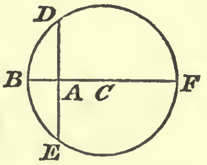
* (297) To illustrate this, let $DBEF$ be a meridian of the earth, divided into two unequal parts, BDE, DFE, by a plane passing through the line DE, perpendicular to the plane of the figure. Through the centre C draw CAB, perpendicular to DE. Suppose now the larger part DFE, attracts the smaller part in the direction BC, with a force represented by $F+f$; and that the part DBE attracts the larger part with the force F only. These forces will not balance each other; on the contrary, the resultant will be the force f, acting in the direction BC, consequently, in this hypothesis, the earth would acquire a motion, in the direction BC, by the mere force of the mutual attraction of its particles, which is absurd.



we shall give the differential equations of the motion of a system of bodies, obeying their mutual attractions, and shall investigate such rigorous integrals as can be obtained. We shall then, in finding the integrals by approximation, make use of those simplifications which depend on the ratios of the distances [391xx] and masses of the heavenly bodies; and shall carry this approximation to such a degree of exactness, as shall be necessary to determine the phenomena of the heavenly bodies with the accuracy required by observations.

CHAPTER II.

ON THE DIFFERENTIAL EQUATIONS OF THE MOTION OF A SYSTEM OF BODIES SUBJECTED TO THEIR MUTUAL ATTRACTIONS.

7. LET m, m', m'', &c., be the masses of the different bodies of the system, considered as so many points; let x, y, z, be the rectangular co-ordinates of the body m; x', y', z', those of m'; &c. The distance from m' to m being [118] $[391^{xxi}]$

$$\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}, \qquad [392]$$

its action on m will, by the law of general gravitation, be equal to

$$\frac{m'}{(x'-x)^2+(y'-y)^2+(z'-z)^2}. \qquad [393]$$

If we resolve this force in directions parallel to the axes of x, y, z, the force parallel to x, in a direction *opposite to the origin of these co-ordinates*, will be* $[393']$

$$\frac{m'.(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}; \qquad [394]$$

or

$$\frac{1}{m}\cdot\left\{\frac{d\cdot\dfrac{m.m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}}{dx}\right\}. \qquad [395]$$

We shall in like manner have

$$\frac{1}{m}\cdot\left\{\frac{d\cdot\dfrac{m.m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}}{dx}\right\} \qquad [396]$$

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* (297a) This is deduced from the formula [13], $S.\frac{(x-a)}{s}$, by writing for S and s, the values [393, 392]; also changing x into x', and a into x. The expression [395] is evidently equivalent to [394], as will appear by developing the differential relative to d.

for the action of m'' on m, resolved in a direction parallel to that of the axis of x, and in the same manner for the rest. Suppose therefore

[397]

$$\lambda = \frac{m \,.\, m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} + \frac{m \,.\, m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}$$
$$+ \frac{m' \,.\, m''}{\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}} + \&c.\,;$$

λ being the sum of the products of the masses m, m', m'', &c., taken two by two, and divided by their respective distances, $\frac{1}{m} \,.\, \left(\frac{d\lambda}{dx}\right)$ will express the sum of the actions of the bodies m', m'', &c., on m, resolved parallel to the [397′] axis of x, in a *direction opposite to the origin* of the co-ordinates. Putting therefore dt for the element of the time, considered as constant, we shall [397″] have, by the principles of dynamics, explained in the preceding book,*

[398]

$$0 = m \,.\, \frac{ddx}{dt^2} - \left(\frac{d\lambda}{dx}\right).$$

We shall likewise have

[399]

$$0 = m \,.\, \frac{ddy}{dt^2} - \left(\frac{d\lambda}{dy}\right);$$
$$0 = m \,.\, \frac{ddz}{dt^2} - \left(\frac{d\lambda}{dz}\right).$$

Differential Equations of the motions of a system of bodies referred to a fixed point.

If we consider in the same manner, the action of the bodies m, m'', &c., on m'; that of the bodies m, m', &c., on m'', and in the same manner for the rest; we shall have the equations

[400]

$$0 = m' \,.\, \frac{ddx'}{dt^2} - \left(\frac{d\lambda}{dx'}\right); \quad 0 = m' \,.\, \frac{ddy'}{dt^2} - \left(\frac{d\lambda}{dy'}\right); \quad 0 = m' \,.\, \frac{ddz'}{dt^2} - \left(\frac{d\lambda}{dz'}\right);$$
$$0 = m'' \,.\, \frac{ddx''}{dt^2} - \left(\frac{d\lambda}{dx''}\right); \quad 0 = m'' \,.\, \frac{ddy''}{dt^2} - \left(\frac{d\lambda}{dy''}\right); \quad 0 = m'' \,.\, \frac{ddz''}{dt^2} - \left(\frac{d\lambda}{dz''}\right);$$
$$\&c.$$

The determination of the motions of m, m', m'', &c., depends on the integration of these differential equations; but this has not yet been done

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* (298) The equations [398, 399, 400], are found by putting in [38], for P, Q, R, &c. their values,

$$\frac{1}{m} \,.\, \left(\frac{d\lambda}{dx}\right); \quad \frac{1}{m} \,.\, \left(\frac{d\lambda}{dy}\right); \quad \frac{1}{m} \,.\, \left(\frac{d\lambda}{dz}\right); \quad \frac{1}{m'} \,.\, \left(\frac{d\lambda}{dx'}\right);\ \&c.$$

completely, except in the simple case where the system is composed of only two bodies. In the other cases, there have been obtained but a few rigorous integrals, which we shall now investigate.

8. For this purpose, we shall first combine the differential equations in x, x', x'', &c., by adding them together, observing that by the nature of the function λ, we have[*]

$$0=\left(\frac{d\lambda}{dx}\right)+\left(\frac{d\lambda}{dx'}\right)+\left(\frac{d\lambda}{dx''}\right)+\&c.\,; \qquad [401]$$

we shall have $0=\Sigma.m.\frac{ddx}{dt^2}$. In like manner $0=\Sigma.m.\frac{ddy}{dt^2}$; $0=\Sigma.m.\frac{ddz}{dt^2}$. [401′]

Let X, Y, Z, be the three co-ordinates of the centre of gravity of the system; [401″] we shall have, by the property of this centre,[†]

$$X=\frac{\Sigma.mx}{\Sigma.m}\,; \qquad Y=\frac{\Sigma.my}{\Sigma.m}\,; \qquad Z=\frac{\Sigma.mz}{\Sigma.m}\,; \qquad [402]$$

therefore we shall have

$$0=\frac{ddX}{dt^2}\,; \qquad 0=\frac{ddY}{dt^2}\,; \qquad 0=\frac{ddZ}{dt^2}\,; \qquad [403]$$

whence by integration

$$X=a+bt\,; \qquad Y=a'+b't\,; \qquad Z=a''+b''t\,; \qquad [404]$$

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[*] (299) This is easily proved by taking any term of λ, [397], and computing its effect on the proposed function. Thus the term $\frac{m'm''}{\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}}$, affects only the terms $\left(\frac{d\lambda}{dx'}\right)$ and $\left(\frac{d\lambda}{dx''}\right)$, the former is $\left(\frac{d\lambda}{dx'}\right)=\frac{m'.m''.(x''-x')}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{3}{2}}}$; the latter $\left(\frac{d\lambda}{dx''}\right)=\frac{-m'.m''.(x''-x')}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{3}{2}}}$, and the sum of both is equal to nothing. The [401a] same thing occurs with all the similar terms, consequently,

$$\left(\frac{d\lambda}{dx}\right)+\left(\frac{d\lambda}{dx'}\right)+\left(\frac{d\lambda}{dx''}\right)+\&c.=0.$$

[†] (300) As in [126, 127]. Taking the second differential of these equations [402], divided by dt^2, and substituting in the second members 0 for $\Sigma.m.\frac{ddx}{dt^2}$, $\Sigma.m.\frac{ddy}{dt^2}$, $\Sigma.m.\frac{ddz}{dt^2}$, [401′], we get [403], which, being integrated twice, gives [404]. Hence we may prove, as in note 74*a*, page 104, that the motion of the centre of gravity is rectilinear and uniform.

a, b, a', b', a'', b''; being the arbitrary constant quantities. Hence we perceive [404′] that the motion of the centre of gravity of the system, is rectilinear and uniform, and that it is not affected by the action of the bodies of the system upon each other; which is conformable to what was proved in the fifth chapter of the first book [159′].

Uniform motion of the centre of gravity of the system.

Let us now resume the differential equations of the motions of these bodies. If we multiply the differential equations in y, y', y'', &c., respectively by x, x', x'', &c., and add these products to those formed by multiplying the differential equations in x, x', x'', &c, by $-y, -y', -y''$, &c., respectively, we shall have

[405]
$$\begin{aligned}0 = m\,.\left(\frac{x\,d\,d\,y - y\,d\,d\,x}{d\,t^2}\right) + m'\,.\left(\frac{x'\,d\,d\,y' - y'\,d\,d\,x'}{d\,t^2}\right) + \&\text{c.}\\ + y\,.\left(\frac{d\,\lambda}{d\,x}\right) + y'\,.\left(\frac{d\,\lambda}{d\,x'}\right) + \&\text{c.}\\ - x\,.\left(\frac{d\,\lambda}{d\,y}\right) - x'\,.\left(\frac{d\,\lambda}{d\,y'}\right) + \&\text{c.}\end{aligned}$$

But the nature of the function λ gives*

[406]
$$\begin{aligned}0 = y\,.\left(\frac{d\,\lambda}{d\,x}\right) + y'\,.\left(\frac{d\,\lambda}{d\,x'}\right) + \&\text{c.}\\ - x\,.\left(\frac{d\,\lambda}{d\,y}\right) - x'\,.\left(\frac{d\,\lambda}{d\,y'}\right) - \&\text{c.};\end{aligned}$$

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* (301) This is proved by an analysis similar to that in note 299. For by substituting in the function $y\,.\left(\frac{d\,\lambda}{d\,x}\right) + y'.\left(\frac{d\,\lambda}{d\,x'}\right) + \&\text{c.} - x.\left(\frac{d\,\lambda}{d\,y}\right) - x'\,.\left(\frac{d\,\lambda}{d\,y'}\right) - \&\text{c.}$, the parts depending on the term of λ, affected by $m'\,.\,m''$, which are

[405a] $$\left(\frac{d\,\lambda}{d\,x'}\right) = \frac{m'\,.\,m''\,.\,(x''-x')}{\left\{(x''-x')^2+(y''-y')^2+(z''-z')^2\right\}^{\frac{3}{2}}};\quad \left(\frac{d\,\lambda}{d\,x''}\right) = \frac{-m'\,.\,m''\,.\,(x''-x')}{\left\{(x''-x')^2+(y''-y')^2+(z''-z')^2\right\}^{\frac{3}{2}}};\ \&\text{c.}$$

they will produce, in the proposed function, the terms,

$$\frac{m'\,.\,m''}{\left\{(x''-x')^2+(y''-y')^2+(z''-z')^2\right\}^{\frac{3}{2}}}\,.\,\left\{y'.(x''-x') - y''\,.\,(x''-x') - x'.\,(y''-y') + x''.(y''-y')\right\}$$

in which the terms between the braces are easily reduced to the form

$$(y'-y'')\,.\,(x''-x') - (y'-y'')\,.\,(x''-x'),$$

which is identically nothing. The same takes place with all the other terms of λ, consequently, the function [406], is equal to nothing. Substituting this in [405], and taking the integral, we get [407]; changing y into z, and then x into y, we get the formulas [408] therefore we shall have, by integrating the preceding equation,

$$c = \Sigma \, . \, m \, . \left(\frac{x\,dy - y\,dx}{dt}\right). \qquad [407]$$

We shall find in a similar manner

$$c' = \Sigma \, . \, m \, . \left(\frac{x\,dz - z\,dx}{dt}\right);$$

$$c'' = \Sigma \, . \, m \, . \left(\frac{y\,dz - z\,dy}{dt}\right); \qquad [408]$$

Preservation of Areas.

c, c', c'', being arbitrary constant quantities. These three integrals comprise the principle of the preservation of areas, explained in the fifth chapter of the first book [167]. [408′]

Lastly, if we multiply the differential equations in x, x', x'', &c., respectively by dx, dx', dx'', &c.; those in y, y', &c., respectively by dy, dy', dy'', &c.; those in z, z', &c., by dz, dz', &c.; and then add all these products together, we shall have*

$$0 = \Sigma \, . \, m \, . \frac{\{dx \, . \, ddx + dy \, . \, ddy + dz \, . \, ddz\}}{dt^2} - d\lambda, \qquad [409]$$

and by integration

$$h = \Sigma \, . \, m \, . \left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) - 2\lambda; \qquad [410]$$

Preservation of the living force.

h being another arbitrary constant quantity. This integral comprises the principle of the preservation of the living forces, explained in Chapter V, Book I, [144]. [410′]

The seven preceding integrals are all the rigorous integrals which have hitherto been discovered: in the case where the system is composed of only two bodies, they reduce the determinations of the motions of these bodies to differential equations of the first order, which may be integrated, as we shall show hereafter; but when the system is formed of three, or of a greater number of bodies, we must necessarily have recourse to methods of approximation. [410″]

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* (301*a*) Putting for $\left(\frac{d\lambda}{dx}\right) . dx + \left(\frac{d\lambda}{dy}\right) . dy + \left(\frac{d\lambda}{dz}\right) . dz + \left(\frac{d\lambda}{dx'}\right) . dx' +$ &c., its value $d\lambda$. [409*a*]

[410'''] 9. As we can only observe the relative motions of the heavenly bodies, it is usual to refer the motions of the planets and comets to the centre of the sun, and the motions of the satellites to the centres of their primary planets. To compare the theory with observations, it will therefore be necessary to determine the relative motions of a system of bodies about one of the bodies, considered as the centre of their motions.

On the relative motions of a system of bodies about one of them.

Let M be this last body, m, m', m'', &c., being the other bodies whose relative motions about M are required; let ξ, Π, and γ, be the rectangular co-ordinates of M; $\xi+x$, $\Pi+y$, $\gamma+z$, those of m; $\xi+x'$, $\Pi+y'$, $\gamma+z'$, those of m', &c.; it is evident that x, y, z, will be the co-ordinates of m, referred to M as a centre; x', y', z', will be those of m', referred to the same body; and in like manner for the others. Let r, r', &c., be the distances of the bodies m, m', &c., from M; so that* [410^iv] [410^v]

[411]
$$r=\sqrt{x^2+y^2+z^2}\,; \qquad r'=\sqrt{x'^2+y'^2+z'^2}\,;$$

and suppose

[412]
$$\lambda=\frac{m\,.\,m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}+\frac{m\,.\,m''}{\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}}$$
$$+\frac{m'\,.\,m''}{\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}}+\text{\&c.}\,;$$

Then the action of m on M, resolved parallel to the axis of x, and taken *in a direction opposite to the origin*, will be $\frac{mx}{r^3}$; that of m' on M, resolved in the same direction, will be $\frac{m'x'}{r'^3}$; and in the same manner for the rest. We shall therefore have, to determine ξ, the differential equation,† [412']

[413]
$$0=\frac{dd\xi}{dt^2}-\Sigma\,.\,\frac{mx}{r^3}\,;$$

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* (302) The values of r [411], are easily deduced from [12], by writing ξ, Π, γ, for a, b, c, and $\xi+x$, $\Pi+y$, $\gamma+z$, for x, y, z, respectively; r', r'', &c. are found in the same manner.

† (303) The action of m upon M is $\frac{m}{r^2}$. Putting this for S in the first formula, [13], and also for s, x, a, writing r, $\xi+x$, ξ, respectively, we shall get $\frac{mx}{r^3}$, for that force resolved

we shall in like manner have

$$0 = \frac{d\,d\,\Pi}{d\,t^2} - \Sigma\,.\,\frac{my}{r^3}\,;$$

$$0 = \frac{d\,d\gamma}{d\,t^2} - \Sigma\,.\,\frac{mz}{r^3}. \qquad [414]$$

The action of M on m, resolved parallel to the axis of x, and taken in a *direction opposite to the origin*, will be* $-\frac{Mx}{r^3}$, and the sum of the actions of the bodies m', m'', &c., on m, resolved in the same direction, will be† $\frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right)$; therefore we shall have‡ [414′]

$$0 = \frac{d\,d\,.\,(\xi + x)}{d\,t^2} + \frac{Mx}{r^3} - \frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right); \qquad [415]$$

---

in a direction parallel to x. In like manner, $\frac{m'x'}{r'^3}$; $\frac{m''x''}{r''^3}$, &c. will represent the similar forces of m', m'', &c. upon M. The sum of all these is $\Sigma\,.\,\frac{mx}{r^3}$. Putting this for P, in [38], and writing $d\,d\,\xi$, for $d\,d\,x$ the second differential of the co-ordinates of M, we get $\frac{d\,d\,\xi}{dt^2} = \Sigma\,.\,\frac{mx}{r^3}$, as in [413]. In like manner the two last formulas of [38] give those of [414].

* (304) This force $-\frac{Mx}{r^3}$, is found in the same manner as the force $\frac{mx}{r^3}$, of the last note, but it must be observed that in the present arrangement of the symbols, the body M may be supposed to be nearer the origin than any of the other bodies, its attraction must therefore tend to draw the other bodies towards that centre, and thus *decrease* the co-ordinates and as the effect of m on M, was supposed positive, this must be put negative.

† (305) This is proved as in [397′], the value of λ, [397] being of the same form as in [412].

‡ (306) This may be deduced from [398], by changing x, y, z, into $\xi + x$, $\Pi + y$, $\gamma + z$, respectively, as in [410^{iv}], and instead of the force $\frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right)$, [397′], substituting the value found in [414′], $\frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right) - \frac{Mx}{r^3}$. Substituting in [415] the value of $\frac{d\,d\,\xi}{dt^2}$, [413], it becomes as in [416]. In like manner, from [414, 399] we get [417, 418].

substituting for $\frac{dd\xi}{dt^2}$ its value [413] $\Sigma.\frac{mx}{r^3}$, we shall have

[416] $$0=\frac{ddx}{dt^2}+\frac{Mx}{r^3}+\Sigma.\frac{mx}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dx}\right); \qquad (1)$$

and in like manner

[417] $$0=\frac{ddy}{dt^2}+\frac{My}{r^3}+\Sigma.\frac{my}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dy}\right); \qquad (2)$$

[418] $$0=\frac{ddz}{dt^2}+\frac{Mz}{r^3}+\Sigma.\frac{mz}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dz}\right). \qquad (3)$$

[418′] If we change successively, in the equations (1), (2), (3), the quantities m, x, y, z, into m', x', y', z'; m'', x'', y'', z'', &c., and the contrary; we shall have the equations of the motions of the bodies m', m'', &c. about M.

If we multiply the differential equation in ξ, by $M+\Sigma.m$; that in x by m; that in x' by m'; and in the same manner for the others; and then add these products together, observing that by the nature of the function λ, we have*

[419] $$0=\left(\frac{d\lambda}{dx}\right)+\left(\frac{d\lambda}{dx'}\right)+\&c.;$$

we shall find†

[420] $$0=(M+\Sigma.m).\frac{dd\xi}{dt^2}+\Sigma.m.\frac{ddx}{dt^2};$$

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* (307) Proved as in [401], the value of λ, [397], being of the same form as in [412].

† (308) Multiplying [416] by m, we get,

$$0=m.\frac{ddx}{dt^2}+M.\frac{mx}{r^3}+m.\Sigma.\frac{mx}{r^3}-\left(\frac{d\lambda}{dx}\right),$$

and the similar equation in x', is

$$0=m'.\frac{ddx'}{dt^2}+M.\frac{mx'}{r^3}+m'.\Sigma.\frac{mx}{r^3}-\left(\frac{d\lambda}{dx'}\right);$$

increasing the accents we shall get the similar equations in x'', x''', &c. The sum of all these, by putting $\left(\frac{d\lambda}{dx}\right)+\left(\frac{d\lambda}{dx'}\right)+\&c.=0$, [419], becomes,

$$0=\Sigma.m.\frac{ddx}{dt^2}+M.\Sigma.\frac{mx}{r^3}+\Sigma.m.\Sigma.\frac{mx}{r^3}.$$

This added to the product of [413], by $M+\Sigma.m$,

$$0=(M+\Sigma.m).\frac{dd\xi}{dt^2}-M.\Sigma.\frac{mx}{r^3}-\Sigma.m.\Sigma.\frac{mx}{r^3},$$

whence by integration

$$\xi = a + b\,t - \frac{\Sigma\,.\,m\,x}{M + \Sigma\,.\,m}\,; \qquad [421]$$

a and b being two arbitrary constant quantities. We shall likewise have

$$\Pi = a' + b'\,t - \frac{\Sigma\,.\,m\,y}{M + \Sigma\,.\,m}\,;$$
$$\gamma = a'' + b''\,t - \frac{\Sigma\,.\,m\,z}{M + \Sigma\,.\,m}\,; \qquad [422]$$

a', b', a'', b'', being arbitrary constant quantities: we shall thus obtain the absolute motion of M in space, when the relative motions of m, m', &c., about it, are known.

If we multiply the differential equation in x [416], by

$$-\,m\,y + m\,.\,\frac{\Sigma\,.\,m\,y}{M + \Sigma\,.\,m}\,; \qquad [423]$$

and the differential equation in y [417], by

$$m\,x - m\,.\,\frac{\Sigma\,.\,m\,x}{M + \Sigma\,.\,m}\,; \qquad [424]$$

Likewise if we multiply the differential equation in x', by

$$-\,m'\,y' + m'\,.\,\frac{\Sigma\,.\,m\,y}{M + \Sigma\,.\,m}\,; \qquad [425]$$

and the differential equation in y' by

$$m'\,x' - m'\,.\,\frac{\Sigma\,.\,m\,x}{M + \Sigma\,.\,m}\,; \qquad [426]$$

and in a similar manner for the rest; and then add together all these products, observing that the nature of the function λ gives*

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becomes as in [420]. Integrating [420], and adding the constant quantity, $(M + \Sigma\,.\,m)\,.\,b$, to complete the integral, we get

$$(M + \Sigma\,.\,m)\,.\,b = (M + \Sigma\,.\,m)\,.\,\frac{d\,\xi}{d\,t} + \Sigma\,.\,m\,.\,\frac{d\,x}{d\,t}.$$

Multiplying this by $d\,t$, and again integrating, putting $(M + \Sigma\,.\,m)\,.\,a$, for the new constant quantity, we get, $(M + \Sigma\,.\,m)\,.\,(a + b\,t) = (M + \Sigma\,.\,m)\,.\,\xi + \Sigma\,.\,m\,x.$ Transposing the last term, and dividing by $M + \Sigma\,.\,m$, we obtain the value of ξ, [421]. By changing ξ, x, into Π, y, and γ, z, we find [422] from [414].

* (309) The first of these equations [427], is proved as in [405a, 406]; the second and third as in [401a], where it is shown that $\Sigma\,.\,m\,.\left(\frac{d\,\lambda}{d\,x}\right) = 0$; and as λ, [412], is symmetrical in x, y, z, we may change in this x into y or z. [427a]

[427]
$$0 = \Sigma . x . \left(\frac{d\lambda}{dy}\right) - \Sigma . y . \left(\frac{d\lambda}{dx}\right) ;$$
$$0 = \Sigma . \left(\frac{d\lambda}{dx}\right) ; \qquad 0 = \Sigma . \left(\frac{d\lambda}{dy}\right) ;$$

we shall have*

[428]
$$0 = \Sigma . m . \frac{\{x\,ddy - y\,ddx\}}{dt^2} - \frac{\Sigma . mx}{M + \Sigma . m} . \Sigma . m . \frac{ddy}{dt^2} + \frac{\Sigma . my}{M + \Sigma . m} . \Sigma . m . \frac{ddx}{dt^2} ;$$

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* (310) Put for brevity $X = \frac{\Sigma . mx}{M + \Sigma . m}$, $Y = \frac{\Sigma . my}{M + \Sigma . m}$. Multiply these by $M + \Sigma . m$, and we shall get $-MX + \Sigma . mx - X . \Sigma . m = 0$, $MY - \Sigma . my + Y . \Sigma . m = 0$. [428a]

Multiply [416], by $-my + mY$, and [417], by $mx - mX$; the sum of these products will be

$$0 = \left\{ \frac{ddx}{dt^2} . (-my + mY) + \frac{ddy}{dt^2} . (mx - mX) \right\}$$
$$+ \frac{M}{r^3} . \left\{ x . (-my + mY) + y . (mx - mX) \right\}$$
$$+ (-my + mY) . \Sigma . \frac{mx}{r^3} + (mx - mX) . \Sigma . \frac{my}{r^3}$$
$$+ \left(\frac{d\lambda}{dx}\right) . (y - Y) + \left(\frac{d\lambda}{dy}\right) . (-x + X).$$

Neglecting the two terms $-mxy + mxy$, in the second line, and changing a little the order of the terms in each line, we get,

$$0 = \left\{ m . \frac{(x\,ddy - y\,ddx)}{dt^2} - X . m . \frac{ddy}{dt^2} + Y . m . \frac{ddx}{dt^2} \right\}$$
$$+ M . \left\{ Y . \frac{mx}{r^3} - X . \frac{my}{r^3} \right\}$$
$$+ \left\{ -my . \Sigma . \frac{mx}{r^3} + Y . m . \Sigma . \frac{mx}{r^3} + mx . \Sigma . \frac{my}{r^3} - X . m . \Sigma . \frac{my}{r^3} \right\}$$
$$- \left\{ x . \left(\frac{d\lambda}{dx}\right) - y . \left(\frac{d\lambda}{dx}\right) + Y . \left(\frac{d\lambda}{dx}\right) - X . \left(\frac{d\lambda}{dy}\right) \right\} .$$

Marking the letters m, x, y, r, without the sign Σ, successively, with one, two, three, &c. accents, we shall obtain the similar equations in x', x'', x''', &c. The sum of all these equations will be,

[428b]
$$0 = \Sigma . m . \frac{(x\,ddy - y\,ddx)}{dt^2} - X . \Sigma . m . \frac{ddy}{dt^2} + Y . \Sigma . m . \frac{ddx}{dt^2}$$
$$+ M . Y . \Sigma . \frac{mx}{r^3} - M . X . \Sigma . \frac{my}{r^3}$$
$$- \Sigma . my . \Sigma . \frac{mx}{r^3} + Y . \Sigma . m . \Sigma . \frac{mx}{r^3} + \Sigma . mx . \Sigma . \frac{my}{r^3} - X . \Sigma . m . \Sigma . \frac{my}{r^3}$$
$$- \left\{ \Sigma . x . \left(\frac{d\lambda}{dx}\right) - \Sigma . y . \left(\frac{d\lambda}{dx}\right) \right\} - Y . \Sigma . \left(\frac{d\lambda}{dx}\right) + X . \left(\frac{d\lambda}{dy}\right).$$

the integral of which is*

$$\text{constant}=\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}-\frac{\Sigma.mx}{M+\Sigma.m}.\Sigma.m.\frac{dy}{dt}+\frac{\Sigma.my}{M+\Sigma.m}.\Sigma.m.\frac{dx}{dt}; \qquad [429]$$

or

$$c=M.\Sigma.m.\frac{(xdy-ydx)}{dt}+\Sigma.mm'.\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\}; \;(4) \qquad [430]$$

The lower line becomes nothing by means of [427]. The coefficients of $\Sigma.\frac{mx}{r^3}$, $\Sigma.\frac{my}{r^3}$, in the second and third lines, are respectively $MY-\Sigma.my+Y.\Sigma.m$, and $-MX+\Sigma.mx-X.\Sigma.m$, which by [428*a*] are nothing; therefore the equation [428*b*], will be reduced to the terms in the first line, which are the same as in [428].

* (311) This is easily proved by taking the differential of the equation [429], divided by dt, and comparing it with [428]. The first term $\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}$, evidently produces the first term of [428], $\Sigma.m.\frac{\{x\,ddy-y\,ddx\}}{dt^2}$. If we neglect, for a moment, the constant factors, $M+\Sigma.m$, dt^2, which occur in the denominators of the two last terms of [429], divided by dt, they become, $-\Sigma.mx.\Sigma.mdy+\Sigma.my.\Sigma.mdx$, the differential of which is

$$-\{\Sigma.mdx.\Sigma.mdy+\Sigma.mx.\Sigma.mddy\}+\{\Sigma.mdy.\Sigma.mdx+\Sigma.my.\Sigma.mddx\},$$

which by reduction is $-\Sigma.mx.\Sigma.mddy+\Sigma.my.\Sigma.mddx$. Resubstituting the factors of the denominator $M+\Sigma.m$, dt^2, it becomes like the two last terms of [428]. Multiply [429] by $M+\Sigma.m$, and put c for the product of the constant term, by $M+\Sigma.m$, we shall get,

$$c=M.\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}+\Sigma.m.\frac{(x\,dy-y\,dx)}{dt}.\Sigma.m$$
$$-\Sigma.mx.\Sigma.m.\frac{dy}{dt}+\Sigma.my.\Sigma.m.\frac{dx}{dt}.$$

Now by [189*a*], we have identically,

$$\Sigma.m.\frac{(xdy-ydx)}{dt}.\Sigma.m=\Sigma.mm'.\left\{\frac{(x'-x).(dy'-dy)-(y'-y).(dx'-dx)}{dt}\right\}$$
$$+\Sigma.mx.\Sigma.m.\frac{dy}{dt}-\Sigma.my.\Sigma.m.\frac{dx}{dt}.$$

Substituting this in the preceding value of c, it becomes as in [430]. Changing in this the terms relative to the axis x, into those relative to the axis z, it becomes as in [431]; and in like manner, by changing the axis of x into the axis of y, in [431], we shall get [432].

c being an arbitrary constant quantity. In the same manner we may obtain the two following equations:

[431] $$c'=M.\Sigma.m.\frac{(x\,dz-z\,dx)}{dt}+\Sigma.mm'.\left\{\frac{(x'-x).(dz'-dz)-(z'-z).(dx'-dx)}{dt}\right\}; \qquad (5)$$

[432] $$c''=M.\Sigma.m.\frac{(y\,dz-z\,dy)}{dt}+\Sigma.mm'.\left\{\frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt}\right\}; \qquad (6)$$

c', c'', being two other arbitrary constant quantities.

If we multiply the differential equation in x [416], by

[433] $$2\,m\,d\,x-2\,m\,.\frac{\Sigma\,.\,m\,d\,x}{M+\Sigma\,.\,m};$$

the differential equation in y [417], by

[434] $$2\,m\,d\,y-2\,m\,.\frac{\Sigma\,.\,m\,d\,y}{M+\Sigma\,.\,m};$$

the differential equation in z [418], by

[435] $$2\,m\,d\,z-2\,m\,.\frac{\Sigma\,.\,m\,d\,z}{M+\Sigma\,.\,m};$$

in like manner, if we multiply the differential equation in x', by

[436] $$2\,m'\,d\,x'-2\,m'\,.\frac{\Sigma\,.\,m\,d\,x}{M+\Sigma\,.\,m};$$

the differential equation in y', by

[437] $$2\,m'\,d\,y'-2\,m'\,.\frac{\Sigma\,.\,m\,d\,y}{M+\Sigma\,.\,m};$$

the differential equation in z', by

[438] $$2\,m'\,d\,z'-2\,m'\,.\frac{\Sigma\,.\,m\,d\,z}{M+\Sigma\,.\,m};$$

and in the same manner for the others; and then add together all these products, observing that*

[439] $$0=\Sigma\,.\left(\frac{d\,\lambda}{d\,x}\right); \qquad 0=\Sigma\,.\left(\frac{d\,\lambda}{d\,y}\right); \qquad 0=\Sigma\,.\left(\frac{d\,\lambda}{d\,z}\right);$$

we shall have†

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* (312) As in [427], or in [427a].

† (313) The product of [416] by $2\,m\,d\,x-2\,m\,.\frac{\Sigma\,.\,m\,d\,x}{M+\Sigma.m}$, is

$$0=2\,m\,d\,x\,.\left\{\frac{d\,d\,x}{d\,t^2}+M.\frac{x}{r^3}+\Sigma.\frac{m\,x}{r^3}-\frac{1}{m}.\left(\frac{d\,\lambda}{d\,x}\right)\right\}$$
$$+2\,m\,.\frac{\Sigma\,.\,m\,d\,x}{M+\Sigma\,.\,m}.\left\{-\frac{d\,d\,x}{d\,t^2}-M.\frac{x}{r^3}-\Sigma.\frac{m\,x}{r^3}+\frac{1}{m}.\left(\frac{d\,\lambda}{d\,x}\right)\right\},$$

$$0 = 2\,.\,\Sigma\,.\,m\,.\,\frac{(dx\,.\,ddx + dy\,.\,ddy + dz\,.\,ddz)}{dt^2} - \frac{2\,.\,\Sigma\,.\,m\,dx}{M+\Sigma\,.\,m}\,.\,\Sigma\,.\,\frac{m\,.\,ddx}{dt^2}$$

$$-\frac{2\,.\,\Sigma\,.\,m\,dy}{M+\Sigma\,.\,m}\,.\,\Sigma\,.\,\frac{m\,.\,ddy}{dt^2} - \frac{2\,.\,\Sigma\,.\,m\,dz}{M+\Sigma\,.\,m}\,.\,\Sigma\,.\,\frac{m\,.\,ddz}{dt^2} + 2M\,.\,\Sigma\,.\,\frac{mdr}{r^2} - 2d\lambda\,; \qquad [440]$$

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and the similar expressions in x', x'', &c. are produced by merely accenting the letters. The sum of all the equations in x, x', x'', &c., thus formed, will be obtained by prefixing the sign Σ; hence, we get

$$0 = 2\,.\,\Sigma\,.\,m\,.\,\frac{dx\,.\,ddx}{dt^2} + 2M\,.\,\Sigma\,.\,m\,.\,\frac{x\,dx}{r^3} + 2\,.\,\Sigma\,.\,\frac{mx}{r^3}\,.\,\Sigma\,.\,m\,dx - 2\,.\,\Sigma\,.\,dx\,.\left(\frac{d\lambda}{dx}\right)$$

$$+\frac{2\,.\,\Sigma\,.\,m\,.\,dx}{M+\Sigma\,.\,m}\,.\left\{-\Sigma\,.\,m\,.\,\frac{ddx}{dt^2} - M\,.\,\Sigma\,.\,\frac{mx}{r^3} - \Sigma\,.\,m\,.\,\Sigma\,.\,\frac{mx}{r^3} + \Sigma\,.\left(\frac{d\lambda}{dx}\right)\right\}$$

in which the factor of $2\,.\,\Sigma\,.\,\frac{mx}{r^3}\,.\,\Sigma\,.\,m\,.\,dx$, is $\left\{1 - \frac{M}{M+\Sigma\,.\,m} - \frac{\Sigma\,.\,m}{M+\Sigma\,.\,m}\right\}$, which vanishes, because these terms mutually destroy each other; also from [439], $\Sigma\left(\frac{d\lambda}{dx}\right) = 0$; lastly, the term $2\,\Sigma\,.\,m\,dx\,.\,\Sigma\,.\,m\,.\,\frac{ddx}{dt^2}$, may be written $d\,.\left(\Sigma\,.\,m\,.\,\frac{dx}{dt}\right)^2$; these values being substituted, we get

$$0 = 2\,\Sigma\,.\,m\,.\,\frac{dx\,.\,ddx}{dt^2} + 2M\,.\,\Sigma\,.\,m\,.\,\frac{x\,dx}{r^3} - 2\,\Sigma\,.\,dx\,.\left(\frac{d\lambda}{dx}\right) - \frac{d\,.\left(\Sigma\,.\,m\,.\,\frac{dx}{dt}\right)^2}{M+\Sigma\,.\,m}.$$

Changing successively, x into y, and into z, we shall obtain the similar products formed from the equations [417, 418], namely,

$$0 = 2\,\Sigma\,.\,m\,.\,\frac{dy\,.\,ddy}{dt^2} + 2M\,.\,\Sigma\,.\,m\,.\,\frac{y\,dy}{r^3} - 2\,\Sigma\,.\,dy\,.\left(\frac{d\lambda}{dy}\right) - \frac{d\,.\,(\Sigma\,.\,m\,dy)^2}{(M+\Sigma\,.\,m)\,.\,dt^2}\,;$$

$$0 = 2\,\Sigma\,.\,m\,.\,\frac{dz\,.\,ddz}{dt^2} + 2M\,.\,\Sigma\,.\,m\,.\,\frac{z\,dz}{r^3} - 2\,\Sigma\,.\,dz\,.\left(\frac{d\lambda}{dz}\right) - \frac{d\,.\,(\Sigma\,.\,m\,dz)^2}{(M+\Sigma\,.\,m)\,.\,dt^2}\,;$$

Adding these three equations together, and reducing, by putting for $2\,\Sigma\,.\,m\,.\,\frac{(x\,dx+y\,dy+z\,dz)}{r^3}$ its value, [411], $2\,\Sigma\,.\,m\,.\,\frac{r\,dr}{r^3} = 2\,\Sigma\,.\,m\,.\,\frac{dr}{r^2}$, and for

$$\Sigma\,.\left\{\left(\frac{d\lambda}{dx}\right).\,dx + \left(\frac{d\lambda}{dy}\right).\,dy + \left(\frac{d\lambda}{dz}\right).\,dz\right\},$$

its value $d\,\lambda$, [409a], we shall obtain the equation [440], or as it may be written,

$$0 = 2\,\Sigma\,.\,m\,.\,\frac{(dx\,.\,ddx + dy\,.\,ddy + dz\,.\,ddz)}{dt^2} - d\,.\,\frac{\left\{(\Sigma\,.\,m\,dx)^2 + (\Sigma\,.\,m\,dy)^2 + (\Sigma\,.\,m\,dz)^2\right\}}{(M+\Sigma\,.\,m)\,.\,dt^2}$$

$$+ 2M\,.\,\Sigma\,.\,\frac{m\,dr}{r^2} - 2\,.\,d\lambda,$$

which gives by integration

[441] $$\text{constant} = \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} - \frac{\{(\Sigma . m dx)^2+(\Sigma . m dy)^2+(\Sigma . m dz)^2\}}{(M+\Sigma . m) . dt^2}$$
$$-2 . M . \Sigma . \frac{m}{r} - 2\lambda ;$$

or

[442] $$h = M . \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} + \Sigma . m m' . \left\{\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2}\right\}$$
$$-\left\{2 M . \Sigma . \frac{m}{r} + 2\lambda\right\} . (M+\Sigma . m) ; \qquad (7)$$

h being an arbitrary constant quantity. We had already found these integrals, in the fifth chapter of the first book, for a system of bodies which act upon each other in any manner whatever ; but on account of the importance of these formulas in the theory of the system of the world, we have thought it best to give this additional demonstration.

10. These integrals are all which have been obtained. In the present state of analysis, we are under the necessity of having recourse to methods [442′] of approximation, making use of those simplifications which arise from the structure of the system of the world. One of the most remarkable of these methods depends on the circumstance that the solar system is divided into several smaller systems, composed of the planets and their satellites. These systems are so formed that the distances of the satellites from their primary planets, are much less than the distances of the planets from the sun ; and [442″] since in this case the action of the sun is nearly the same on the planet and on the satellites, it follows that the satellites must move, about the planet,

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whose integral is evidently of the form [441]. Multiply [441] by $(M+\Sigma . m)$, and put h for the term produced by the constant quantity of the first member, it becomes,

$$h = M . \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2} + \Sigma . m . \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2}$$
$$-\left(\Sigma . \frac{m dx}{dt}\right)^2 - \left(\Sigma . \frac{m dy}{dt}\right)^2 - \left(\Sigma . \frac{m dz}{dt}\right)^2 - \left\{2 M . \Sigma . \frac{m}{r} + 2\lambda\right\} . (M+\Sigma . m),$$

which, by substituting for $\Sigma . m . \Sigma . m . \frac{(dx^2+dy^2+dz^2)}{dt^2}$, its identical value [190a], becomes of the form [442].

in nearly the same manner as they would if they were attracted by the planet
alone. Hence we derive this remarkable property, that the motion of the
centre of gravity of a planet and its satellites, is nearly the same as it would [442''']
be if all the bodies were united together in this centre.

To demonstrate this, suppose that the distances of the bodies m, m', m'',
&c., from each other, are very small in comparison with the distance of their [442^iv]
common centre of gravity from the body M. Put

$$\begin{array}{lll} x = X + x_{\prime}\,; & y = Y + y_{\prime}\,; & z = Z + z_{\prime}\,; \\ x' = X + x'_{\prime}\,; & y' = Y + y'_{\prime}\,; & z' = Z + z'_{\prime}\,; \\ \&c.\,; & & \end{array} \qquad [443]$$

X, Y, Z, being the co-ordinates of the centre of gravity of the system of
bodies m, m', m'', &c.; the origin of these co-ordinates, as well as that of [443']
the co-ordinates x, y, z, x', y', z', &c., being at the centre of M. It is evident
that $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, $x'_{\prime}$, &c., will be the co-ordinates of m, m', &c., referred to their
common centre of gravity; we shall suppose these co-ordinates to be very
small quantities of the first order, in comparison with X, Y, Z. This being [443'']
premised, we shall find, as has been shown in the first book,* that the force
acting upon the centre of gravity of the system, parallel to any right line
whatever, is equal to the sum of the forces which act on the body parallel to
that right line, multiplied respectively by their masses, and divided by the [443''']
sum of the masses. We have seen also, in the same book, that the mutual
actions of bodies connected together in any manner, does not alter the motion
of the centre of gravity of the system;† and by § 8 [404'], the mutual
attraction of the bodies does not affect this motion; it is therefore sufficient,
in estimating the forces acting on the centre of gravity of the system,
to have regard to the action of the body M, which is foreign from the [443^iv]
system.

The action of M upon m, resolved parallel to the axis of x, and taken in a

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* (314) This results from the equations [155], and the remarks which follow immediately after them, taken in connexion with what is said on the same subject, in the note in page 103.

† (315) This appears from the equations [157], taken in connexion with the remarks immediately following them.

[443v] direction opposite to the origin, is* $-\frac{M.x}{r^3}$; therefore the whole force which acts on the centre of gravity of the system of bodies m, m', m'', &c., in that direction, is [443‴]

[444] $$-\frac{M.\Sigma.\frac{mx}{r^3}}{\Sigma.m}.$$

By substituting for x and r their values [443, 411], we have

[445] $$\frac{x}{r^3}=\frac{X+x_{,}}{\{(X+x_{,})^2+(Y+y_{,})^2+(Z+z_{,})^2\}^{\frac{3}{2}}}.$$

[445′] If we neglect the very small quantities of the second order, that is, the squares and products of the variable quantities $x_{,}$, $y_{,}$, $z_{,}$, $x'_{,}$, &c.; and put R equal to the distance $\sqrt{X^2+Y^2+Z^2}$ of the centre of gravity of the system from the body M; we shall find†

[446] $$\frac{x}{r^3}=\frac{X}{R^3}+\frac{x_{,}}{R^3}-\frac{3.X.\{Xx_{,}+Yy_{,}+Zz_{,}\}}{R^5}.$$

[446′] By marking successively with one accent, two accents, &c., the letters $x_{,}$, $y_{,}$, $z_{,}$, in the second member of this equation, we shall have the values of $\frac{x'}{r'^3}$, $\frac{x''}{r''^3}$; &c.; but we have, by the nature of the centre of gravity,‡

[447] $$0=\Sigma.mx_{,}\,; \qquad 0=\Sigma.my_{,}\,; \qquad 0=\Sigma.mz_{,}\,;$$

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* (315a) The force $-\frac{Mx}{r^3}$ is found as in [414′]. The similar forces on m', m'', &c. are $-\frac{Mx'}{r'^3}$, $\frac{Mx''}{r''^3}$, &c. Multiply these respectively by m, m', m'', &c. and add the products together; then divide the sum by $\Sigma.m$, as in [443‴], and we shall obtain the expression [444].

† (316) The denominator of [445], putting $X^2+Y^2+Z^2=R^2$, is nearly

$$\{R^2+2.(Xx_{,}+Yy_{,}+Zz_{,})\}^{-\frac{3}{2}}=R^{-3}.\left\{1+\frac{2.(Xx_{,}+Yy_{,}+Zz_{,})}{R^2}\right\}^{-\frac{3}{2}}$$
$$=R^{-3}.\left\{1-\frac{3.(Xx_{,}+Yy_{,}+Zz_{,})}{R^2}\right\}=\frac{1}{R^3}-\frac{3.(Xx_{,}+Yy_{,}+Zz_{,})}{R^5},$$

hence the value of $\frac{x}{r^3}$, [446], is easily deduced.

‡ (317) These are the same as the equations [124]. Now if we multiply [446] by m, we shall have $\frac{mx}{r^3}$, and by accenting the letters we shall get $\frac{m'x'}{r'^3}$, &c. The sum of all these is

hence we shall have, by neglecting quantities of the second order,

$$-\frac{M.\Sigma.\frac{mx}{r^3}}{\Sigma.m}=-\frac{M.X}{R^3};$$ [448]

therefore the centre of gravity of the system is attracted in a direction
parallel to the axis of x, by the action of the body M, in the same manner as [448′]
if all the bodies of the system were united in that centre.* The same
result evidently takes place with respect to the axes of y and z; so that
the forces with which the centre of gravity of the system is urged, parallel
to these axes, by the action of M, will be $-\frac{M.Y}{R^3}$, and $-\frac{M.Z}{R^3}$. [448″]

When we consider the relative motion of the centre of gravity of the
system about M, we must transfer to that centre, in a contrary direction, the
force which acts on the body M. This force, resulting from the actions of
m, m', m'', &c., on M, resolved in a direction parallel to the axis of x, and
tending to increase the co-ordinates, is† $\Sigma.\frac{mx}{r^3}$; and if we neglect quantities [448‴]
of the second order, this function will be, by what precedes [448], equal to

$$\frac{X.\Sigma.m}{R^3};$$ [449]

In like manner, the forces with which M is urged, by the action of the other

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$$\Sigma.\frac{mx}{r^3}=\frac{X}{R^3}.\Sigma.m+\frac{1}{R^3}.\Sigma.mx_{,}-\frac{3.X^2}{R^5}.\Sigma.mx_{,}-\frac{3.XY}{R^5}.\Sigma.my_{,}-\frac{3.XZ}{R^5}.\Sigma.mz_{,},$$

which by means of [447], becomes simply, $\Sigma.\frac{mx}{r^3}=\frac{X}{R^3}.\Sigma.m$. Substituting this in [444], [447a]
we shall get [448].

* (318) The action of the body M on that centre, resolved in a direction parallel to the axis of x, is found in the same manner as in [414′], where the action of M upon m was computed to be $-\frac{Mx}{r^3}$, and by changing x, r, into X, R, we obtain the force corresponding to a *point* placed in the centre of gravity of the system $-\frac{MX}{R^3}$.

† (318a) This is proved in [412′], and in [447a], it is shown that $\Sigma.\frac{mx}{r^3}=\frac{X}{R^3}.\Sigma.m$, as in [449]. The formulas [450] are found in the same manner, merely changing X into Y, Z.

bodies of the system, parallel to the axes of y and z, and in directions contrary to their origin, are

[450] $$\frac{Y.\Sigma.m}{R^3}, \qquad \text{and} \qquad \frac{Z.\Sigma.m}{R^3}.$$

Hence we see, that the action of the system on the body M, is nearly the same as if all the bodies of the system were united in their common centre of gravity.* By transferring to this centre, and with a contrary sign, the three preceding forces, this point will be urged parallel to the axes of x, y, and z, in its relative motion about M, by the three following forces, [448, 449, &c.] [450′]

[451] $$-(M+\Sigma.m).\frac{X}{R^3}; \qquad -(M+\Sigma.m).\frac{Y}{R^3}; \qquad -(M+\Sigma.m).\frac{Z}{R^3}.$$

[451′] These forces are the same as if all the bodies m, m', m'', &c., were united in their common centre of gravity;† this centre therefore moves as if all the bodies were united in it, neglecting quantities of the second order.

Hence it follows, that if there are several systems whose centres of gravity are very far from each other, in comparison with the mutual distances of the bodies of any system from each other, these centres of gravity will be moved in nearly the same manner, as if the bodies of each system were united in their respective centres; for the action of the first system upon each body of the second system, is, by what has been said, very nearly the same as if all the bodies of the first system were united in their common centre of gravity; the action of the first system upon the centre of gravity of the second, will therefore, by what precedes, be the same as in that hypothesis; hence we may in general conclude that the reciprocal action of the different [451″]

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* (318b) If all the bodies m, m', &c. were collected in their centre of gravity, their action upon the centre of the body M would be represented by $\frac{\Sigma.m}{R^2}$, and this resolved in the direction parallel to X, will be $\frac{X.\Sigma.m}{R^3}$, [13]; this is equal to the expression [449], which represents the sum of all the separate forces of m, m', &c. upon M. The formulas [450] furnish the same result.

† (319) Because the terms $x_{,}$, $y_{,}$, $z_{,}$, $x'_{,}$, &c. are not found in [451] which is the same as if $x_{,}=0$, $y_{,}=0$, &c.

systems on their respective centres of gravity, is the same as if the bodies of each system were united in these centres; consequently these centres must move in the same manner as if the masses were thus united. $[451''']$ It is evident that this result takes place, whether the bodies are free, or connected together in any manner whatever; because their mutual action has no effect on the motion of their common centre of gravity [443''', &c.]

On the attraction of a system of bodies upon another system.

The system of a planet and its satellites acts therefore very nearly on the other bodies of the solar system, as if the planet and its satellites were $[451^{\mathrm{iv}}]$ united in their common centre of gravity; and this centre is attracted by the different bodies of the solar system, as in this hypothesis.

Every heavenly body is formed by the union of an infinite number of particles, endowed with an attractive power, and as the dimensions of the body are very small in comparison with the distances of the other bodies $[451^{\mathrm{v}}]$ of the system of the world, the centre of gravity of the body will be attracted in nearly the same manner, as if all its mass was collected in this centre; and its action on the other bodies of the system will be nearly the same as in that supposition. Therefore we may, in the investigation of the motions of the centres of gravity of the heavenly bodies, consider these bodies as massive points, placed in their centres of gravity. This supposition, which is very near the truth, is rendered much more exact by the spherical $[451^{\mathrm{vi}}]$ form of the planets and satellites. For these bodies may be considered as being formed very nearly of spherical strata, whose densities vary according to any law; and we shall now show that the action of a spherical stratum upon a body placed without it, is the same as if its mass was united at its centre. To prove this, we shall give some general propositions on the $[451^{\mathrm{vii}}]$ attractions of spheroids, which will be very useful in the course of this work.

On the attraction of Spheroids.

11. Let x, y, z, be the three co-ordinates of the attracted point, which we shall denote by m; dM a particle of the spheroid, and x', y', z', the $[451^{\mathrm{viii}}]$ co-ordinates of this particle; if we put ρ for its density, ρ being a function of $[451^{\mathrm{ix}}]$ x', y', z', independent of x, y, z; we shall have*

$$dM=\rho\,.\,dx'\,.\,dy'\,.\,dz'. \qquad [452]$$

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* (320) The attracting particle is supposed to be a parallelopiped whose sides are dx', dy', dz', and density ρ. Its attraction upon the point m, is evidently equal to the mass dM,

The action of dM upon m, resolved parallel to the axis of x, and directed towards the origin of the co-ordinates, will be

[453] $$\frac{\rho\,.\,dx'\,.\,dy'\,.\,dz'\,.\,(x-x')}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{3}{2}}}\,;$$

consequently it will be equal to

[454] $$-\left\{\frac{d\,.\,\dfrac{\rho\,.\,dx'\,.\,dy'\,.\,dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{dx}\right\}\,;$$

putting therefore

[455] $$V=\int\frac{\rho\,.\,dx'\,.\,dy'\,.\,dz'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}\,;$$

the integral being taken so as to include the whole mass of the spheroid;*

[455′] we shall have $-\left(\frac{dV}{dx}\right)$, for the whole action of the spheroid on the point m, resolved parallel to the axis of x, and *directed towards the origin of the co-ordinates.*

[455″] V is the sum of the particles of the spheroid, divided by their respective distances from the attracted point. To obtain the attraction of the spheroid upon this point, parallel to any right line whatever, we must suppose V to be a function of three rectangular co-ordinates, one of which is parallel to

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divided by the square of the distance [392], $(x-x')^2+(y-y')^2+(z-z')^2$. This

[455a] being multiplied by $\frac{x-x'}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}}$, gives, as in [394], the force resolved in a direction parallel to the axis of x, and directed *towards the origin* of the co-ordinates, agreeing with [453]. The direction of the force in [455′] is different from that in [393′], and this is indicated by the term $(x'-x)$, in the numerator of [394], being changed to $[x-x']$ in [453]. This last formula might be illustrated by a figure, as in page 8, in which the attracting particle dM, whose co-ordinates are x', y', z', should be placed at A, while the attracted point m is placed at c, whose co-ordinates are x, y, z; so that x', y', z', would be less than x, y, z, respectively. We may observe that the formula [453] is, by development, equivalent to [454].

* (321) In finding the integral [455], x, y, z, are considered as constant quantities; therefore $-\left(\frac{dV}{dx}\right)$, deduced from [455], must be equivalent to the whole of the forces [454], corresponding to the whole mass of the spheroid.

the proposed right line, and we must then take the differential of this function relative to that co-ordinate; *the coefficient of this differential,* [455‴] *taken with a contrary sign, will be the expression of the attraction of the spheroid, parallel to the proposed line, and directed towards the origin of the co-ordinate to which it is parallel.**

If we put

$$\beta = \{(x-x')^2+(y-y')^2+(z-z')^2\}^{-\frac{1}{2}}\,; \qquad [455^{iv}]$$

we shall have

$$V = \int \beta\,.\,\rho\,.\,dx'\,.\,dy'\,.\,dz'. \qquad [456]$$

The integration referring only to x', y', z', it is evident that we shall have†

$$\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right)=\int\rho\,.\,dx'.\,dy'.\,dz'.\left\{\left(\frac{dd\beta}{dx^2}\right)+\left(\frac{dd\beta}{dy^2}\right)+\left(\frac{dd\beta}{dz^2}\right)\right\}. \qquad [457]$$

But we have‡

$$0=\left(\frac{dd\beta}{dx^2}\right)+\left(\frac{dd\beta}{dy^2}\right)+\left(\frac{dd\beta}{dz^2}\right)\,; \qquad [458]$$

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* (322) As V is equal to the sum of the particles of the spheroid, divided by their respective distances from the attracted point, [455″], the values of V must be independent of the situation of the co-ordinates, which may therefore be changed, so as to have one of them parallel to the right line, in the direction of which the force of attraction is required to be computed, and by naming this new right line X, the attraction will be represented by $-\left(\frac{dV}{dX}\right)$, as is evident from [455′], changing x into X.

† (323) For ρ, dx', dy', dz', being independent of x, y, z, [451^{ix}], we shall have $\left(\frac{ddV}{dx^2}\right)=\int\rho\,.\,dx'\,.\,dy'\,.\,dz'\,.\left(\frac{dd\beta}{dx^2}\right)$, and by changing, successively, x into y and z, we shall obtain similar expressions of $\left(\frac{ddV}{dy^2}\right)$, $\left(\frac{ddV}{dz^2}\right)$, whose sum is as in [457].

‡ (324) $\beta=\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{-\frac{1}{2}}$, [455^{iv}], gives $\left(\frac{d\beta}{dx}\right)=-(x-x')\,.\left\{(x-x')^2+(y-y')^2+(z-z')^2\right\}^{-\frac{3}{2}}$, or $\left(\frac{d\beta}{dx}\right)=-(x-x')\,.\,\beta^3$. The differential of this relative to x, is $\left(\frac{dd\beta}{dx^2}\right)=-\beta^3-3\,.\,(x-x')\,.\,\beta^2\,.\left(\frac{d\beta}{dx}\right)$, and by substituting the value of $\left(\frac{d\beta}{dx}\right)$, we shall get $\left(\frac{dd\beta}{dx^2}\right)=\beta^5\,.\left\{-\beta^{-2}+3\,.\,(x-x')^2\right\}$. Changing, successively, x into y and z, &c. we shall find

$$\left(\frac{dd\beta}{dy^2}\right)=\beta^5\,.\,\{-\beta^{-2}+3\,.\,(y-y')^2\}, \qquad \left(\frac{dd\beta}{dz^2}\right)=\beta^5\,.\,\{-\beta^2+3\,.\,(z-z')^2\}.$$

therefore we shall have

[459] $$0=\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right). \qquad (A)$$

Important Equation for computing the attractions of Spheroids and the figures of the Heavenly Bodies.

This remarkable equation will be of the greatest use to us, in the theory of the figures of the heavenly bodies. We may put it under other forms which are more convenient on several occasions. For example, suppose we draw from the origin of the co-ordinates to the attracted point, a radius which we
[459′] shall call r; let θ be the angle which this radius makes with the axis of x, and ϖ the angle which the plane formed by r and by this axis, makes with the plane of x and y;[*] we shall have

[460] $$x=r\,.\,\cos.\,\theta\,; \qquad y=r\,.\,\sin.\,\theta\,.\,\cos.\,\varpi\,; \qquad z=r\,.\,\sin.\,\theta\,.\,\sin.\,\varpi\,;$$

The sum of these three quantities is

$$\left(\frac{dd\beta}{dx^2}\right)+\left(\frac{dd\beta}{dy^2}\right)+\left(\frac{dd\beta}{dz^2}\right)=\beta^5\,.\left\{-3\,.\,\beta^{-2}+3\,.\,[(x-x')^2+(y-y')^2+(z-z')^2]\right\};$$

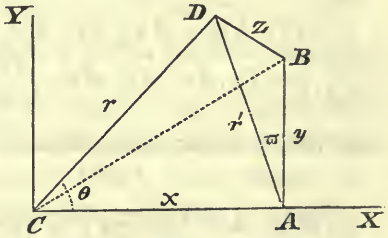
the second member of which is evidently equal to nothing, because

$$(x-x')^2+(y-y')^2+(z-z')^2=\beta^{-2}, \; [455^{iv}];$$

therefore each term of the integral, [457],

$$\int\rho\,.\,dx'\,.\,dy'\,.\,dz'\,.\left\{\left(\frac{dd\beta}{dx^2}\right)+\left(\frac{dd\beta}{dy^2}\right)+\left(\frac{dd\beta}{dz^2}\right)\right\},$$

must vanish, consequently, $\left(\frac{ddV}{dx^2}\right)+\left(\frac{ddV}{dy^2}\right)+\left(\frac{ddV}{dz^2}\right)=0$, as in [459].



[*] (325) In the adjoined figure, let C be the origin of the co-ordinates, D the attracted point, CA the axis of x; AB, parallel to CY the axis of y; and BD parallel to the axis of z; making $CA=x$, $AB=y$, $BD=z$, $CD=r$, the angles $ACD=\theta$, and $DAB=\varpi$, Then

$$CA=CD\,.\cos.\,ACD,$$

gives $x=r\,.\cos.\theta$, [460], $AD=CD\,.\sin.\,ACD=r\,.\sin.\,\theta$, and in the rectangular triangle DBA, we have $AB=AD\,.\cos.\,DAB$; $BD=AD\,.\sin.\,DAB$; hence $y=r\,.\sin.\,\theta\,.\cos.\,\varpi$; $z=r\,.\sin.\,\theta\,.\sin.\,\varpi$, as in [460]. It is also evident that $CD=\sqrt{CA^2+AB^2+BD^2}$, $\cos.\,ACD=\frac{CA}{CD}$, $\text{tang.}\,DAB=\frac{BD}{AB}$, whence we easily obtain, geometrically, the expressions of r, $\cos.\,\theta$, $\text{tang.}\,\varpi$, [461], which are also easily proved to be correct, by substituting the values of x, y, z, [460], in the second members of [461]; since by reduction they will become like the first members of [461.]

from which we get

$$r=\sqrt{x^2+y^2+z^2}; \qquad \cos.\theta=\frac{x}{\sqrt{x^2+y^2+z^2}}; \qquad \text{tang.}\,\varpi=\frac{z}{y}; \qquad [461]$$

Hence we may obtain the partial differentials of r, θ, and ϖ, relative to the variable quantities x, y, z; and may thence deduce the values of $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dy^2}\right)$, $\left(\frac{ddV}{dz^2}\right)$, in partial differentials of V, relative to the variable quantities r, θ, ϖ. As we shall often use these transformations of partial differentials, it will be useful to recall to mind the principles of this calculation. [461′] Considering V first as a function of x, y, z, and then as a function of the variable quantities r, θ, ϖ, we shall have

$$\left(\frac{dV}{dx}\right)=\left(\frac{dV}{dr}\right).\left(\frac{dr}{dx}\right)+\left(\frac{dV}{d\theta}\right).\left(\frac{d\theta}{dx}\right)+\left(\frac{dV}{d\varpi}\right).\left(\frac{d\varpi}{dx}\right). \qquad [462]$$

To obtain the partial differentials $\left(\frac{dr}{dx}\right)$, $\left(\frac{d\theta}{dx}\right)$, $\left(\frac{d\varpi}{dx}\right)$, we must suppose x only to vary in the preceding expressions of r, $\cos.\theta$, and $\text{tang.}\,\varpi$, [461]; taking therefore the differentials of these expressions, we shall have*

$$\left(\frac{dr}{dx}\right)=\cos.\theta; \qquad \left(\frac{d\theta}{dx}\right)=-\frac{\sin.\theta}{r}; \qquad \left(\frac{d\varpi}{dx}\right)=0; \qquad [463]$$

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* (326) The partial differentials of $r=\sqrt{x^2+y^2+z^2}$, [461], relative to x, y, z, are

$$\left(\frac{dr}{dx}\right)=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}=\cos.\theta; \qquad \left(\frac{dr}{dy}\right)=\frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{y}{r}=\sin.\theta.\cos.\varpi; \qquad [463a]$$

$$\left(\frac{dr}{dz}\right)=\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{r}=\sin.\theta.\sin.\varpi. \qquad [463b]$$

Again from $\cos.\theta=\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$, we get $\sin.\theta=\frac{\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}}$, and since the differential of $\cos.\theta$ is $-d\theta.\sin.\theta$. we shall find, by substituting the value of $\sin.\theta$,

$$d\theta=\frac{-\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}.d.\frac{x}{\sqrt{x^2+y^2+z^2}};$$

hence we shall get its partial differentials, which may be reduced, by observing that

$$\sqrt{y^2+z^2}=r.\sin.\theta, \ [460]. \qquad [463c]$$

$$\left(\frac{d\theta}{dx}\right)=\frac{-\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}.\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}=-\frac{\sqrt{y^2+z^2}}{x^2+y^2+z^2}=-\frac{\sin.\theta}{r}, \qquad [465a]$$

$$\left(\frac{d\theta}{dy}\right)=\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}.\frac{xy}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{xy}{\sqrt{y^2+z^2}.(x^2+y^2+z^2)}=\frac{xy}{r^2.\sqrt{y^2+z^2}}=\frac{\cos.\theta.\cos.\varpi}{r}, \qquad [465b]$$

[460].

hence we get

[464] $$\left(\frac{dV}{dx}\right)=\cos.\theta.\left(\frac{dV}{dr}\right)-\frac{\sin.\theta}{r}.\left(\frac{dV}{d\theta}\right).$$

We shall thus have the partial differential $\left(\frac{dV}{dx}\right)$, expressed in partial

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[465c] In a similar way $\left(\frac{d\theta}{dz}\right)=\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}.\frac{xz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, which is equal to the preceding value of $\left(\frac{d\theta}{dy}\right)$, multiplied by $\frac{z}{y}=\text{tang.}\,\varpi$, [461], hence $\left(\frac{d\theta}{dz}\right)=\frac{\cos.\theta.\sin.\varpi}{r}$. Again from [461], $\varpi=\text{arc.}\left(\text{tang.}\frac{z}{y}\right)$, its differential, by 51 Int. is

[465d] $$d\varpi=\frac{d.\frac{z}{y}}{1+\frac{z^2}{y^2}}=\frac{y\,dz-z\,dy}{y^2+z^2}.$$

Hence,

$$\left(\frac{d\varpi}{dx}\right)=0;\qquad \left(\frac{d\varpi}{dy}\right)=-\frac{z}{y^2+z^2},\qquad \left(\frac{d\varpi}{dz}\right)=\frac{y}{y^2+z^2},$$

and by substituting the values y, z, [460], y^2+z^2, [463c], they will become,

[465e] $$\left(\frac{d\varpi}{dx}\right)=0;\qquad \left(\frac{d\varpi}{dy}\right)=\frac{-\sin.\varpi}{r.\sin.\theta};\qquad \left(\frac{d\varpi}{dz}\right)=\frac{\cos.\varpi}{r.\sin.\theta}.$$

The preceding quantities being substituted in $\left(\frac{dV}{dx}\right)$, [462], and in the similar expressions of $\left(\frac{dV}{dy}\right)$, $\left(\frac{dV}{dz}\right)$, will give their values under the required form, [464]. It now remains to obtain, from these, the values of $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dy^2}\right)$, $\left(\frac{ddV}{dz^2}\right)$. Putting for brevity $\left(\frac{dV}{dr}\right)=V'$, $\left(\frac{dV}{d\theta}\right)=V''$, $\left(\frac{dV}{d\varpi}\right)=V'''$, the formula [462] becomes,

[465f] $$\left(\frac{dV}{dx}\right)=V'.\left(\frac{dr}{dx}\right)+V''.\left(\frac{d\theta}{dx}\right)+V'''.\left(\frac{d\varpi}{dx}\right),$$

whose partial differential relative to x, is

$$\left(\frac{ddV}{dx^2}\right)=\left(\frac{dV'}{dx}\right).\left(\frac{dr}{dx}\right)+V'.\left(\frac{ddr}{dx^2}\right)+\left(\frac{dV''}{dx}\right).\left(\frac{d\theta}{dx}\right)$$
$$+V''.\left(\frac{dd\theta}{dx^2}\right)+\left(\frac{dV'''}{dx}\right).\left(\frac{d\varpi}{dx}\right)+V'''.\left(\frac{dd\varpi}{dx^2}\right).$$

The terms $\left(\frac{dV'}{dx}\right)$, $\left(\frac{dV''}{dx}\right)$, $\left(\frac{dV'''}{dx}\right)$, may be deduced from those of $\left(\frac{dV}{dx}\right)$, [462], by changing V into $\left(\frac{dV}{dr}\right).\left(\frac{dV}{d\theta}\right).\left(\frac{dV}{d\varpi}\right)$, respectively; hence differentials of V, taken with respect to the variable quantities r, θ, ϖ. Taking again the differential of $\left(\frac{dV}{dx}\right)$, we shall obtain the partial differential [464']

$$\left(\frac{dV'}{dx}\right)=\left(\frac{ddV}{dr^2}\right)\cdot\left(\frac{dr}{dx}\right)+\left(\frac{ddV}{drd\theta}\right)\cdot\left(\frac{d\theta}{dx}\right)+\left(\frac{ddV}{drd\varpi}\right)\cdot\left(\frac{d\varpi}{dx}\right);$$

$$\left(\frac{dV''}{dx}\right)=\left(\frac{ddV}{drd\theta}\right)\cdot\left(\frac{dr}{dx}\right)+\left(\frac{ddV}{d\theta^2}\right)\cdot\left(\frac{d\theta}{dx}\right)+\left(\frac{ddV}{d\theta d\varpi}\right)\cdot\left(\frac{d\varpi}{dx}\right);$$

$$\left(\frac{dV'''}{dx}\right)=\left(\frac{ddV}{drd\varpi}\right)\cdot\left(\frac{dr}{dx}\right)+\left(\frac{ddV}{d\theta d\varpi}\right)\cdot\left(\frac{d\theta}{dx}\right)+\left(\frac{ddV}{d\varpi^2}\right)\cdot\left(\frac{d\varpi}{dx}\right);$$

which being substituted in $\left(\frac{ddV}{dx^2}\right)$, it becomes

$$\begin{aligned}\left(\frac{ddV}{dx^2}\right)=&\left(\frac{ddV}{dr^2}\right)\cdot\left(\frac{dr}{dx}\right)^2+\left(\frac{ddV}{d\theta^2}\right)\cdot\left(\frac{d\theta}{dx}\right)^2+\left(\frac{ddV}{d\varpi^2}\right)\cdot\left(\frac{d\varpi}{dx}\right)^2\\&+\left(\frac{dV}{dr}\right)\cdot\left(\frac{ddr}{dx^2}\right)+\left(\frac{dV}{d\theta}\right)\cdot\left(\frac{dd\theta}{dx^2}\right)+\left(\frac{dV}{d\varpi}\right)\cdot\left(\frac{dd\varpi}{dx^2}\right)\\&+2.\left(\frac{ddV}{drd\theta}\right)\cdot\left(\frac{dr}{dx}\right)\cdot\left(\frac{d\theta}{dx}\right)+2.\left(\frac{ddV}{drd\varpi}\right)\cdot\left(\frac{dr}{dx}\right)\cdot\left(\frac{d\varpi}{dx}\right)+2.\left(\frac{ddV}{d\theta d\varpi}\right)\cdot\left(\frac{d\theta}{dx}\right)\cdot\left(\frac{d\varpi}{dx}\right).\end{aligned}$$ [465g]

By changing in this, successively, x into y, and z, we obtain $\left(\frac{ddV}{dy^2}\right)$; $\left(\frac{ddV}{dz^2}\right)$; and the sum of all three of these expressions being put equal to nothing, gives the transformation of the proposed equation, [459]. We shall examine, separately, the coefficients of each of the nine terms of which this equation is composed. *First*, The coefficient of $\left(\frac{ddV}{dr^2}\right)$, is

$$\left(\frac{dr}{dx}\right)^2+\left(\frac{dr}{dy}\right)^2+\left(\frac{dr}{dz}\right)^2=\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2+\left(\frac{z}{r}\right)^2=\frac{x^2+y^2+z^2}{r^2}=1.$$ [465h]

Second, The coefficient of $\left(\frac{ddV}{d\theta^2}\right)$, is

$$\begin{aligned}\left(\frac{d\theta}{dx}\right)^2+\left(\frac{d\theta}{dy}\right)^2+\left(\frac{d\theta}{dz}\right)^2&=\left(\frac{-\sin.\theta}{r}\right)^2+\left(\frac{\cos.\theta\,.\cos.\varpi}{r}\right)^2+\left(\frac{\cos.\theta\,.\sin.\varpi}{r}\right)^2\\&=\frac{\sin.^2\theta+\cos.^2\theta\,.(\cos.^2\varpi+\sin.^2\varpi)}{r^2}=\frac{\sin.^2\theta+\cos.^2\theta}{r^2}=\frac{1}{r^2}.\end{aligned}$$ [465i]

Third, The coefficient of $\left(\frac{ddV}{d\varpi^2}\right)$, is

$$\left(\frac{d\varpi}{dx}\right)^2+\left(\frac{d\varpi}{dy}\right)^2+\left(\frac{d\varpi}{dz}\right)^2=\frac{\sin.^2\varpi+\cos.^2\varpi}{(r\,.\sin.\theta)^2}=\frac{1}{r^2.\sin.^2\theta}.$$ [465k]

Fourth, The coefficient of $\left(\frac{dV}{dr}\right)$ is $\left(\frac{ddr}{dx^2}\right)+\left(\frac{ddr}{dy^2}\right)+\left(\frac{ddr}{dz^2}\right)$. Now we have $\left(\frac{dr}{dx}\right)=\frac{x}{\sqrt{x^2+y^2+z^2}}$, [463a], and its partial differential relative to x, is

$$\left(\frac{ddr}{dx^2}\right)=\frac{1}{\sqrt{x^2+y^2+z^2}}-\frac{x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{1}{r}-\frac{x^2}{r^3};$$ [465l]

[464″] $\left(\frac{d\,d\,V}{d\,x^2}\right)$, in partial differentials of V, taken with respect to the variable quantities r, θ, and ϖ. We may find, by the same method, the values of $\left(\frac{d\,d\,V}{d\,y^2}\right)$, $\left(\frac{d\,d\,V}{d\,z^2}\right)$.

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and in like manner, by changing x into y, and into z, we get

$$\left(\frac{d\,d\,r}{d\,y^2}\right)=\frac{1}{r}-\frac{y^2}{r^3}; \qquad \left(\frac{d\,d\,r}{d\,z^2}\right)=\frac{1}{r}-\frac{z^2}{r^3}.$$

The sum of these three expressions is $\frac{3}{r}-\frac{(x^2+y^2+z^2)}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{2}{r}$, therefore

[465m] $$\left(\frac{d\,d\,r}{d\,x^2}\right)+\left(\frac{d\,d\,r}{d\,y^2}\right)+\left(\frac{d\,d\,r}{d\,z^2}\right)=\frac{2}{r},$$

and this term becomes $\frac{2}{r}\cdot\left(\frac{d\,V}{d\,r}\right)$.

Fifth, The coefficient of $\left(\frac{d\,V}{d\,\theta}\right)$ is $\left(\frac{d\,d\,\theta}{d\,x^2}\right)+\left(\frac{d\,d\,\theta}{d\,y^2}\right)+\left(\frac{d\,d\,\theta}{d\,z^2}\right)$;

and if we take the partial differentials of

$$\left(\frac{d\,\theta}{d\,x}\right)=\frac{-\sqrt{y^2+z^2}}{x^2+y^2+z^2}, \quad \left(\frac{d\,\theta}{d\,y}\right)=\frac{x\,y}{(x^2+y^2+z^2)\cdot\sqrt{y^2+z^2}}, \quad \left(\frac{d\,\theta}{d\,z}\right)=\frac{x\,z}{(x^2+y^2+x^2)\cdot\sqrt{y^2+z^2}},$$

[465a—c], relative to x, y, z, and put afterwards for brevity $\sqrt{x^2+y^2+z^2}=r$, and $\sqrt{y^2+z^2}=s$, we shall get,

[465n] $$\left(\frac{d\,d\,\theta}{d\,x^2}\right)=\frac{2\,s\,x}{r^4}, \qquad \left(\frac{d\,d\,\theta}{d\,y^2}\right)=\frac{x}{r^2\,s}-\frac{2\,x\,y^2}{r^4\,s}-\frac{x\,y^2}{r^2\,s^3}; \qquad \left(\frac{d\,d\,\theta}{d\,z^2}\right)=\frac{x}{r^2\,s}-\frac{2\,x\,z^2}{r^4\,s}-\frac{x\,z^2}{r^2\,s^3}.$$

The sum of these three expressions is

[465o] $$\frac{2\,s\,x}{r^4}+\frac{2\,x}{r^2\,s}-\frac{2\,x\,.\,(y^2+z^2)}{r^4\,s}-\frac{x\,.\,(y^2+z^2)}{r^2\,s^3}=\frac{2\,s\,x}{r^4}+\frac{2\,x}{r^2\,s}-\frac{2\,s\,x}{r^4}-\frac{x}{r^2\,s}=\frac{x}{r^2\,s},$$

and using $s=r\,.\,\sin.\,\theta$, [463c], it becomes $\frac{\cos.\,\theta}{r^2\,.\,\sin.\,\theta}$, therefore this term becomes $\frac{\cos.\,\theta}{r^2\,.\,\sin.\,\theta}\cdot\left(\frac{d\,V}{d\,\theta}\right)$.

Sixth, The coefficient of $\left(\frac{d\,V}{d\,\varpi}\right)$ is $\left(\frac{d\,d\,\varpi}{d\,x^2}\right)+\left(\frac{d\,d\,\varpi}{d\,y^2}\right)+\left(\frac{d\,d\,\varpi}{d\,z^2}\right)$;

and if we take the partial differentials of the preceding values [465d—e], we shall get

[465p] $$\left(\frac{d\,d\,\varpi}{d\,x^2}\right)=0, \qquad \left(\frac{d\,d\,\varpi}{d\,y^2}\right)=\frac{2\,y\,z}{(y^2+z^2)^2}, \qquad \left(\frac{d\,d\,\varpi}{d\,z^2}\right)=\frac{-2\,y\,z}{(y^2+z^2)^2},$$

whose sum is nothing, therefore the coefficient of $\left(\frac{d\,V}{d\,\varpi}\right)$ is nothing.

Seventh, The coefficient of $2\,.\left(\frac{d\,d\,V}{d\,r\,d\,\theta}\right)$ is

In this way, the equation (*A*) [459] may be transformed into the following.

$$0=\left(\frac{d d V}{d \theta^2}\right)+\frac{\cos.\theta}{\sin.\theta}\cdot\left(\frac{d V}{d \theta}\right)+\frac{\left(\frac{d d V}{d \varpi^2}\right)}{\sin.^2\theta}+r\cdot\left(\frac{d d\,.\,r V}{d r^2}\right);\qquad (B)$$ [465]

and if we put $\cos.\theta=\mu$, this last equation will become* [465′]

$$\left(\frac{d r}{d x}\right)\cdot\left(\frac{d \theta}{d x}\right)+\left(\frac{d r}{d y}\right)\cdot\left(\frac{d \theta}{d y}\right)+\left(\frac{d r}{d z}\right)\cdot\left(\frac{d \theta}{d z}\right)$$ [465q]

$$=\frac{-\cos.\theta\,.\sin.\theta}{r}+\frac{(\sin.\theta\,.\cos.\varpi)\,.\,(\cos.\theta\,.\cos.\varpi)}{r}+\frac{(\sin.\theta\,.\sin.\varpi)\,.\,(\cos.\theta\,.\sin.\varpi)}{r}$$

$$=\frac{\cos.\theta\,.\sin.\theta}{r}\,.\,\left\{-1+\cos.^2\varpi+\sin.^2\varpi\right\}=0.$$

Eighth, The coefficient of $2\,.\left(\frac{d d V}{d r d \varpi}\right)$ is

$$\left(\frac{d r}{d x}\right)\cdot\left(\frac{d \varpi}{d x}\right)+\left(\frac{d r}{d y}\right)\cdot\left(\frac{d \varpi}{d y}\right)+\left(\frac{d r}{d z}\right)\cdot\left(\frac{d \varpi}{d z}\right),$$

of which the first term vanishes, because $\left(\frac{d \varpi}{d x}\right)=0$, [465*e*], and the other two become

$$(\sin.\theta\,.\cos.\varpi)\,.\,\frac{(-\sin.\varpi)}{r\,.\sin.\theta}+(\sin.\theta\,.\sin.\varpi)\,.\,\frac{\cos.\varpi}{r\,.\sin.\theta}=0.$$ [465r]

Ninth, The coefficient of $2\,.\left(\frac{d d V}{d \theta d \varpi}\right)$ is

$$\left(\frac{d \theta}{d x}\right)\cdot\left(\frac{d \varpi}{d x}\right)+\left(\frac{d \theta}{d y}\right)\cdot\left(\frac{d \varpi}{d y}\right)+\left(\frac{d \theta}{d z}\right)\cdot\left(\frac{d \varpi}{d z}\right),$$

the first term of which vanishes, because $\left(\frac{d \varpi}{d x}\right)=0$, and the other two terms become $\frac{(\cos.\theta\,.\cos.\varpi)}{r}\cdot\frac{(-\sin.\varpi)}{r\,.\sin.\theta}+\frac{(\cos.\theta\,.\sin.\varpi)}{r}\cdot\frac{\cos.\varpi}{r\,.\sin.\theta}=0.$ Now connecting together all [465s] these terms of the equation, [465*g*, &c.] we shall obtain the following transformation of [459].

$$0=\left(\frac{d d V}{d r^2}\right)+\frac{1}{r^2}\cdot\left(\frac{d d V}{d \theta^2}\right)+\frac{1}{r^2.\sin.^2\theta}\cdot\left(\frac{d d V}{d \varpi^2}\right)+\frac{2}{r}\cdot\left(\frac{d V}{d r}\right)+\frac{\cos.\theta}{r^2.\sin.\theta}\;\left(\frac{d V}{d \theta}\right).$$ [465t]

Multiply this by r^2, and substitute $r\,.\left(\frac{d d\,.\,r V}{d r^2}\right)$ for $2r\,.\left(\frac{d V}{d r}\right)+r^2\,.\left(\frac{d d V}{d r^2}\right)$, which [465u] are easily proved to be identical, by development, we shall obtain the formula [465].

* (327) Considering V first as a function of θ, and then as a function of μ, we shall have $\left(\frac{d V}{d \theta}\right)=\left(\frac{d V}{d \mu}\right)\cdot\left(\frac{d \mu}{d \theta}\right)$, $\left(\frac{d d V}{d \theta^2}\right)=\left(\frac{d d V}{d \mu^2}\right)\cdot\left(\frac{d \mu}{d \theta}\right)^2+\left(\frac{d V}{d \mu}\right)\cdot\left(\frac{d d \mu}{d \theta^2}\right)$. Hence,

$$\left(\frac{d d V}{d \theta^2}\right)+\frac{\cos.\theta}{\sin.\theta}\cdot\left(\frac{d V}{d \theta}\right)=\left(\frac{d d V}{d \mu^2}\right)\cdot\left(\frac{d \mu}{d \theta}\right)^2+\left(\frac{d V}{d \mu}\right)\cdot\left\{\left(\frac{d d \mu}{d \theta^2}\right)+\frac{\cos.\theta}{\sin.\theta}\cdot\left(\frac{d \mu}{d \theta}\right)\right\}.$$ [465w]

[466] $$0=\left\{\frac{d.\left\{(1-\mu^2).\left(\frac{dV}{d\mu}\right)\right\}}{d\mu}\right\}+\frac{\left(\frac{ddV}{d\varpi^2}\right)}{1-\mu^2}+r.\left(\frac{dd.rV}{dr^2}\right). \qquad (C)$$

12. Suppose now that the spheroid is a spherical stratum, whose centre
Attraction of Spheres. is at the origin of the co-ordinates; it is evident that V will depend solely
[466'] on r, and will not contain either μ or ϖ; the equation (C) [466] will then
become

[467] $$0=\left(\frac{dd.rV}{dr^2}\right);$$

whence by integration*

[468] $$V=A+\frac{B}{r},$$

A and B being two arbitrary constant quantities. Hence we shall have

[469] $$-\left(\frac{dV}{dr}\right)=\frac{B}{r^2}.$$

[469'] $-\left(\frac{dV}{dr}\right)$ expresses, by what has been said [455'''], the action of the spherical stratum upon the point m, resolved in the direction of the radius r, and tending towards the centre of the stratum; now it is evident that the whole action of the stratum must be in that direction;† therefore

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[466a] Now $\mu=\cos.\theta$, [465'], gives $\sin.\theta=\sqrt{1-\mu^2}$; $\left(\frac{d\mu}{d\theta}\right)=-\sin.\theta=-\sqrt{1-\mu^2}$;
[466b] $\left(\frac{d^2\mu}{d\theta^2}\right)=\frac{\mu.\left(\frac{d\mu}{d\theta}\right)}{\sqrt{1-\mu^2}}=-\mu.$ Hence, $\frac{\cos.\theta}{\sin.\theta}.\left(\frac{d\mu}{d\theta}\right)=-\mu,$ and the preceding
expression becomes $\left(\frac{ddV}{d\mu^2}\right).(1-\mu^2)-2\mu.\left(\frac{dV}{d\mu}\right),$ which is evidently equal to
$\left(\frac{d.\left\{(1-\mu^2).\left(\frac{dV}{d\mu}\right)\right\}}{d\mu}\right).$ This being substituted in [465], gives [466].

* (328) Multiply [467] by dr, and integrate, it becomes $d.\left(\frac{rV}{dr}\right)=A.$ Multiplying again by dr, and integrating, we find $rV=Ar+B,$ whence $V=A+\frac{B}{r}.$

† (329) For the stratum being spherical and homogeneous, its attracting particles must be equal, and similarly situated on every side of the line, drawn from the attracted point to the

$-\left(\frac{dV}{dr}\right)$ expresses the whole action of the spherical stratum upon the point *m*.

Suppose, in the first place, that this point is placed within the stratum. If it was situated precisely at the centre, the action of the stratum would be nothing ;* therefore, when $r=0$, we shall have $-\left(\frac{dV}{dr}\right)=0$, or [469′] $\frac{B}{r^2}=0$, which gives $B=0$; consequently $-\left(\frac{dV}{dr}\right)=0$, whatever may be the value of r. Hence it follows, *that a point placed within a spherical* [469″] *stratum is not affected by it*; *or, which is the same thing, it is equally attracted in every direction.*

If the point *m* is placed without the stratum ; it is evident that by supposing the point to be infinitely distant from the centre of the stratum, the action of the stratum will be the same as if the whole mass was collected in that centre ;† putting therefore M for the mass of the stratum ; $-\left(\frac{dV}{dr}\right)$ or [469ⁱᵛ] $\frac{B}{r^2}$ will become, in this case, equal to $\frac{M}{r^2}$, which gives $B=M$; therefore in general, for points situated without the stratum, we shall have

$$-\left(\frac{dV}{dr}\right)=\frac{M}{r^2}\,; \qquad [470]$$

that is, *a spherical stratum attracts any point situated without it, as if all the* [470′] *mass of the stratum was united at its centre.*

A sphere being a spherical stratum, the radius of whose interior surface

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centre of the stratum; and it is evident there cannot be any attraction, in a direction [469*a*] perpendicular to this line, because there is no reason why it should deviate on the one side rather than on the other. The whole attraction must therefore, be in the direction of the line connecting the centre of the stratum and the attracted point.

[*] (330) Because the attraction of the particles, situated on opposite sides of the stratum, would exactly counterbalance each other, and the whole result would be nothing.

[†] (331) The dimensions of the body, being infinitely small in comparison with the distance, its whole mass may be considered as collected in a point.

[470″] is nothing, it is evident that its attraction on a point placed at its surface, or without it, is the same as if the whole mass was collected at its centre.

The same thing takes place also with globes composed of concentrical strata, varying in density from the centre to the surface, according to any law, since it is true for each of these strata; now the sun, the planets and [470‴] the satellites may be considered as very nearly like globes of this nature; they will therefore attract external bodies in nearly the same manner as if their masses were united at their centres of gravity; which is in conformity to what we have found by observation in § 5. It is true that the figures of [470ⁱᵛ] the heavenly bodies vary a little from a sphere; but the difference is very small, and the error resulting from the preceding supposition, is of the same [470ᵛ] order as that difference, as it respects points near their surface;* but for

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* (332) Let D be the distance of a body or massive particle M, from the centre of [470a] gravity G, of a system of particles, m, m', m'', &c., and r, r', r'', &c., the distances of these particles, m, m', &c., from their common centre of gravity G. Then it has been proved [470b] in [445′—450′], that the action of the system, m, m', &c., upon the body M, would be the same as if all the particles, m, m', &c., were collected at G, neglecting quantities of the order $\frac{r^2}{D^2}$. The same would hold true, if the particles, m, m', &c., composed a solid [470c] body, or spheroid S, and its attraction on the body M, neglecting terms of the order $\frac{r^2}{D^2}$, would therefore be the same as if its whole mass were collected in G. Suppose now the body S to be nearly spherical, so that it may be considered as being composed of an inscribed [470d] sphere, whose radius is ρ, and centre G, and of an external spheroidal shell, of very small thickness $\alpha\,\rho$, having the same centre of gravity G, α being a very small quantity, depending on the thickness of this shell, in various parts of its surface. Then by [470″], the sphere would attract the external particle M, *exactly* as if all its mass were united in the centre G; and the spheroidal shell, by what has been shown, [470c], would attract the same particle M, as if all its mass were collected in the same centre G, neglecting the part of the attraction of this shell, corresponding to the terms of the order $\frac{r^2}{D^2}$, and as the mass of this shell is of the order α, in comparison with that of the sphere, it follows that this neglected part would be of the order $\alpha \cdot \frac{r^2}{D^2}$, in comparison with the whole attraction of the spheroid, as is asserted in [470ᵛⁱ].

Again, since the attraction of the internal sphere is as its mass μ, divided by the square of the distance of the attracted point from the centre G, if that distance should vary by

distant points, the error is of the same order as the product of this difference,
by the square of the ratio of the radius of the attracting body to its distance [470^{vi}]
from the attracted point; for we have seen, in § 10, that the circumstance
of the distance of the attracted point, renders the error of the preceding [470^{vii}]
supposition, of the same order as the square of this ratio. The heavenly
bodies attract therefore, in nearly the same manner, as if their masses were
united at their centres of gravity, not only because their distances from each [470^{viii}]
other are very great with respect to their dimensions; but also because their
figures are very nearly spherical.

The property posessed by spheres, in the law of nature, of attracting as if
their masses were united in their centres, is very remarkable; and it is an
object of curiosity, to ascertain whether the same thing takes place in other [470^{ix}]
laws of attraction. For this purpose, we shall observe that if the law of
gravity is such that a homogeneous sphere attracts a point placed without its
surface, as if all its mass was united at its centre, the same result ought to
take place in a stratum of uniform thickness. For if we take from a sphere a
spherical stratum of uniform thickness, we shall form another sphere of a
less radius, which will have, like the preceding, the property of attracting
as if all its mass was united at the centre; now it is evident that these two
spheres cannot have this common property, unless it also appertains to the
spherical stratum formed by the difference of these two spheres. The
problem is therefore reduced to the determination of the laws of attraction,
by which a spherical stratum, of an infinitely small but uniform thickness, [470^{x}]
attracts an external point, as if all its mass was united at the centre of the
stratum.

Let r be the distance from the attracted point to the centre of the
spherical stratum, u the radius of this stratum, and du its thickness. Let θ [470^{xi}]

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quantities of the order α, in different points of the surface of the spheroid, so that the distance of one point should be ρ, and of another point $\rho.(1+\alpha)$, the attraction of this sphere would change from $\frac{\mu}{\rho^2}$ to $\frac{\mu}{\rho^2.(1+\alpha)^2}$ or $\frac{\mu}{\rho^2}.(1-2\alpha)$, and it would therefore vary by terms of the order α, consequently the attraction of the whole spheroid, upon bodies placed on or near its surface, would vary by terms of that order, as is asserted in [470^{v}]. This subject is fully treated of in Book III, particularly in § 25, [1647].

be the angle which the radius u makes with the right line r; ϖ the angle
[470^{xii}] which the plane passing through the right lines r and u makes with a fixed plane passing through the right line r; the element of the spherical stratum
[470^{xiii}] will be* $u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\, \theta$. If we then put f for the distance of this element from the attracted point, we shall have†

[471] $$f^2 = r^2 - 2ru \,.\, \cos.\, \theta + u^2.$$

[471′] Let us denote by $\varphi(f)$, the law of attraction at the distance f; the action of the element of the stratum, upon the attracted point,‡ resolved in a

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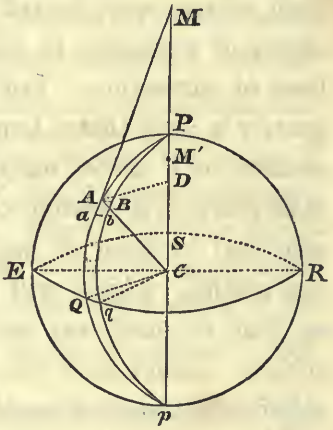
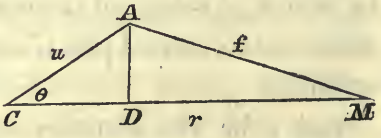


* (333) In the adjoined figure, which is similar to that in page 181, M is the attracted point, situated on the continuation of the diameter pCP; C the centre of the spherical stratum $PEpR$; PEp the great circle from which the angle $\varpi = ECQ$ is counted; $PAaQp$, $PBbqp$, two great circles drawn through the poles P, p, infinitely near to each other; $EQqRSE$ the great circle, whose poles are P, p; AB, ab, arcs of circles parallel to Qq. $CA = u$, $CM = r$, angle $ACM = \theta$, $Qq = u\,d\varpi$, $AB = u\,d\varpi \,.\, \sin.\, \theta$, $Aa = u\,d\theta$; hence the space $ABba = u^2 \,.\, d\varpi \,.\, d\theta \,.\, \sin.\, \theta$; this multiplied by the thickness of the stratum du, gives its mass,

$$u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\, \theta, \qquad \text{as in } [470^{xiii}].$$



† (333*a*) In the plane triangle CAM, which is here drawn separately from the preceding figure, if we let fall from A, the perpendicular AD on CM, and put $AM = f$, we shall have,

$$CD = CA \,.\, \cos.\, ACM = u \,.\, \cos.\, \theta, \qquad AD = CA \,.\, \sin.\, ACM = u \,.\, \sin.\, \theta,$$
$$MD = CM - CD = r - u \,.\, \cos.\, \theta,$$

and since $AM^2 = AD^2 + MD^2$, we shall get

$$f^2 = (u \,.\, \sin.\, \theta)^2 + (r - u \,.\, \cos.\, \theta)^2 = r^2 - 2ru \,.\, \cos.\, \theta + u^2,$$

as in [471]. This is the formula 63 of the Introduction.

‡ (334) The mass of the particle $u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\, \theta$, [$470^{xiii}$], being multiplied by the force $\varphi(f)$, gives the attraction of the particle in the direction MA. This multiplied by $\frac{MD}{AM} = \frac{r - u \,.\, \cos.\, \theta}{f}$, will give its force, [11], in the direction MC, as in [472].

direction parallel to r, and tending towards the centre of the stratum, will be

$$u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\, \frac{(r-u\,.\cos.\theta)}{f} \,.\, \varphi(f)\,;$$ [472]

but we have*

$$\frac{r-u\,.\cos.\theta}{f}=\left(\frac{df}{dr}\right);$$ [473]

therefore the preceding quantity may be put under the form

$$u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\left(\frac{df}{dr}\right).\,\varphi(f)\,;$$ [474]

and if we put

$$\int df \,.\, \varphi(f)=\varphi_{,}(f),$$ [474′]

we shall obtain the whole action of the spherical stratum, upon the attracted point, by means of the integral $u^2 du \,.\int d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\, \varphi_{,}(f)$, taking its differential relative to r, and dividing it by dr.† [474″]

This integral ought to be taken relative to ϖ, from $\varpi=0$ to $\varpi=$ the circumference of the circle; after this integration it becomes

$$2\pi \,.\, u^2 \,.\, du \,.\int d\theta \,.\, \sin.\theta \,.\, \varphi_{,}(f)\,;$$ [475]

π being the ratio of the semi-circumference of a circle to its radius. If we take the differential of the value of f, with respect to θ, we shall have‡ [475′]

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* (335) Since $f=\sqrt{r^2-2ru\,.\cos.\theta+u^2}$, [471], we have,

$$\left(\frac{df}{dr}\right)=\frac{r-u\,.\cos.\theta}{\sqrt{r^2-2ru\,.\cos.\theta+u^2}}=\frac{r-u\,.\cos.\theta}{f}.$$

† (336) By [474′], $\varphi_{,}(f)=\int df \,.\, \varphi(f)$, its partial differential relative to r, is $\left(\frac{d\,.\,\varphi_{,}(f)}{dr}\right)=\left(\frac{df}{dr}\right).\,\varphi(f)$, hence the element of attraction [474], becomes,

$$u^2 du \,.\, d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\left(\frac{d\,.\,\varphi_{,}(f)}{dr}\right).$$

Taking its integral relative to the whole surface of the stratum, it becomes

$$u^2 du \,.\int d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\left(\frac{d\,.\,\varphi_{,}(f)}{dr}\right),$$

and since ϖ, θ, are independent of r, if we put $\int d\varpi \,.\, d\theta \,.\, \sin.\theta \,.\, \varphi_{,}(f)=F$, this expression will become $u^2 \,.\, du \,.\left(\frac{dF}{dr}\right)$, as in [474″]. [474a]

‡ (337) In the integral $\int d\theta \,.\, \sin.\theta \,.\, \varphi_{,}(f)$, [475], the quantities f, θ, are considered as variable, r, u, as constant, because r, u are the same for all the particles $ABab$, of the

[476] $$d\theta \,.\, \sin.\theta = \frac{f\,df}{r\,u};$$

consequently

[476′] $$2\pi \,.\, u^2 du \,.\, \int d\theta \,.\, \sin.\theta \,.\, \varphi_{\prime}(f) = 2\pi \,.\, \frac{u\,du}{r} \,.\, \int f\,df \,.\, \varphi_{\prime}(f).$$

[476″] The integral relative to θ being taken from $\theta = 0$, to $\theta = \pi$; and at these two limits, we have* $f = r - u$, and $f = r + u$; therefore the integral relative to f, ought to be taken from $f = r - u$, to $f = r + u$; suppose therefore

[476‴] $$\int f\,df \,.\, \varphi_{\prime}(f) = \psi(f);$$

we shall have

[477] $$\frac{2\pi \,.\, u\,du}{r} \int f\,df \,.\, \varphi_{\prime}(f) = \frac{2\pi \,.\, u\,du}{r} \,.\, \{\psi(r+u) - \psi(r-u)\}.$$

The coefficient of $d\,r$, in the differential of the second member of this
[477′] equation, taken with respect to r, will give the attraction of the spherical stratum upon the attracted point [474″]; hence it is easy to perceive that in
[477″] the case of nature, where $\varphi(f) = \frac{1}{f^2}$, this attraction is equal to† $-\frac{4\pi \,.\, u^2 du}{r^2}$;

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spherical surface. Now the differential of f^2, [471], taken in this hypothesis, is

$$2\,f\,df = 2\,r\,u \,.\, d\theta \,.\, \sin.\theta, \qquad \text{hence} \qquad d\theta \,.\, \sin.\theta = \frac{f\,df}{r\,u},$$

which being substituted in $\int d\theta \,.\, \sin.\theta \,.\, \varphi_{\prime}(f)$, it becomes $\frac{1}{r\,u} \,.\, \int f\,df \,.\, \varphi_{\prime}(f)$, the constant quantities r, u, being brought from under the sign $\int$. This being substituted in the expression of the attraction [475], it becomes as in [476′].

* (338) The integral relative to ϖ, being taken in [476′], from $\varpi = 0$ to $\varpi =$ the circumference 2π, it will represent the attraction of an annulus of the stratum, formed by the revolution of the arch $A\,a$, about the diameter $P\,p$. In order therefore to embrace the whole surface of the spherical stratum, it will be necessary that the point A should move along the semi-circle $P\,A\,p$, from P to the opposite point of the axis p, or from $\theta = 0$ to $\theta =$ the semi-circumference π; at the first limit, the point A falls in P, and f becomes $MP = r - u$, and at the last limit the point A falls in p, where $\theta = \pi$, and the value of f becomes $Mp = r + u$.

† (339) $\varphi(f) = \frac{1}{f^2}$ gives [474′], $\varphi_{\prime}(f) = \int df \,.\, \varphi(f) = \int \frac{df}{f^2} = -\frac{1}{f}$; and by [476‴], $\psi(f) = \int f\,df \,.\, \varphi_{\prime}(f) = -\int df = -f$; hence,

that is, it is the same as if all the mass of the spherical stratum was united at
its centre, which furnishes another demonstration of the property we have [477''']
before spoken of [470''] relative to the attraction of spheres.

Let us now ascertain the form of $\varphi(f)$, upon the supposition that the
attraction of the stratum is the same as if its mass was united at its centre.
This mass is equal to $4\pi.u^2du$, [275b], and if it was united at its centre, [477^{iv}]
its action upon the attracted point, would be $4\pi.u^2du.\varphi(r)$, we shall
therefore have [477', 477^{iv}]

$$2\pi.u\,du.\left\{\frac{d.\left\{\frac{1}{r}.(\psi[r+u]-\psi[r-u])\right\}}{dr}\right\}=4\pi.u^2du.\varphi(r)\,; \qquad (D)$$ [478]

By taking the integral relative to dr, we shall have*

$$\psi(r+u)-\psi(r-u)=2ru.\int dr.\varphi(r)+rU\,;$$ [479]

U being a function of u, and constant quantities, added to the integral
$2u.\int dr.\varphi(r)$. If we represent $\psi(r+u)-\psi(r-u)$, by R,† we shall have, [479']

$$\psi(r+u)-\psi(r-u)=-(r+u)+(r-u)=-2u, \qquad \text{therefore}$$
$$\frac{2\pi.u\,du}{r}.\{\psi(r+u)-\psi(r-u)\}=-\frac{4\pi.u^2du}{r},$$
the differential taken relative to r, and divided by dr, expresses the whole attraction of the stratum [477'], $\frac{4\pi.u^2du}{r^2}$, as in [477'']; but, by [275b], the mass dm of a spherical stratum, writing u for R, is $4\pi.u^2du$, and if this mass was collected in the centre of the stratum, its attraction on the proposed point would be $\frac{4\pi.u^2du}{r^2}$, which is equal to the preceding expression.

[*] (340) Dividing the equation [478] by $\frac{2\pi.u\,du}{dr}$, we get
$$\left\{\frac{d.\left\{\frac{1}{r}.(\psi[r+u]-\psi[r-u])\right\}}{dr}\right\}.dr=2u.dr.\varphi(r),$$
which, by integration, relative to r, gives $\frac{1}{r}.(\psi[r+u]-\psi[r-u])=2u\int dr.\varphi(r)+U$. This multiplied by r gives [479].

[†] (341) The equation [479], by putting R for its first member, becomes
$R=2ru.\int dr.\varphi r+rU$; hence $\left(\frac{dR}{dr}\right)=2u.\int dr.\varphi r+2ur\varphi(r)+U,$

by taking the differential of the preceding equation,

[480]
$$\left(\frac{ddR}{dr^2}\right)=4u\,.\,\varphi(r)+2ru\,.\,\frac{d\,.\,\varphi(r)}{dr}\,;$$
$$\left(\frac{ddR}{du^2}\right)=r\,.\left(\frac{ddU}{du^2}\right);$$

but by the nature of the function R, we have*

[481]
$$\left(\frac{ddR}{dr^2}\right)=\left(\frac{ddR}{du^2}\right);$$

therefore†

[482]
$$2u\,.\left\{2\varphi(r)+r\,.\,\frac{d\,.\,\varphi(r)}{dr}\right\}=r\,.\left(\frac{ddU}{du^2}\right);$$

or

[482′]
$$\frac{2\varphi(r)}{r}+\frac{d\,.\,\varphi(r)}{dr}=\frac{1}{2u}\,.\left(\frac{ddU}{du^2}\right).$$

Now as the first member of this equation is independent of u, and the second
[482″] member is independent of r, each member ought to be equal to an arbitrary
constant quantity, which we shall denote by $3A$; therefore we shall have

[483]
$$\frac{2\,.\,\varphi(r)}{r}+\frac{d\,.\,\varphi(r)}{dr}=3A\,;$$

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and $\left(\frac{ddR}{dr^2}\right)=4u\,.\,\varphi(r)+2ru\,.\left(\frac{d\,.\,\varphi(r)}{dr}\right)$. Also, $\left(\frac{dR}{du}\right)=2r\,.\int dr\,.\,\varphi(r)+r\,.\left(\frac{dU}{du}\right)$, and $\left(\frac{ddR}{du^2}\right)=r\,.\left(\frac{ddU}{du^2}\right)$, as in [480].

* (342) Since $R=\psi(r+u)-\psi(r-u)$, we have

[480a] $\left(\frac{dR}{dr}\right)=\psi'(r-u)-\psi'(r-u)$, and $\left(\frac{ddR}{dr^2}\right)=\psi''(r-u)-\psi''(r+u)$,

denoting, for brevity, $\left(\frac{d\,.\,\psi(f)}{df}\right)$, $\left(\frac{dd\,.\,\psi(f)}{df^2}\right)$, by $\psi'(f)$, and $\psi''(f)$, respectively. In a similar manner we have,

$\left(\frac{dR}{du}\right)=\psi'(r+u)+\psi'(r-u)$, and $\left(\frac{ddR}{du^2}\right)=\psi''(r+u)-\psi''(r-u)$,

consequently $\left(\frac{ddR}{dr^2}\right)=\left(\frac{ddR}{du^2}\right)$, as in [481].

† (343) Substituting the values of $\left(\frac{ddR}{dr^2}\right)$, $\left(\frac{ddR}{du^2}\right)$, [480], in [481], we get $4u\,.\,\varphi(r)+2ru\,.\,\frac{d\,.\,\varphi(r)}{dr}=r\,.\left(\frac{ddU}{du^2}\right)$, as in [482]. Dividing this by $2ru$, it becomes as in [482′].

whence by integration*

$$\varphi(r) = A r + \frac{B}{r^2};$$ [484]

Laws of Attraction in which a Sphere acts as if all its mass was collected at its centre.

B being another arbitrary constant quantity. All the laws of attraction in which a sphere acts upon a point placed without its surface, at the distance r from its centre, as if all its mass was united at its centre, are therefore comprised in this general formula

$$A r + \frac{B}{r^2};$$ [485]

and it is easy to prove that in fact this expression satisfies the equation (*D*) [478], whatever values are taken for A and B.†

If we suppose $A = 0$, we shall have the law of nature; and it appears, in the infinite number of laws, which render the attraction very small at great distances, that the law of nature only possesses the property of making the attraction of spheres the same as if their masses were united at their centres. [485']

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* (344) Multiplying [483] by $r^2 d r$, it becomes

$$2 r d r . \varphi(r) + r^2 d r . \left(\frac{d . \varphi(r)}{d r}\right) = 3 A . r^2 d r,$$

whose integral is $r^2 . \varphi(r) = A r^3 + B$; dividing this by r^2 we get, $\varphi(r) = A r + \frac{B}{r^2}$, [484].

† (345) If we put $\varphi(r) = A r + \frac{B}{r^2}$, we shall have [474'],

$$\varphi_{,}(f) = \int df . \varphi(f) = \int df . \left(A f + \frac{B}{f^2}\right) = \tfrac{1}{2} A f^2 - \frac{B}{f} + C,$$ [486a]

C being a constant quantity. Hence by [476'''],

$$\psi(f) = \int f df . \varphi_{,}(f) = \int \{\tfrac{1}{2} A f^3 df - B df + C f df\} = \tfrac{1}{8} A f^4 - B f + \tfrac{1}{2} C f^2 + D,$$ [486b]

D being another constant quantity. This gives

$$\psi(r+u) - \psi(r-u) = \tfrac{1}{8} A . \{(r+u)^4 - (r-u)^4\} - B . \{(r+u) - (r-u)\}$$
$$+ \tfrac{1}{2} C . \{(r+u)^2 - (r-u)^2\},$$

which by reduction becomes $A . (r^3 u + r u^3) - 2 B u + 2 C r u$, hence

$$\frac{1}{r} . (\psi[r+u] - \psi[r-u]) = A . (r^2 u + u^3) - \frac{2 B u}{r} + 2 C u,$$ [486c]

and its differential relative to r, being taken, and divided by $d r$, becomes $2 A r u + \frac{2 B u}{r^2}$, Substitute this in the first member of [478], also φr, [484], in its second member, and the equation will become identical, leaving A and B indeterminate.

This is also the only law, in which a body situated within a spherical stratum, of uniform thickness, is equally attracted in every direction. It [485''] follows, from the preceding analysis, that the attraction of a spherical stratum whose thickness is du, on a point situated within it, is expressed by*

[486] $$2\pi . u\,du . \left\{ \frac{d . \frac{1}{r} . \{\psi(u+r) - \psi(u-r)\}}{dr} \right\}.$$

To make this function nothing, we must have

[487] $$\psi(u+r) - \psi(u-r) = r . U;$$

[487'] U being a function of u independent of r, and it is easy to show that this [487''] equation is satisfied in the law of nature, in which we have $\varphi(f) = \frac{B}{f^2}$.† But to prove that it is satisfied by no other law, we shall denote by $\psi'(f)$,

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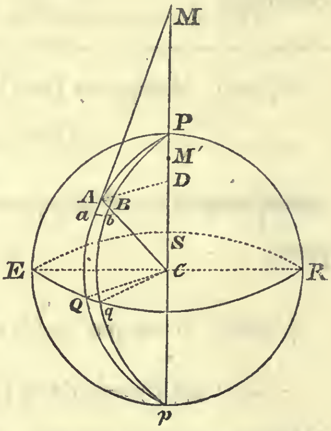

* (346) When the point M falls within the stratum, as at M', the limits of the integral $\psi(f)$ in the equation [477], will be

$$f = PM' = u - r, \qquad \text{and} \qquad f = pM' = u + r,$$

and the integral of the first member of [477] becomes, in this case, $\frac{2\pi . u\,du}{r} . \{\psi(u+r) - \psi(u-r)\}$. To render the force nothing, we must put the differential of this expression, taken relative to dr, equal to nothing [474''], which will give, by neglecting the constant factor, $2\pi . u\,du$,

$$\left(\frac{d . \frac{1}{r} . \{\psi(u+r) - \psi(u-r)\}}{dr} \right) = 0.$$

Multiplying by dr and integrating, we get $\frac{1}{r} . \{\psi(u+r) - \psi(u-r)\} = U$, hence $\psi(u+r) - \psi(u-r) = r\,U$, as in [487].

† (347) Put $A = 0$, and $r = f$, in [484], and it becomes $\varphi(f) = \frac{B}{f^2}$, as in the law of nature [487''], and then $\psi(f)$, [486b], becomes $\psi(f) = -Bf + \frac{1}{2}Cf^2 + D$; hence, $\psi(u+r) - \psi(u-r) = -B . \{(u+r) - (u-r)\} + \frac{1}{2}C . \{(u+r)^2 - (u-r)^2\} = -2Br + 2Cur$, hence by [487], $U = -2B + 2Cu$, and since this value of U is independent of r, as is required in [487'], the assumed value of $\varphi(f) = \frac{B}{f^2}$, must satisfy the proposed equation [487].

the differential of $\psi(f)$, divided by df; and by $\psi''(f)$, the differential of $\psi'(f)$, divided by df, and in the same manner for others; we shall then [487'''] have, by taking, twice in succession, the differential of the preceding equation with respect to r,*

$$\psi''(u+r)-\psi''(u-r)=0. \qquad [488]$$

As this equation exists for all values of u and r, it follows that $\psi''(f)$ must be equal to a constant quantity, whatever be the value of f, consequently [488'] $\psi'''(f)=0$; now by what precedes we have†

$$\psi'(f)=f\,.\,\varphi_{,}(f); \qquad [489]$$

hence we deduce

$$\psi'''(f)=2\,.\,\varphi(f)+f\,.\,\varphi'(f); \qquad [490]$$

therefore we shall have

$$0=2\,.\,\varphi(f)+f\,.\,\varphi'(f); \qquad [491]$$

which gives by integration,‡ $\varphi(f)=\frac{B}{f^2}$, corresponding to the law of [491'] nature.

13. Let us now resume the equation (C) of § 11, [466]. If we could

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* (348) The first differential of [487], gives $\psi'(u+r)+\psi'(u-r)=U$, its second differential is as in [488], from which we get $\psi''(u+r)=\psi''(u-r)$, whence it would follow, as in [482''], that we must have generally $\psi''(u+r)$, or $\psi''(f)$, equal to a constant quantity.

† (349) By [476'''], we have $\psi(f)=\int f\,df\,.\,\varphi_{,}(f)$, its differential divided by df, gives $\psi'(f)=f\,.\,\varphi_{,}(f)$. Again, taking the differential and dividing by df, we get $\psi''(f)$, and by writing, for brevity, $\varphi_{,}'(f)$, $\varphi_{,}''(f)$, for the differential of $\varphi_{,}(f)$, divided by df, and that of $\varphi_{,}'(f)$, divided by df, we shall have, $\psi''(f)=\varphi_{,}(f)+f\,.\,\varphi_{,}'(f)$; again taking its differential and dividing by df, $\psi'''(f)=2\varphi_{,}'(f)+f\,.\,\varphi_{,}''(f)$; but $\varphi_{,}(f)=\int df\,.\varphi(f)$, [474'], gives $\varphi_{,}'(f)=\varphi(f)$, and $\varphi_{,}''(f)=\varphi'(f)$, hence, $\psi'''(f)=2\varphi\,.\,(f)+f\,.\,\varphi'(f)$, as in [490], and as $\psi'''(f)=0$, [488'], we shall have $0=2\varphi(f)+f\,.\,\varphi'(f)$, as in [491].

‡ (350) Multiplying [491] by $f\,df$, it becomes $0=2f\,df\,.\,\varphi(f)+f^2\,df\,.\,\varphi'(f)$ or $0=2f\,df\,.\,\varphi(f)+f^2\,.\,d\,.\,\big(\varphi(f)\big)$, whose integral is $B=f^2\,.\,\varphi(f)$, whence $\varphi(f)=\frac{B}{f^2}$, as above.

obtain the general integral of this equation, we should have an expression of V which would contain two arbitrary functions, which might be determined by seeking the attraction of the spheroid upon some point selected, so as to simplify the calculation, and then comparing this attraction with the general expression. But the integration of the equation (C) [466] is impossible, except in some particular cases, such as that in which the attracting spheroid is a sphere, which reduces the equation to common differentials; it is also possible, when the spheroid is a cylinder, whose base is an oval, or re-entering curve, and whose length is infinite; we shall see in the third book [2075], that this particular case includes the theory of Saturn's rings. [491'']

Let us take the origin of r on the axis of the cylinder, which we shall suppose to be infinitely extended on both sides of this origin. Putting r' for the distance of the attracted point from the axis; we shall have* [491''']

[492] $$r' = r\sqrt{1-\mu^2}.$$

It is evident that V depends solely on r' and ϖ, since it is the same for all points in which these two variable quantities are the same; it does not

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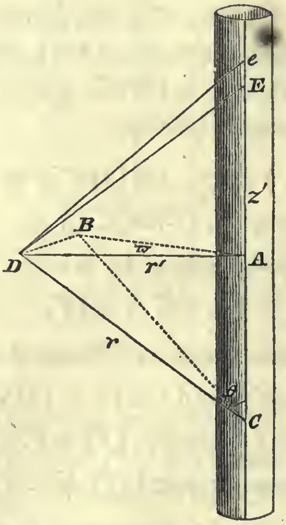
* (351) In the annexed figure $CAEc$ is the axis of the cylinder, taken as the axis of x; D the attracted point, C the origin of the co-ordinates; $CA = x$, $AB = y$, $BD = z$, $CD = r$, angle $DCA = \theta$, $DAB = \varpi$, and

[493a] $$DA = r' = r\,.\,\sin.\,\theta = r\,.\,\sqrt{1-\mu^2}, \quad [465'],$$

as in [492]. Hence we get

[493b] $$\left(\frac{dr'}{dr}\right) = \sqrt{1-\mu^2}, \qquad \left(\frac{dr'}{d\mu}\right) = -\frac{r\mu}{\sqrt{1-\mu^2}},$$

which are used in [493, 493c].



As V does not contain r, except through r', [492], we shall have

[493c] $$\left(\frac{dV}{dr}\right) = \left(\frac{dV}{dr'}\right)\,.\,\left(\frac{dr'}{dr}\right) = \left(\frac{dV}{dr'}\right).\sqrt{1-\mu^2},$$

and in a similar manner $\left(\frac{ddV}{dr^2}\right) = \left(\frac{ddV}{dr'^2}\right).(1-\mu^2)$. Now the equation [466] becomes by developing the terms affected by the sign d,

$$0 = -2\mu\,.\left(\frac{dV}{d\mu}\right) + (1-\mu^2)\,.\left(\frac{ddV}{d\mu^2}\right) + \frac{\left(\frac{ddV}{d\varpi^2}\right)}{1-\mu^2} + 2r\,.\left(\frac{dV}{dr}\right) + r^2\,.\left(\frac{ddV}{dr^2}\right).$$

therefore contain μ, except by means of r', considered as a function of that quantity; this gives [493*b*]

$$\left(\frac{dV}{d\mu}\right)=\left(\frac{dV}{dr'}\right).\left(\frac{dr'}{d\mu}\right)=-\frac{r\mu}{\sqrt{1-\mu^2}}.\left(\frac{dV}{dr'}\right);$$ [493]
$$\left(\frac{ddV}{d\mu^2}\right)=\frac{r^2\mu^2}{1-\mu^2}.\left(\frac{ddV}{dr'^2}\right)-\frac{r}{(1-\mu^2)^{\frac{3}{2}}}.\left(\frac{dV}{dr'}\right);$$

therefore the equation (*C*) [466] becomes

$$0=r'^2.\left(\frac{ddV}{dr'^2}\right)+\left(\frac{ddV}{d\varpi^2}\right)+r'.\left(\frac{dV}{dr'}\right);$$ [494]

hence by integration[*]

$$V=\varphi.\{r'.\cos.\varpi+r'.\sqrt{-1}.\sin.\varpi\}+\psi\{r'.\cos.\varpi-r'.\sqrt{-1}.\sin.\varpi\};$$ [495]

Substituting the values [493, 493*c*], we get

$$0=\frac{2r\mu^2}{\sqrt{1-\mu^2}}.\left(\frac{dV}{dr'}\right)+(1-\mu^2).\left\{\frac{r^2\mu^2}{1-\mu^2}.\left(\frac{ddV}{dr'^2}\right)-\frac{r}{(1-\mu^2)^{\frac{3}{2}}}.\left(\frac{dV}{dr'}\right)\right\}$$
$$+\frac{\left(\frac{ddV}{d\varpi^2}\right)}{1-\mu^2}+2r\sqrt{1-\mu^2}.\left(\frac{dV}{dr'}\right)+r^2.(1-\mu^2).\left(\frac{ddV}{dr'^2}\right).$$

Reducing and multiplying by $1-\mu^2$, we shall get

$$r^2.(1-\mu^2).\left(\frac{ddV}{dr'^2}\right)+\left(\frac{ddV}{d\varpi^2}\right)+r.\sqrt{1-\mu^2}.\left(\frac{dV}{dr'}\right),$$

which, by substituting the value of r', [492], becomes as in [494].

[*] (353) It is easy to prove that the value here assumed for V satisfies the proposed equation. For, if we notice only the function φ, which may be done, because the demonstration is the same for φ as for ψ, and put for brevity, φ instead of $\varphi.\}r'.\cos.\varpi+r'\sqrt{-1}.\sin.\varpi\}$, we shall have $\left(\frac{dV}{dr'}\right)=(\cos.\varpi+\sqrt{-1}.\sin.\varpi).\varphi'$;

$$\left(\frac{dV}{d\varpi}\right)=(-r'.\sin.\varpi+r'.\sqrt{-1}.\cos.\varpi).\varphi'; \qquad \left(\frac{ddV}{dr'^2}\right)=(\cos.\varpi+\sqrt{-1}.\sin.\varpi)^2.\varphi'';$$
$$\left(\frac{ddV}{d\varpi^2}\right)=-r'(\cos.\varpi+\sqrt{-1}.\sin.\varpi).\varphi'+(-r'.\sin.\varpi+r'.\sqrt{-1}.\cos.\varpi)^2.\varphi''.$$

these values being substituted in [494], it becomes,

$$\cdot\,{}^2.(\cos.\varpi+\sqrt{-1}.\sin.\varpi)^2.\varphi''-r'.(\cos.\varpi+\sqrt{-1}.\sin.\varpi).\varphi'$$
$$+(-r'.\sin.\varpi+r'.\sqrt{-1}.\cos.\varpi)^2.\varphi''+r'.(\cos.\varpi+\sqrt{-1}.\sin.\varpi).\varphi';$$

which is identically nothing, the first term being balanced by the third, and the second by the fourth. The same thing takes place with the function ψ, by changing φ into ψ, and writing $-\sqrt{-1}$ for $\sqrt{-1}$. Therefore the value V, [495], satisfies the proposed equation, and as it contains two arbitrary functions, it must be the complete integral.

The equation [494], is of the second order of partial differentials, and is a simple case

$\varphi(r')$ and $\psi(r')$ being arbitrary functions of r', which may be determined, by seeking the attraction of the cylinder, when ϖ is nothing, and when it is a right angle.

If the base of the cylinder is a circle, V will evidently be a function of r', independent of ϖ; the preceding equation of partial differentials [494] will then become

[496] $$0 = r'^2 \cdot \left(\frac{d\,d\,V}{d\,r'^2}\right) + r' \cdot \left(\frac{d\,V}{d\,r'}\right);$$

which gives by integration*

[497] $$-\left(\frac{d\,V}{d\,r'}\right) = \frac{H}{r'},$$

H being a constant quantity. To determine it, we shall suppose r' to be extremely great with respect to the radius of the base of the cylinder, which permits us to consider the cylinder as a right line infinitely long. Let the [497′] base be A, and put z for the distance from any point of the axis of the cylinder to the point where this axis is intersected by r'; the action of the cylinder considered as concentrated on its axis, will be in a direction parallel to r', equal to†

[498] $$\int \frac{A\,r'\,.\,d\,z}{(r'^2 + z^2)^{\frac{3}{2}}},$$

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of a much more extensive class of equations, which has been treated of by several mathematicians. It comes under the form of that given by La Croix, in § 750, edit. 1, or § 756, edit. 2, of his "Traité du calcul différentiel, &c." For by putting $V = z$, $r' = x$, $\varpi = y$, in [494], it becomes $0 = x^2 \cdot \left(\frac{d\,d\,z}{d\,x^2}\right) + \left(\frac{d\,d\,z}{d\,y^2}\right) + x \cdot \left(\frac{d\,z}{d\,x}\right)$. That treated of by La Croix being

[496a] $$0 = R \cdot \left(\frac{d\,d\,z}{d\,x^2}\right) + S \cdot \left(\frac{d\,d\,z}{d\,x\,d\,y}\right) + T \cdot \left(\frac{d\,d\,z}{d\,y^2}\right) + P \cdot \left(\frac{d\,z}{d\,x}\right) + Q \cdot \left(\frac{d\,z}{d\,y}\right) + N\,z = M;$$

which corresponds to the present example by putting $R = x^2$, $S = 0$, $T = 1$, $P = x$, $Q = 0$, $N = 0$, $M = 0$; and by following the method he has given, we should obtain for z or V, the expression [495], which we have demonstrated synthetically; this method being used for brevity.

* (354) Multiplying [496] by $-\frac{d\,r'}{r'}$, it becomes $0 = -r'\,d\,r' \cdot \left(\frac{d\,d\,V}{d\,r'^2}\right) - d\,r' \cdot \left(\frac{d\,V}{d\,r'}\right)$, whose integral is $-r' \cdot \left(\frac{d\,V}{d\,r'}\right) = H$, whence $-\left(\frac{d\,V}{d\,r'}\right) = \frac{H}{r'}$, as in [497].

† (355) Suppose the whole mass of the cylinder to be collected in the axis $CAEe$, and put $AE = z'$, $Ee = d\,z'$. Then the mass of matter in the space Ee will be

the integral being taken from $z=-\infty$ to $z=\infty$, which reduces it to* $\frac{2A}{r'}$; this is the expression of $-\left(\frac{dV}{dr'}\right)$, when r' is very great. By comparing it with the preceding, we have $H=2A$, and we find that whatever be r', *the action of the cylinder on a point placed without its surface, is* $\frac{2A}{r'}$. [498']

If the attracted point is placed within a circular cylindrical stratum, of uniform thickness, and infinitely long, we shall also have $-\left(\frac{dV}{dr'}\right)=\frac{H}{r'}$, [497]; and as the attraction is nothing when the attracted point is on the axis of the stratum, we shall have $H=0$; consequently, *a point placed within such a circular stratum, is equally attracted in every direction.* [498'']

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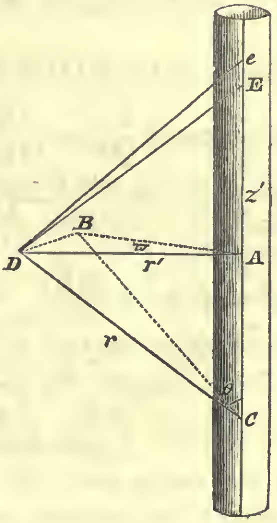
represented by $A.dz'$. Dividing this by DE^2, we get the attraction in the direction DE. Multiplying this by $\frac{DA}{DE}$, gives the attraction in the direction DA equal to $\frac{Ar'.dz'}{(r'^2+z'^2)^{\frac{3}{2}}}$, [498a] whose integral gives the whole attraction as in [498]; the letter z' being accented, to distinguish it from the co-ordinate $BD=z$. This integral is to be taken through the whole length of the cylinder from $z'=-\infty$, to $z'=\infty$.



* (356) Putting the angle $ADE=\varepsilon$, we shall get $z'=r'.\text{tang.}\,\varepsilon$, hence $r'^2+z'^2=\frac{r'^2}{\cos.^2\varepsilon}$, and $dz'=\frac{r'd\varepsilon}{\cos.^2\varepsilon}$, r' being constant. Hence

$$\int\frac{Ar'.dz'}{(r'^2+z'^2)^{\frac{3}{2}}}=\int\frac{A.d\varepsilon.\cos.\varepsilon}{r'}=\frac{A.\sin.\varepsilon}{r'}+\text{constant}.$$

Now at the first limit of this integral, where $z'=-\infty$, $\varepsilon=-\frac{1}{2}\pi$, π being the semi-circumference of a circle whose radius is 1, this becomes $0=-\frac{A}{r'}+\text{constant}$, hence the corrected integral is $\frac{A.\sin.\varepsilon}{r'}+\frac{A}{r'}$. This, when $z'=\infty$, and $\varepsilon=\frac{1}{2}\pi$, becomes $\frac{A}{r'}+\frac{A}{r'}=\frac{2A}{r'}$, as above. Putting this equal to $\frac{H}{r'}$, [497], we shall get, as above, $H=2A$.

14. The equations (*A*), (*B*), (*C*), of § 11, [459, 465, 466], may also be applied to the motion of a body; and an equation of condition may be obtained from them, which will be very useful in proving the calculations made by the theory, or in verifying the theory of universal gravity itself. The equations (1), (2), (3), of § 9 [416—418] by which the relative motion of m about M is determined, may be put under this form*

[499] $$\frac{ddx}{dt^2}=\left(\frac{dQ}{dx}\right); \qquad \frac{ddy}{dt^2}=\left(\frac{dQ}{dy}\right); \qquad \frac{ddz}{dt^2}=\left(\frac{dQ}{dz}\right); \qquad (i)$$

in which

[499′] $$Q=\frac{M+m}{r}-\Sigma\,.\,\frac{m'(xx'+yy'+zz')}{r'^3}+\frac{\lambda}{m};$$

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* (357) The assumed value of Q, [499′], gives

[498a′] $$\left(\frac{dQ}{dx}\right)=-\frac{Mx}{r^3}-\frac{mx}{r^3}-\Sigma\,.\,\frac{m'x'}{r'^3}+\frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right),$$

because $\left(\frac{d\,.\,\frac{M+m}{r}}{dx}\right)=\frac{-(M+m)}{r^2}\,.\left(\frac{dr}{dx}\right)=\frac{-(M+m)\,.\,x}{r^3}$, and the terms $-\Sigma\,.\,\frac{m'xx'}{r'^3}$ produce $-\Sigma\,.\,\frac{m'x'}{r'^3}$, in which $\frac{mx}{r^3}$ is not included. If we therefore include the terms

[498b] $-\frac{mx}{r^3}$ under the sign Σ, we shall have, $\left(\frac{dQ}{dx}\right)=-\frac{Mx}{r^3}-\Sigma\,.\,\frac{mx}{r^3}+\frac{1}{m}\,.\left(\frac{d\lambda}{dx}\right)$, which being substituted in [416], it becomes $\frac{ddx}{dt^2}=\left(\frac{dQ}{dx}\right)$, and the equations [417, 418], give in like manner $\frac{ddy}{dt^2}=\left(\frac{dQ}{dy}\right)$, $\frac{ddz}{dt^2}=\left(\frac{dQ}{dz}\right)$. If we compare the equation $0=\frac{ddx}{dt^2}+P$, [364], with $0=\frac{ddx}{dt^2}-\left(\frac{dQ}{dx}\right)$, [499], it will be evident that the force P, acting on the body, in a direction parallel to the ordinate x, and tending towards the origin of the force, [363^{vi}] is equivalent to $-\left(\frac{dQ}{dx}\right)$, in the motion of m about M; therefore $\left(\frac{dQ}{dx}\right)$ will express the force acting on the body m in its relative orbit about M, in the direction parallel to x, and tending to increase the co-ordinate x. And as the ordinate x is arbitrary, we may say

[499a] generally that $\left(\frac{dQ}{dy}\right)$, $\left(\frac{dQ}{dz}\right)$, $\left(\frac{dQ}{dr}\right)$, represent the forces acting on the body m in its relative motion about M, resolved in the direction of the lines y, z, r, respectively, and tending to increase those lines. This agrees with the remarks made in the note page 253.

and it is easy to prove that*

$$0=\left(\frac{ddQ}{dx^2}\right)+\left(\frac{ddQ}{dy^2}\right)+\left(\frac{ddQ}{dz^2}\right); \qquad (E)$$ [500]

supposing the variable quantities x', y', z', x'', &c., *contained in* Q, *to be independent of* x, y, z. [500′]

We shall transform the variable quantities x, y, z, into others, more convenient for astronomical purposes. Putting r for the radius drawn from the centre of M to that of m, v the angle which the projection of this radius [500″]

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* [358) Taking the partial differential of $\left(\frac{dQ}{dx}\right)$, [498$a'$], relative to x, we get

$$\left(\frac{ddQ}{dx^2}\right)=-\frac{(M+m)}{r^3}+\frac{3.(M+m).x}{r^4}\cdot\left(\frac{dr}{dx}\right)+\frac{1}{m}\cdot\left(\frac{dd\lambda}{dx^2}\right),$$

because $\Sigma.\frac{m'x'}{r'^3}$ does not contain x, [500′]. Substituting for $\left(\frac{dr}{dx}\right)$ its value $\frac{x}{r}$, [463a], it becomes $\left(\frac{ddQ}{dx^2}\right)=-\frac{(M+m)}{r^3}+\frac{3.(M+m).x^2}{r^5}+\frac{1}{m}\cdot\left(\frac{dd\lambda}{dx^2}\right)$; and by changing [500$a$] successively x into y and z, we find $\left(\frac{ddQ}{dy^2}\right)=-\frac{(M+m)}{r^3}+\frac{3.(M+m).y^2}{r^5}+\frac{1}{m}\cdot\left(\frac{dd\lambda}{dy^2}\right)$; $\left(\frac{ddQ}{dz^2}\right)=-\frac{(M+m)}{r^3}+\frac{3.(M+m).z^2}{r^5}+\frac{1}{m}\cdot\left(\frac{dd\lambda}{dz^2}\right)$. Adding these three equations together and putting for $x^2+y^2+z^2$, its value r^2, the terms multiplied by $(M+m)$ will destroy each other, and the sum will become,

$$\left(\frac{ddQ}{dx^2}\right)+\left(\frac{ddQ}{dy^2}\right)+\left(\frac{ddQ}{dz^2}\right)=\frac{1}{m}\cdot\left\{\left(\frac{dd\lambda}{dx^2}\right)+\left(\frac{dd\lambda}{dy^2}\right)+\left(\frac{dd\lambda}{dz^2}\right)\right\}.$$ [500b]

Now each of the terms of which λ, [412], is composed, being substituted in the second member of this equation, renders it equal to nothing. For example, the first term of λ, [412], by using the value of β, [455^{iv}], becomes $mm'\beta$, which produces, in the second member of the preceding equation, the terms $m'.\left\{\left(\frac{dd\beta}{dx^2}\right)+\left(\frac{dd\beta}{dy^2}\right)+\left(\frac{dd\beta}{dz^2}\right)\right\}$, which by [458] is nothing, and the same would take place if we put

$$\beta=\left\{(x-x'')^2+(y-y'')^2+(z-z'')^2\right\}^{-\frac{1}{2}}$$

Hence in general, $\left(\frac{dd\lambda}{dx^2}\right)+\left(\frac{dd\lambda}{dy^2}\right)+\left(\frac{dd\lambda}{dz^2}\right)=0$; consequently

$$\left(\frac{ddQ}{dx^2}\right)+\left(\frac{ddQ}{dy^2}\right)+\left(\frac{ddQ}{dz^2}\right)=0;$$ [500c]

upon the plane of x, y, makes with the axis of x; and θ the inclination of r above the same plane; we shall have*

[501]
$$x = r\,.\cos.\theta\,.\cos.v\,;$$
$$y = r\,.\cos.\theta\,.\sin.v\,;$$
$$z = r\,.\sin.\theta.$$

The equation (E) [500], referred to these new variable quantities, will be, by § 11,†

[502]
$$0 = r^2.\left(\frac{ddQ}{dr^2}\right) + 2r\,.\left(\frac{dQ}{dr}\right) + \frac{\left(\frac{ddQ}{dv^2}\right)}{\cos.^2\theta} + \left(\frac{ddQ}{d\theta^2}\right) - \frac{\sin.\theta}{\cos.\theta}\,.\left(\frac{dQ}{d\theta}\right). \qquad (F)$$

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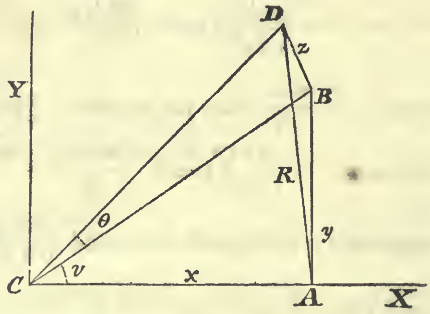
* (359) Let C be the place of the body M, D that of m, CA the axis of x, AB, BD, lines drawn parallel to the axes of y, z; making $CA = x$, $AB = y$, $BD = z$, $CD = r$, angle $ACB = v$, angle $BCD = \theta$. Then $CB = CD\,.\cos.BCD = r\,.\cos.\theta$, this being substituted in $CA = CB\,.\cos.ACB$, $AB = CB\,.\sin.ACB$, we get

[501a] $x = r\,.\cos.\theta\,.\cos.v$, $y = r\,.\cos.\theta\,.\sin.v$. Lastly, $BD = CD\,.\sin.BCD$, hence, $z = r\,.\sin.\theta$, agreeing with [501].



† (360) It is proved in [465g—u], that by putting $x = r\,.\cos.\theta$, $y = r\,.\sin.\theta\,.\cos.\varpi$, $z = r\,.\sin.\theta\,.\sin.\varpi$, [460], the equation [459] would change into [465]. And as the equation [459] would not vary, by writing z for x, x for y, and y for z, it will follow that if we had put $z = r\,.\cos.\theta$, $x = r\,.\sin.\theta\,.\cos.\varpi$, $y = r\,.\sin.\theta\,.\sin.\varpi$, the equation [459] would change into [465]. If in these values of x, y, z, we write v for ϖ, and $90 - \theta$ for θ, they will become $x = r\,.\cos.\theta\,.\cos.v$, $y = r\,.\cos.\theta\,.\sin.v$, $z = r\,.\sin.\theta$, which agree with those in [501], and the result from substituting these last values in the equation $0 = \left(\frac{ddQ}{dx^2}\right) + \left(\frac{ddQ}{dy^2}\right) + \left(\frac{ddQ}{dz^2}\right)$, will be obtained by writing, in the equation [465], Q for V, v for ϖ, $90 - \theta$ for θ, therefore $-d\theta$ for $d\theta$, and it will then become

$$0 = \left(\frac{ddQ}{d\theta^2}\right) - \frac{\sin.\theta}{\cos.\theta}\,.\left(\frac{dQ}{d\theta}\right) + \frac{\left(\frac{ddQ}{dv^2}\right)}{\cos.^2\theta} + r\,.\left(\frac{dd.rQ}{dr^2}\right),$$

and by substituting, as in [465u], $2r\,.\left(\frac{dQ}{dr}\right) + r^2\,.\left(\frac{ddQ}{dr^2}\right)$, for the term $r\,.\left(\frac{dd.rQ}{dr^2}\right)$ it becomes as in [502].

If we multiply the first of the equations (i) [499], by $\cos.\theta.\cos.v$; the second by $\cos.\theta.\sin.v$; the third by $\sin.\theta$; and for brevity put*

$$M'=\frac{ddr}{dt^2}-\frac{r.dv^2}{dt^2}.\cos.^2\theta-\frac{r.d\theta^2}{dt^2};$$ [503]

we shall have, by adding these products,

$$M'=\left(\frac{dQ}{dr}\right).$$ [504]

Likewise, if we multiply the first of the equations (i) [499], by

$-r.\cos.\theta.\sin.v$; the second by $r.\cos.\theta.\cos.v$;

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* (361) Considering in the first place, Q as a function of r, θ, v, and then of x, y, z, we shall have $\left(\frac{dQ}{dr}\right)=\left(\frac{dQ}{dx}\right).\left(\frac{dx}{dr}\right)+\left(\frac{dQ}{dy}\right).\left(\frac{dy}{dr}\right)+\left(\frac{dQ}{dz}\right).\left(\frac{dz}{dr}\right)$. Now from [501], [502a] we get $\left(\frac{dx}{dr}\right)=\cos.\theta.\cos.v$, $\left(\frac{dy}{dr}\right)=\cos.\theta.\sin.v$, $\left(\frac{dz}{dr}\right)=\sin.\theta$. Hence

$$\left(\frac{dQ}{dr}\right)=\cos.\theta.\cos.v.\left(\frac{dQ}{dx}\right)+\cos.\theta.\sin.v.\left(\frac{dQ}{dy}\right)+\sin.\theta.\left(\frac{dQ}{dz}\right);$$ [502b]

and by means of the equations [499], this becomes

$$\left(\frac{dQ}{dr}\right)=\frac{\cos.\theta}{dt^2}.\left\{ddx.\cos.v+ddy.\sin.v\right\}+\sin.\theta.\frac{ddz}{dt^2}.$$ [503a]

In finding ddx, ddy, we shall, for brevity, put $r.\cos.\theta=R$, which gives, by [501], $x=R.\cos.v$, $y=R.\sin v$. whence $dx=dR.\cos.v-Rdv.\sin.v$; [503b]

$$ddx=(ddR-R.dv^2).\cos.v-(2dR.dv+Rddv).\sin.v,$$ [503c]

and in a similar way,

$$ddy=(ddR-Rdv^2).\sin.v+(2dR.dv+Rddv).\cos.v.$$ [503d]

The former being multiplied by $\cos.v$, and the latter by $\sin.v$, and the products added, we shall get

$$ddx.\cos.v+ddy.\sin.v=ddR-Rdv^2.$$ [503e]

Substitute this in [503a], and it becomes $\left(\frac{dQ}{dr}\right)=\left(\frac{ddR}{dt^2}-\frac{Rdv^2}{dt^2}\right).\cos.\theta+\frac{ddz}{dt^2}.\sin.\theta$. [503f]

Again from $R=r.\cos.\theta$, we get $ddR=(ddr-rd\theta^2).\cos.\theta-(2dr.d\theta+rdd\theta).\sin.\theta$, also, $ddz=(ddr-rd\theta^2).\sin.\theta+(2dr.d\theta+rdd\theta).\cos.\theta$. These values being [503g] easily deduced from those of ddx, ddy, [503c, d], by writing r, θ, for R, v, respectively. Hence, $ddR.\cos.\theta+ddz.\sin.\theta=ddr-rd\theta^2$, this being substituted in [503f], we shall get, $\left(\frac{dQ}{dr}\right)=\frac{ddr}{dt^2}-r.\frac{d\theta^2}{dt^2}-R.\cos.\theta.\frac{dv^2}{dt^2}$, and by putting for R its value $r.\cos.\theta$, it becomes as in [504].

and add these products, supposing

[505] $$N' = \frac{d.\left(r^2.\frac{dv}{dt}.\cos.^2\theta\right)}{dt},$$

we shall have*

[506] $$N' = \left(\frac{dQ}{dv}\right).$$

Lastly, if we multiply the first of the equations (i) [499], by $-r.\sin.\theta.\cos.v$; the second by $-r.\sin.\theta.\sin.v$; the third by $r.\cos.\theta$; and add the products, putting

[507] $$P' = r^2.\frac{dd\theta}{dt^2} + r^2.\frac{dv^2}{dt^2}.\sin.\theta.\cos.\theta + \frac{2r.dr.d\theta}{dt^2};$$

we shall find†

[508] $$P' = \left(\frac{dQ}{d\theta}\right).$$

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* (362) In the same manner as $\left(\frac{dQ}{dr}\right)$ was found in the preceding note we shall find $\left(\frac{dQ}{dv}\right) = \left(\frac{dQ}{dx}\right).\left(\frac{dx}{dv}\right) + \left(\frac{dQ}{dy}\right).\left(\frac{dy}{dv}\right) + \left(\frac{dQ}{dz}\right).\left(\frac{dz}{dv}\right)$; and from [501] we get

[506a] $$\left(\frac{dx}{dv}\right) = -r.\cos.\theta.\sin.v; \qquad \left(\frac{dy}{dv}\right) = r.\cos.\theta.\cos.v; \qquad \left(\frac{dz}{dv}\right) = 0;$$

Hence, $\left(\frac{dQ}{dv}\right) = -r.\cos.\theta.\sin.v.\left(\frac{dQ}{dx}\right) + r.\cos.\theta.\cos.v.\left(\frac{dQ}{dy}\right)$, and by substituting [499], it becomes $-r.\cos.\theta.\sin.v.\frac{ddx}{dt^2} + r.\cos.\theta.\cos.v.\frac{ddy}{dt^2}$, or as it may be written $\left(\frac{dQ}{dv}\right) = \frac{r.\cos.\theta}{dt^2}.\left\{-\sin.v.ddx + \cos.v.ddy\right\}$. Substituting the values ddx, ddy, [503c, d], in the expression $-\sin.v.ddx+\cos.v.ddy$, it becomes $2dR.dv+Rddv$, or $\frac{d.(R^2dv)}{R}$, which is changed into $\frac{d.(r^2.dv.\cos.^2\theta)}{r.\cos.\theta}$, by using R, [503b]. Hence we have $\left(\frac{dQ}{dv}\right) = \frac{d.(r^2.dv.\cos.^2\theta)}{dt^2} = \frac{d.\left(r^2.\frac{dv}{dt}.\cos.^2\theta\right)}{dt} = N'$, [505], as in [506].

† (363) As in the two preceding notes, we find,

$$\left(\frac{dQ}{d\theta}\right) = \left(\frac{dQ}{dx}\right).\left(\frac{dx}{d\theta}\right) + \left(\frac{dQ}{dy}\right).\left(\frac{dy}{d\theta}\right) + \left(\frac{dQ}{dz}\right).\left(\frac{dz}{d\theta}\right);$$

[507a] $$\left(\frac{dx}{d\theta}\right) = -r.\sin.\theta.\cos.v; \qquad \left(\frac{dy}{d\theta}\right) = -r.\sin.\theta.\sin.v; \qquad \left(\frac{dz}{d\theta}\right) = r.\cos.\theta.$$

The values of r, v, θ, contain six arbitrary quantities, introduced by the integrations.* Let us consider any three of these arbitrary quantities, which we shall denote by a, b, c; the equation $M' = \left(\frac{dQ}{dr}\right)$, will give the three following equations :†

$$\begin{aligned}
&\left(\frac{ddQ}{dr^2}\right) . \left(\frac{dr}{da}\right) + \left(\frac{ddQ}{dr\,dv}\right) . \left(\frac{dv}{da}\right) + \left(\frac{ddQ}{dr\,d\theta}\right) . \left(\frac{d\theta}{da}\right) = \left(\frac{dM'}{da}\right);\\
&\left(\frac{ddQ}{dr^2}\right) . \left(\frac{dr}{db}\right) + \left(\frac{ddQ}{dr\,dv}\right) . \left(\frac{dv}{db}\right) + \left(\frac{ddQ}{dr\,d\theta}\right) . \left(\frac{d\theta}{db}\right) = \left(\frac{dM'}{db}\right);\\
&\left(\frac{ddQ}{dr^2}\right) . \left(\frac{dr}{dc}\right) + \left(\frac{ddQ}{dr\,dv}\right) . \left(\frac{dv}{dc}\right) + \left(\frac{ddQ}{dr\,d\theta}\right) . \left(\frac{d\theta}{dc}\right) = \left(\frac{dM'}{dc}\right).
\end{aligned}$$ [509]

We may obtain from these equations, the value of $\left(\frac{ddQ}{dr^2}\right)$, and if we make

$$\begin{aligned}
m &= \left(\frac{dv}{db}\right) . \left(\frac{d\theta}{dc}\right) - \left(\frac{dv}{dc}\right) . \left(\frac{d\theta}{db}\right);\\
n &= \left(\frac{dv}{dc}\right) . \left(\frac{d\theta}{da}\right) - \left(\frac{dv}{da}\right) . \left(\frac{d\theta}{dc}\right);\\
p &= \left(\frac{dv}{da}\right) . \left(\frac{d\theta}{db}\right) - \left(\frac{dv}{db}\right) . \left(\frac{d\theta}{da}\right);
\end{aligned}$$ [510]

These and the equations [499], give

$$\begin{aligned}
\left(\frac{dQ}{d\theta}\right) &= -r . \sin. \theta . \cos. v . \frac{ddx}{dt^2} - r . \sin. \theta . \sin. v . \frac{ddy}{dt^2} + r . \cos. \theta . \frac{ddz}{dt^2}\\
&= -\frac{r.\sin.\theta}{dt^2} . \left\{ ddx . \cos. v + ddy . \sin. v \right\} + r . \cos. \theta . \frac{ddz}{dt^2}.
\end{aligned}$$

Substituting the value of $ddx . \cos. v + ddy . \sin. v$, [503e], it becomes

$$\left(\frac{dQ}{d\theta}\right) = \frac{r}{dt^2} . \left\{ -ddR . \sin. \theta + ddz . \cos. \theta \right\} + \frac{r.\sin.\theta}{dt^2} . R\,dv^2.$$

Now from ddR, ddz, [503g], we get $-ddR . \sin.\theta + ddz . \cos.\theta = 2dr.d\theta + r\,dd\theta$, which being substituted, and also $r . \cos. \theta$, for R, we shall find

$$\left(\frac{dQ}{d\theta}\right) = r^2 . \frac{dd\theta}{dt^2} + r^2 . \frac{dv^2}{dt^2} . \sin. \theta . \cos. \theta + \frac{2r.dr.d\theta}{dt^2}.$$

being the same as P', [507], which agrees with [508].

* (364) Each of the three equations [504, 506, 508], is a differential of the second order in r, v, θ; their integrals must therefore contain six arbitrary constant quantities.

† (365) The first equation is found by taking the differential of $M' = \left(\frac{dQ}{dr}\right)$, [504], relative to a, considering Q as a function of r, v, θ, and these quantities as functions of a, b, c. The other equations are found by changing a into b and c successively.

[510']
$$\beta=\left(\frac{dr}{da}\right)\cdot\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{dc}\right)-\left(\frac{dr}{da}\right)\cdot\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{db}\right)+\left(\frac{dr}{db}\right)\cdot\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{da}\right)$$
$$-\left(\frac{dr}{db}\right)\cdot\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{dc}\right)+\left(\frac{dr}{dc}\right)\cdot\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{db}\right)-\left(\frac{dr}{dc}\right)\cdot\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{da}\right);$$

we shall have*

[511]
$$\beta\cdot\left(\frac{ddQ}{dr^2}\right)=m\cdot\left(\frac{dM'}{da}\right)+n\cdot\left(\frac{dM'}{db}\right)+p\cdot\left(\frac{dM'}{dc}\right).$$

In like manner, if we put

[512]
$$m'=\left(\frac{dr}{dc}\right)\cdot\left(\frac{d\theta}{db}\right)-\left(\frac{dr}{db}\right)\cdot\left(\frac{d\theta}{dc}\right);$$
$$n'=\left(\frac{dr}{da}\right)\cdot\left(\frac{d\theta}{dc}\right)-\left(\frac{dr}{dc}\right)\cdot\left(\frac{d\theta}{da}\right);$$
$$p'=\left(\frac{dr}{db}\right)\cdot\left(\frac{d\theta}{da}\right)-\left(\frac{dr}{da}\right)\cdot\left(\frac{d\theta}{db}\right);$$

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* (366) If we multiply the three equations [509] by $m, n, p,$ respectively and add the products together, we shall find,

[511a]
$$\left(\frac{ddQ}{dr^2}\right)\cdot\left\{m\cdot\left(\frac{dr}{da}\right)+n\cdot\left(\frac{dr}{db}\right)+p\cdot\left(\frac{dr}{dc}\right)\right\}+\left(\frac{ddQ}{drdv}\right)\cdot\left\{m\cdot\left(\frac{dv}{da}\right)+n\cdot\left(\frac{dv}{db}\right)+p\cdot\left(\frac{dv}{dc}\right)\right\}$$
$$+\left(\frac{ddQ}{drd\theta}\right)\cdot\left\{m\cdot\left(\frac{d\theta}{da}\right)+n\cdot\left(\frac{d\theta}{db}\right)+p\cdot\left(\frac{d\theta}{dc}\right)\right\}=m\cdot\left(\frac{dM'}{da}\right)+n\cdot\left(\frac{dM'}{db}\right)+p\cdot\left(\frac{dM'}{dc}\right).$$

Substituting the values of $m, n, p,$ [510], in the coefficient of $\left(\frac{ddQ}{dr^2}\right)$, it becomes equal to the quantity denoted by β, [510']. The coefficient of $\left(\frac{ddQ}{drdv}\right)$ becomes

$$\left(\frac{dv}{da}\right)\cdot\left\{\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{dc}\right)-\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{db}\right)\right\}+\left(\frac{dv}{db}\right)\cdot\left\{\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{da}\right)-\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{dc}\right)\right\}$$
$$+\left(\frac{dv}{dc}\right)\cdot\left\{\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{db}\right)-\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{da}\right)\right\};$$

and of the six terms of which this is composed, the first is destroyed by the fourth, the second by the fifth, and the third by the sixth, and thus the coefficient is reduced to nothing. The coefficient of $\left(\frac{ddQ}{drd\theta}\right)$ becomes

$$\left(\frac{d\theta}{da}\right)\cdot\left\{\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{dc}\right)-\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{db}\right)\right\}+\left(\frac{d\theta}{db}\right)\cdot\left\{\left(\frac{dv}{dc}\right)\cdot\left(\frac{d\theta}{da}\right)-\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{dc}\right)\right\}$$
$$+\left(\frac{d\theta}{dc}\right)\cdot\left\{\left(\frac{dv}{da}\right)\cdot\left(\frac{d\theta}{db}\right)-\left(\frac{dv}{db}\right)\cdot\left(\frac{d\theta}{da}\right)\right\}.$$

In which the first term is destroyed by the sixth; the second by the third, and the fourth by the fifth, thus reducing it to nothing. Consequently the equation [511a] becomes as in [511].

the equation $N' = \left(\frac{d\,Q}{d\,v}\right)$, will give*

$$\beta . \left(\frac{d\,d\,Q}{d\,v^2}\right) = m' . \left(\frac{d\,N'}{d\,a}\right) + n' . \left(\frac{d\,N'}{d\,b}\right) + p' . \left(\frac{d\,N'}{d\,c}\right).$$ [513]

Lastly, if we put

$$\begin{aligned} m'' &= \left(\frac{d\,r}{d\,b}\right) . \left(\frac{d\,v}{d\,c}\right) - \left(\frac{d\,r}{d\,c}\right) . \left(\frac{d\,v}{d\,b}\right); \\ n'' &= \left(\frac{d\,r}{d\,c}\right) . \left(\frac{d\,v}{d\,a}\right) - \left(\frac{d\,r}{d\,a}\right) . \left(\frac{d\,v}{d\,c}\right); \\ p'' &= \left(\frac{d\,r}{d\,a}\right) . \left(\frac{d\,v}{d\,b}\right) - \left(\frac{d\,r}{d\,b}\right) . \left(\frac{d\,v}{d\,a}\right); \end{aligned}$$ [514]

the equation $P' = \left(\frac{d\,Q}{d\,\theta}\right)$, will give†

$$\beta . \left(\frac{d\,d\,Q}{d\,\theta^2}\right) = m'' . \left(\frac{d\,P'}{d\,a}\right) + n'' . \left(\frac{d\,P'}{d\,b}\right) + p'' . \left(\frac{d\,P'}{d\,c}\right).$$ [515]

The equation (F) [502] will thus become‡

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* (367) Taking the differential of the equation [506], relative to a, b, c, we shall obtain three equations similar to [509], and which may be deduced from them, by writing v for r, r for v, N' for M'. This change being made in m, n, p, [510], they will become respectively, $-m'$, $-n'$, $-p'$, [512]; also β, [510'] will change into $-\beta$. These quantities being substituted in [511] it will become

$$-\beta . \left(\frac{d\,d\,Q}{d\,v^2}\right) = -m' . \left(\frac{d\,N'}{d\,a}\right) - n' . \left(\frac{d\,N'}{d\,b}\right) - p' . \left(\frac{d\,N'}{d\,c}\right),$$

and by changing the signs of all the terms, we shall obtain the equation [513].

† (368) The equation [508] gives three equations similar to [509], by taking the differentials relatively to a, b, c; and these equations may be deduced from [509], by changing r, θ, M' into θ, r, P', respectively. By these changes the values of m, n, p, [510], become $-m''$, $-n''$, $-p''$, [514], respectively, and β, [510'], becomes $-\beta$. These changes being made in [511], it becomes

$$-\beta . \left(\frac{d\,d\,Q}{d\,\theta^2}\right) = -m'' . \left(\frac{d\,P'}{d\,a}\right) - n'' . \left(\frac{d\,P'}{d\,b}\right) - p'' . \left(\frac{d\,P'}{d\,c}\right),$$

and by changing the signs of the terms we obtain [515].

‡ (369) The equation [502], being multiplied by $\beta . \cos.^2 \theta$, becomes, by arranging the terms in a different order,

$$\begin{aligned} 0 = r^2 . \cos.^2 \theta . \beta . \left(\frac{d\,d\,Q}{d\,r^2}\right) + \beta . \left(\frac{d\,d\,Q}{d\,v^2}\right) + \cos.^2 \theta . \beta . \left(\frac{d\,d\,Q}{d\,\theta^2}\right) \\ + \beta . \left\{ 2\,r . \cos.^2 \theta . \left(\frac{d\,Q}{d\,r}\right) - \sin.\theta . \cos.\theta . \left(\frac{d\,Q}{d\,\theta}\right) \right\}, \end{aligned}$$

$$0 = m r^2 . \cos.^2 \theta . \left(\frac{d M'}{d a}\right) + n . r^2 . \cos.^2 \theta . \left(\frac{d M'}{d b}\right) + p . r^2 . \cos.^2 \theta . \left(\frac{d M'}{d c}\right)$$

[516]
$$+ m' . \left(\frac{d N'}{d a}\right) + n' . \left(\frac{d N'}{d b}\right) + p' . \left(\frac{d N'}{d c}\right) \qquad (G)$$

$$+ m'' . \cos.^2 \theta . \left(\frac{d P'}{d a}\right) + n'' . \cos.^2 \theta . \left(\frac{d P'}{d b}\right) + p'' . \cos.^2 \theta . \left(\frac{d P'}{d c}\right)$$

$$+ \beta . \{2 r M' . \cos.^2 \theta - P' . \sin. \theta . \cos. \theta\}.$$

In the theory of the moon, we neglect the perturbations produced by the moon's action upon the relative motion of the sun about the earth, which amounts to the same thing as to suppose the moon's mass to be infinitely small. In this case the variable quantities x', y', z', corresponding to the sun, would be independent of x, y, z, which correspond to the moon; and the
[516'] equation (G) [516] would take place in this theory [500']; therefore the values found for r, v, θ, ought to satisfy this equation; which furnishes a method of verifying these values. If the observed equations in the motion of the moon, result from the mutual attractions of the three bodies, the sun, earth, and moon, it must necessarily follow, that the values of r, v, and θ, deduced from observations, would satisfy the equation (G) [516]; which furnishes a method of verifying the theory of universal gravitation; for the
[516''] mean longitudes of the moon, the perigee, and the ascending node, enter into these values, and we may take a, b, c, for these longitudes.

In like manner, in the theory of the planets, if we neglect the square of the disturbing forces, which may almost always be done, and then put x, y, z, for the co-ordinates of the planet whose orbit is to be computed, we may
[516'''] suppose the co-ordinates x', y', z', x'', &c., of the other planets to correspond to their elliptical motions,* and they will therefore be independent of x, y, z;

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and by substituting the values of the terms computed in [511, 513, 515, 504, 508], it becomes as in [516].

* (370) By neglecting the terms multiplied by m', m'', &c., in Q, [499'], the equations [517] will give the elliptical motion of the body m. The neglected terms will be of the order $m' x'$, $m' y'$, &c. Now any one of the co-ordinates x', y', &c. may be supposed to be divided into two parts, the one depending on the elliptical motion, the other on the disturbing forces of m, m'', &c. This last part being of the order $m x'$, $m y'$, &c.; and it must evidently produce in Q terms of the order $m m'$, &c., or of the square of the disturbing forces.

hence it would follow from [500′] that the equation (*G*) [516] would also take place in this theory of the planets.

15. The differential equations of the preceding article*

$$\left.\begin{aligned}&\frac{ddr}{dt^2}-\frac{r.dv^2}{dt^2}.\cos.^2\theta-r.\frac{d\theta^2}{dt^2}=\left(\frac{dQ}{dr}\right);\\ &\frac{d.\left(r^2.\frac{dv}{dt}.\cos.^2\theta\right)}{dt}=\left(\frac{dQ}{dv}\right);\\ &r^2.\frac{dd\theta}{dt^2}+r^2.\frac{dv^2}{dt^2}.\sin.\theta.\cos.\theta+\frac{2rdr.d\theta}{dt^2}=\left(\frac{dQ}{d\theta}\right);\end{aligned}\right\}\quad(H)\qquad[517]$$

are merely combinations of the differential equations (*i*) [499] of the same article; but they are more convenient, and better adapted to the use of astronomers. We may put them under other forms, which may be useful on several occasions.

Instead of the variable quantities r and θ, let us use u and s, putting

$$u=\frac{1}{r.\cos.\theta},\qquad[517']$$

or u equal to unity divided by the projection of the radius vector upon the plane of x, y, and

$$s=\text{tang.}\,\theta,\qquad[517'']$$

or s equal to the tangent of the latitude of m above that plane. If we multiply the second of the equations (*H*) [517] by $r^2\,dv.\cos.^2\theta$, and then take the integral,† we shall have

$$\left(\frac{dv}{u^2dt}\right)^2=h^2+2.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2};\qquad[518]$$

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* (370*a*) The values of *M′*, [503, 504], being put equal to each other, we get the first of the equations [517]. In like manner the values of *N′*, [505, 506] give the second, and those of *P′*, [507, 508], give the third of the equations [517].

† (371) The product thus formed is

$$\left(r^2.\frac{dv}{dt}.\cos.^2\theta\right).d.\left(r^2.\frac{dv}{dt}.\cos.^2\theta\right)=\left(\frac{dQ}{dv}\right).r^2.dv.\cos.^2\theta,$$

or by putting, as in [517′], $r.\cos.\theta=\frac{1}{u}$, $\left(\frac{dv}{u^2dt}\right).d.\left(\frac{dv}{u^2dt}\right)=\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$,

which, being multiplied by 2, and integrated, gives $\left(\frac{dv}{u^2dt}\right)^2=h^2+2.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}$, as in [518].

h being an arbitrary constant quantity, hence we have

[519] $$dt=\frac{dv}{u^2\cdot\sqrt{h^2+2\cdot\int\left(\frac{dQ}{dv}\right)\cdot\frac{dv}{u^2}}}.$$

If we add the first of the equations (H) [517] multiplied by $-\cos.\theta$, to the third multiplied by $\frac{\sin.\theta}{r}$, we shall find*

[520] $$-\frac{d^2\cdot\frac{1}{u}}{dt^2}+\frac{1}{u}\cdot\frac{dv^2}{dt^2}=u^2\cdot\left(\frac{dQ}{du}\right)+us\cdot\left(\frac{dQ}{ds}\right);$$

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* (372) The products, being added together, make the following sum

[520a] $$\frac{1}{dt^2}\cdot\left\{-ddr.\cos.\theta+2dr.d\theta.\sin.\theta+r.dd\theta.\sin.\theta+r.d\theta^2.\cos.\theta+r.dv^2.\cos.\theta.(\cos.^2\theta+\sin.^2\theta)\right\}$$
$$=-\cos.\theta.\left(\frac{dQ}{dr}\right)+\frac{\sin.\theta}{r}\cdot\left(\frac{dQ}{d\theta}\right).$$

Now $\frac{1}{u}=r.\cos.\theta$, [517′], gives

$$-d^2.\left(\frac{1}{u}\right)=-d^2.(r.\cos.\theta)=-ddr.\cos.\theta+2dr.d\theta.\sin.\theta+r.dd\theta.\sin.\theta+r.d\theta^2.\cos.\theta,$$

the second member of this expression contains the four first terms of [520a], and by substituting $-d^2.\left(\frac{1}{u}\right)$ for those terms and $\frac{1}{u}$ for $r.\cos.\theta$, in the last term, that equation

[520b] becomes $-\frac{d^2.\frac{1}{u}}{dt^2}+\frac{1}{u}\cdot\frac{dv^2}{dt^2}=-\cos.\theta.\left(\frac{dQ}{dr}\right)+\frac{\sin.\theta}{r}\cdot\left(\frac{dQ}{d\theta}\right)$. Now if we consider Q as a function of r, θ, and then as a function of u, s, we shall have

$$\left(\frac{dQ}{dr}\right)=\left(\frac{dQ}{du}\right)\cdot\left(\frac{du}{dr}\right)+\left(\frac{dQ}{ds}\right)\cdot\left(\frac{ds}{dr}\right);\qquad\left(\frac{dQ}{d\theta}\right)=\left(\frac{dQ}{du}\right)\cdot\left(\frac{du}{d\theta}\right)+\left(\frac{dQ}{ds}\right)\cdot\left(\frac{ds}{d\theta}\right).$$

But $u=\frac{1}{r.\cos.\theta}$, and $s=\text{tang.}\,\theta$ give $\left(\frac{du}{dr}\right)=-\frac{1}{r^2.\cos.\theta}$; $\left(\frac{du}{d\theta}\right)=\frac{\sin.\theta}{r.\cos.^2\theta}$; $\left(\frac{ds}{dr}\right)=0$; $\left(\frac{ds}{d\theta}\right)=\frac{1}{\cos.^2\theta}$. Hence $\left(\frac{dQ}{dr}\right)=-\frac{1}{r^2.\cos.\theta}\cdot\left(\frac{dQ}{du}\right)$;

[520c] $$\left(\frac{dQ}{d\theta}\right)=\frac{\sin.\theta}{r.\cos.^2\theta}\cdot\left(\frac{dQ}{du}\right)+\frac{1}{\cos.^2\theta}\cdot\left(\frac{dQ}{ds}\right).$$

These being substituted in the second member of [520b], it becomes

$$\left(\frac{dQ}{du}\right)\cdot\left\{\frac{1}{r^2}+\frac{\sin.^2\theta}{r^2.\cos.^2\theta}\right\}+\frac{\sin.\theta}{r.\cos.^2\theta}\cdot\left(\frac{dQ}{ds}\right),$$

and by reduction it is equal to

$$\left(\frac{dQ}{du}\right)\cdot\frac{1}{(r.\cos.\theta)^2}+\frac{\text{tang.}\,\theta}{r.\cos.\theta}\cdot\left(\frac{dQ}{ds}\right),\qquad\text{or}\qquad\left(\frac{dQ}{du}\right).u^2+us.\left(\frac{dQ}{ds}\right).$$

hence we deduce

$$d.\left(\frac{du}{u^2dt}\right)+\frac{dv^2}{u\,dt}=u^2.dt.\left\{\left(\frac{dQ}{du}\right)+\frac{s}{u}.\left(\frac{dQ}{ds}\right)\right\}. \qquad [521]$$

Substituting for dt its value [519], supposing dv to be constant, we shall have* [521′]

$$0=\frac{ddu}{dv^2}+u+\frac{\left(\frac{dQ}{dv}\right).\frac{du}{u^2dv}-\left(\frac{dQ}{du}\right)-\frac{s}{u}.\left(\frac{dQ}{ds}\right)}{h^2+2.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}}. \qquad [522]$$

The third of the equations (H) [517] becomes, in the same manner,† supposing dv to be constant, [523]

$$0=\frac{dds}{dv^2}+s+\frac{\frac{ds}{dv}.\left(\frac{dQ}{dv}\right)-(1+ss).\left(\frac{dQ}{ds}\right)-us.\left(\frac{dQ}{du}\right)}{u^2.\left\{h^2+2.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}\right\}}. \qquad [524]$$

Hence the expression [520*b*] becomes as in [520]; and as $-d.\frac{1}{u}=\frac{du}{u^2}$, if we substitute this in [520], multiplied by dt, we shall obtain [521].

* (373) Put for brevity, $\sqrt{h^2+2.\int\left(\frac{dQ}{dv}\right).\frac{dv}{u^2}}=A$. This being squared and its differential taken gives $dA=\left(\frac{dQ}{dv}\right).\frac{dv}{Au^2}$. Substitute this value of A in [519] [522*a*] and we shall get $dt=\frac{dv}{Au^2}$, hence, [522*b*]

$$d.\left(\frac{du}{u^2dt}\right)=d.\left(\frac{Adu}{dv}\right)=A.\frac{ddu}{dv}+\frac{du}{dv}.dA=A.\frac{ddu}{dv}+\left(\frac{dQ}{dv}\right).\frac{du}{Au^2},$$

dv being constant. This and the value of dt, being substituted in [521] it becomes

$$A.\frac{ddu}{dv}+\left(\frac{dQ}{dv}\right).\frac{du}{Au^2}+Au\,dv=\frac{dv}{A}.\left\{\left(\frac{dQ}{du}\right)+\frac{s}{u}.\left(\frac{dQ}{ds}\right)\right\}.$$

Dividing by Adv, transposing the two last terms, and substituting for A^2 its value, we shall obtain the equation [522].

† (374) The third of the equation [517] may be thus written,

$$\frac{d.\left(r^2.\frac{d\theta}{dt}\right)}{dt}+r^2.\frac{dv^2}{dt^2}.\cos.^2\theta.\,\text{tang.}\,\theta=\left(\frac{dQ}{d\theta}\right). \qquad [522c]$$

Now $s=\text{tang.}\,\theta$, [517″], gives $ds=\frac{d\theta}{\cos.^2\theta}$, or $d\theta=ds.\cos.^2\theta$, hence

We shall therefore have, instead of the three differential equations (H) [517], the following expressions in which $d\,v$ is constant:

[525]
$$\begin{aligned}
d\,t &= \frac{d\,v}{u^2 \cdot \sqrt{h^2 + 2 \cdot \int \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,v}{u^2}}}\,; \\
0 &= \frac{d\,d\,u}{d\,v^2} + u + \frac{\left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,u}{u^2\,d\,v} - \left(\frac{d\,Q}{d\,u}\right) - \frac{s}{u} \cdot \left(\frac{d\,Q}{d\,s}\right)}{h^2 + 2 \cdot \int \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,v}{u^2}}\,; \qquad (K) \\
0 &= \frac{d\,d\,s}{d\,v^2} + s + \frac{\frac{d\,s}{d\,v} \cdot \left(\frac{d\,Q}{d\,v}\right) - u\,s \cdot \left(\frac{d\,Q}{d\,u}\right) - (1+s\,s) \cdot \left(\frac{d\,Q}{d\,s}\right)}{u^2 \cdot \left\{ h^2 + 2 \cdot \int \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,v}{u^2} \right\}}\,;
\end{aligned}$$

If we wish to avoid fractions and radicals, we may put these equations under the following forms:*

$$r^2 \cdot \frac{d\,\theta}{d\,t} = r^2 \cdot \cos.^2 \theta \cdot \frac{d\,s}{d\,t} = \frac{1}{u^2} \cdot \frac{d\,s}{d\,t}, \quad [517'],$$

and as $\frac{1}{u^2\,d\,t} = \frac{\mathcal{A}}{d\,v}$, [522*b*], it becomes, $\frac{\mathcal{A}\,d\,s}{d\,v}$, whose differential, considering $d\,v$ as constant, is $\frac{\mathcal{A}\,d\,d\,s + d\,\mathcal{A}\,d\,s}{d\,v}$, or $\mathcal{A} \cdot \frac{d\,d\,s}{d\,v} + \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,s}{\mathcal{A}\,u^2}$, [522*a*]. This divided by $d\,t$, or $\frac{d\,v}{\mathcal{A}\,u^2}$, gives the first term of the preceding equation, [522*c*], $u^2\,\mathcal{A}^2 \cdot \frac{d\,d\,s}{d\,v^2} + \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,s}{d\,v}$. Also $r^2 \cdot \cos.^2 \theta \cdot \text{tang.}\, \theta \cdot \frac{d\,v^2}{d\,t^2} = \frac{s}{u^2} \cdot \frac{d\,v^2}{d\,t^2} = \frac{s\,d\,v^2}{u^2} \cdot \left(\frac{u^2\,\mathcal{A}}{d\,v}\right)^2 = u^2\,s \cdot \mathcal{A}^2$. Again by [520*c*] we have $\left(\frac{d\,Q}{d\,\theta}\right) = \frac{\sin.\,\theta}{r \cdot \cos.^2 \theta} \cdot \left(\frac{d\,Q}{d\,u}\right) + \frac{1}{\cos.^2 \theta} \cdot \left(\frac{d\,Q}{d\,s}\right)$, or as it may be written

$$\left(\frac{d\,Q}{d\,\theta}\right) = \frac{\text{tang.}\,\theta}{r \cdot \cos.\,\theta} \cdot \left(\frac{d\,Q}{d\,u}\right) + (1 + \text{tang.}^2 \theta) \cdot \left(\frac{d\,Q}{d\,s}\right) = u\,s \cdot \left(\frac{d\,Q}{d\,u}\right) + (1 + s\,s) \cdot \left(\frac{d\,Q}{d\,s}\right).$$

These values being substituted in [522*c*], it becomes

$$u^2\,\mathcal{A}^2 \cdot \frac{d\,d\,s}{d\,v^2} + \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,s}{d\,v} + u^2\,s\,\mathcal{A}^2 = u\,s \cdot \left(\frac{d\,Q}{d\,u}\right) + (1 + s\,s) \cdot \left(\frac{d\,Q}{d\,s}\right).$$

Transposing the two last terms, and dividing by $u^2\,\mathcal{A}^2$, we obtain the expression [524.] Collecting together [519, 522, 524], we obtain the equations [525].

* (375) Using the value $\mathcal{A}$, [522*a*], as in the two preceding notes, the first equation [525] will give $u^2\,d\,t \cdot \mathcal{A} = d\,v$, and its differential, considering $d\,v$ as constant, and substituting the value of $d\,\mathcal{A}$, [522*a*], will be

$$2\,u \cdot d\,u \cdot d\,t \cdot \mathcal{A} + u^2 \cdot d\,d\,t \cdot \mathcal{A} + u^2 \cdot d\,t \cdot \left(\frac{d\,Q}{d\,v}\right) \cdot \frac{d\,v}{\mathcal{A}\,u^2} = 0\,;$$

$$0=\frac{d\,d\,t}{d\,v^2}+\frac{2\,d\,u.d\,t}{u\,d\,v^2}+u^2.\left(\frac{d\,Q}{d\,v}\right).\frac{d\,t^3}{d\,v^3};$$

$$0=\left(\frac{d\,d\,u}{d\,v^2}+u\right).\left\{1+\frac{2}{h^2}.\int\left(\frac{d\,Q}{d\,v}\right).\frac{d\,v}{u^2}\right\}$$

$$+\frac{1}{h^2}.\left\{\left(\frac{d\,Q}{d\,v}\right).\frac{d\,u}{u^2\,d\,v}-\left(\frac{d\,Q}{d\,u}\right)-\frac{s}{u}.\left(\frac{d\,Q}{d\,s}\right)\right\}; \qquad (L)$$ [526]

$$0=\left(\frac{d\,d\,s}{d\,v^2}+s\right).\left\{1+\frac{2}{h^2}.\int\left(\frac{d\,Q}{d\,v}\right).\frac{d\,v}{u^2}\right\}$$

$$+\frac{1}{h^2.u^2}.\left\{\frac{d\,s}{d\,v}.\left(\frac{d\,Q}{d\,v}\right)-u\,s.\left(\frac{d\,Q}{d\,u}\right)-(1+s\,s).\left(\frac{d\,Q}{d\,s}\right)\right\}.$$

By using other co-ordinates, we may form new systems of differential
equations. Suppose, for example, that we change the co-ordinates x and y
of the equations (i), § 14 [499], into others, relative to two moveable axes,
situated in the plane of the co-ordinates x, y; so that the first of these new [526′]
axes may correspond to the mean longitude of the body m, whilst the other is
perpendicular to it. Let $x_{,}$ and $y_{,}$ be the co-ordinates of m, referred to these
axes, and $n\,t+\varepsilon$, the mean longitude of m, or the angle which the moveable [526″]
axis of $x_{,}$ makes with the axis of x; we shall have*

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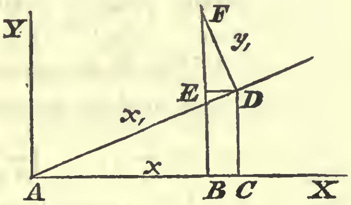
and, by multiplying by A, $\quad (u^2.d\,d\,t+2\,u.d\,u.d\,t).A^2+d\,t.d\,v.\left(\frac{d\,Q}{d\,v}\right)=0.$

Substituting the value of $A=\frac{d\,v}{u^2\,d\,t}$, [522*b*], it becomes

$$(u^2.d\,d\,t+2\,u.d\,u.d\,t).\frac{d\,v^2}{u^4\,d\,t^2}+d\,t.d\,v.\left(\frac{d\,Q}{d\,v}\right)=0.$$

This multiplied by $\frac{u^2\,d\,t^2}{d\,v^4}$, and reduced, corresponds to the first of the equations [526]. The second and third of these equations may be deduced from the second and third of the equations [525], respectively, by multiplying by $\frac{A^2}{h^2}$, or $1+\frac{2}{h^2}.\int\left(\frac{d\,Q}{d\,v}\right).\frac{d\,v}{u^2}$, [522*a*], and reducing.



* (376) In the adjoined figure let the rectangular co-ordinates of the point F be either $AB=x$, $BF=y$, or $AD=x_{,}$, and $DF=y_{,}$. Draw $D\,C$ perpendicular to $A\,B$, and $D\,E$ parallel to $A\,B$. The angle $C\,A\,D=D\,F\,E=n\,t+\varepsilon$, and in the right angled triangles $D\,E\,F$, $A\,C\,D$, we have

[527]
$$x = x_{,} \,.\, \cos.\,(nt+\varepsilon) - y_{,} \,.\, \sin.\,(nt+\varepsilon)\,;$$
$$y = x_{,} \,.\, \sin.\,(nt+\varepsilon) + y_{,} \,.\, \cos.\,(nt+\varepsilon)\,;$$

whence we deduce, by supposing dt constant,*

[528]
$$ddx \,.\, \cos.\,(nt+\varepsilon) + ddy \,.\, \sin.\,(nt+\varepsilon) = ddx_{,} - n^2 x_{,} \,.\, dt^2 - 2n\,dy_{,} . dt\,;$$
$$ddy \,.\, \cos.\,(nt+\varepsilon) - ddx \,.\, \sin.\,(nt+\varepsilon) = ddy_{,} - n^2 y_{,} \,.\, dt^2 + 2n\,dx_{,} \,.\, dt.$$

By substituting the preceding values of x and y in Q, we shall have†

[529]
$$\left(\frac{dQ}{dx}\right) = \left(\frac{dQ}{dx_{,}}\right) . \cos.\,(nt+\varepsilon) - \left(\frac{dQ}{dy_{,}}\right) . \sin.\,(nt+\varepsilon)\,;$$
$$\left(\frac{dQ}{dy}\right) = \left(\frac{dQ}{dx_{,}}\right) . \sin.\,(nt+\varepsilon) + \left(\frac{dQ}{dy_{,}}\right) . \cos.\,(nt+\varepsilon)\,;$$

$AC = AD . \cos.\,CAD = x_{,} . \cos.(nt+\varepsilon)$; $CD (= BE) = AD . \sin.\,CAD = x_{,} . \sin.(nt+\varepsilon)$;
$FE = DF . \cos.\,DFE = y_{,} . \cos.(nt+\varepsilon)$; $DE (= BC) = DF . \sin.\,DFE = y_{,} . \sin.(nt+\varepsilon)$.

Hence
$$x = AC - BC = x_{,} . \cos.\,(nt+\varepsilon) - y_{,} . \sin.\,(nt+\varepsilon)\,;$$
$$y = BE + FE = x_{,} . \sin.\,(nt+\varepsilon) + y_{,} . \cos.\,(nt+\varepsilon)\,;$$
as in [527].

* (377) The differential of x, [527], is
$$dx = dx_{,} . \cos.\,(nt+\varepsilon) - nx_{,} . dt . \sin.\,(nt+\varepsilon) - dy_{,} . \sin.\,(nt+\varepsilon) - ny_{,} . dt . \cos.\,(nt+\varepsilon).$$
Its second differential, supposing dt constant, is

[528a]
$$ddx = \{ddx_{,} - n^2 x_{,} . dt^2 - 2n . dy_{,} . dt\} . \cos.\,(nt+\varepsilon)$$
$$- \{ddy_{,} - n^2 y_{,} . dt^2 + 2n . dx_{,} . dt\} . \sin.\,(nt+\varepsilon).$$

Now by writing $\varepsilon - \tfrac{1}{2}\pi$ for ε, the expression of x, [527], becomes that of y, the same change being made in ddx, gives

[528b] $$ddy = \{ddx_{,} - n^2 x_{,} . dt^2 - 2n\,dy_{,} . dt\} . \sin.(nt+\varepsilon) + \{ddy_{,} - n^2 y_{,} . dt^2 + 2n\,dx_{,} . dt\} . \cos.\,(nt+\varepsilon).$$

Multiplying [528a] by $\cos.\,(nt+\varepsilon)$, and [528b] by $\sin.\,(nt+\varepsilon)$, and adding the products we shall obtain the first of the equations [528]. Also, multiplying [528a] by $-\sin.\,(nt+\varepsilon)$, and [528b] by $\cos.\,(nt+\varepsilon)$, and taking their sum, we shall get the second of the equations [528].

† (378) Considering Q as a function of x, y, and then as a function of $x_{,}$, $y_{,}$, we have

[529a] $$\left(\frac{dQ}{dx}\right) = \left(\frac{dQ}{dx_{,}}\right) . \left(\frac{dx_{,}}{dx}\right) + \left(\frac{dQ}{dy_{,}}\right) . \left(\frac{dy_{,}}{dx}\right); \qquad \left(\frac{dQ}{dy}\right) = \left(\frac{dQ}{dx_{,}}\right) . \left(\frac{dx_{,}}{dy}\right) + \left(\frac{dQ}{dy_{,}}\right) . \left(\frac{dy_{,}}{dy}\right).$$

Multiplying the first of the equations [527] by $\cos.\,(nt+\varepsilon)$, the second by $\sin.\,(nt+\varepsilon)$, and adding these products we shall get $x_{,} = x . \cos.\,(nt+\varepsilon) + y . \sin.\,(nt+\varepsilon)$, hence

[529b] $\left(\frac{dx_{,}}{dx}\right) = \cos.\,(nt+\varepsilon)$; $\left(\frac{dx_{,}}{dy}\right) = \sin.\,(nt+\varepsilon)$. If we had multiplied the first of the equations [527] by $-\sin.\,(nt+\varepsilon)$, the second by $\cos.\,(nt+\varepsilon)$, the sum would have

This being premised, the differential equations (*i*) [499] will give the three following equations*

$$0=\frac{d^2x_{,}}{dt^2}-n^2x_{,}-2n\cdot\frac{dy_{,}}{dt}-\left(\frac{dQ}{dx_{,}}\right);$$
$$0=\frac{d^2y_{,}}{dt^2}-n^2y_{,}+2n\cdot\frac{dx_{,}}{dt}-\left(\frac{dQ}{dy_{,}}\right);\qquad (M)\qquad [530]$$
$$0=\frac{d^2z}{dt^2}-\left(\frac{dQ}{dz}\right).$$

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been $y_{,}=-x\cdot\sin.(nt+\varepsilon)+y\cdot\cos.(nt+\varepsilon)$, hence $\left(\frac{dy_{,}}{dx}\right)=-\sin.(nt+\varepsilon)$, [529*c*]
$\left(\frac{dy_{,}}{dy}\right)=\cos.(nt+\varepsilon)$. These values being substituted in [529*a*], they will become as in [529].

* (379) Substitute the values [529] in the equations [499], and we shall get

$$\frac{ddx}{dt^2}=\left(\frac{dQ}{dx_{,}}\right)\cdot\cos.(nt+\varepsilon)-\left(\frac{dQ}{dy_{,}}\right)\cdot\sin.(nt+\varepsilon);$$
$$\frac{ddy}{dt^2}=\left(\frac{dQ}{dx_{,}}\right)\cdot\sin.(nt+\varepsilon)+\left(\frac{dQ}{dy_{,}}\right)\cdot\cos.(nt+\varepsilon).\qquad [529d]$$

Multiplying the first of these equations by $\cos.(nt+\varepsilon)$, and adding it to the second, multiplied by $\sin.(nt+\varepsilon)$, we get $\frac{ddx}{dt^2}\cdot\cos.(nt+\varepsilon)+\frac{ddy}{dt^2}\cdot\sin.(nt+\varepsilon)=\left(\frac{dQ}{dx_{,}}\right)$. [529*e*]
Again, the first of the preceding equations, multiplied by $-\sin.(nt+\varepsilon)$, added to the second multiplied by $\cos.(nt+\varepsilon)$, gives

$$\frac{ddy}{dt^2}\cdot\cos.(nt+\varepsilon)-\frac{ddx}{dt^2}\cdot\sin.(nt+\varepsilon)=\left(\frac{dQ}{dy_{,}}\right).\qquad [529f]$$

In the equations thus obtained, we must substitute the values of the first members [528], and they will become like the two first of the equations [530]. The third of these equations is like the third of the equations [499].

The calculation in this part of the work, and in the two preceding notes might have been done in rather a more simple manner as follows. First, we evidently have

$$\left(\frac{dQ}{dx_{,}}\right)=\left(\frac{dQ}{dx}\right)\cdot\left(\frac{dx}{dx_{,}}\right)+\left(\frac{dQ}{dy}\right)\cdot\left(\frac{dy}{dx_{,}}\right);\qquad \left(\frac{dQ}{dy_{,}}\right)=\left(\frac{dQ}{dx}\right)\cdot\left(\frac{dx}{dy_{,}}\right)+\left(\frac{dQ}{dy}\right)\cdot\left(\frac{dy}{dy_{,}}\right);$$

and from [527] we get $\left(\frac{dx}{dx_{,}}\right)=\cos.(nt+\varepsilon)$; $\left(\frac{dy}{dx_{,}}\right)=\sin.(nt+\varepsilon)$; $\left(\frac{dx}{dy_{,}}\right)=-\sin.(nt+\varepsilon)$; $\left(\frac{dy}{dy_{,}}\right)=\cos.(nt+\varepsilon)$. Substituting these values, and $\left(\frac{dQ}{dx}\right)$, $\left(\frac{dQ}{dy}\right)$, given by the equations [499], we shall have

After having given the differential equations of the motions of a system of bodies mutually attracting each other, and having deduced from them all the complete integrals which have yet been discovered; it now remains to integrate these equations by successive approximations. In the solar system, the heavenly bodies move in nearly the same manner as if they strictly obeyed the principal force which acts on them, and the disturbing forces are very small; we may therefore, in the first approximation, consider only the [530′] mutual action of two bodies, namely, that of a planet or a comet and the sun, in the theory of the planets and comets; and the mutual action of a satellite and its planet, in the theory of the satellites. We shall therefore begin with an exact computation of the motions of two bodies, which attract each other: this first approximation will lead to a second, in which we shall notice the first power of the disturbing forces; then we shall consider the squares and products of these forces; and by continuing in this manner, we [530″] shall determine the motions of the heavenly bodies with all the precision required by observation.

[529g]

$$\left(\frac{dQ}{dx_{,}}\right)=\frac{ddx}{dt^2}\,.\cos.(nt+\varepsilon)+\frac{ddy}{dt^2}\,.\sin.(nt+\varepsilon)\,;$$

$$\left(\frac{dQ}{dy_{,}}\right)=-\frac{ddx}{dt^2}\,.\sin.(nt+\varepsilon)+\frac{ddy}{dt^2}\,.\cos.(nt+\varepsilon).$$

Substituting, in the second members of these equations, the values [528], we shall obtain, by a small reduction, the two first of the equations [530].

CHAPTER III.

FIRST APPROXIMATION OF THE MOTIONS OF THE HEAVENLY BODIES, OR THEORY OF THE ELLIPTICAL MOTION.

16. We have already shown in the first chapter [380^{iv}], that a body attracted towards a fixed point, by a force in the inverse duplicate ratio of the distance, describes a conic section. Now in the relative motion of m about M, this last body being supposed at rest, we must transfer to m, in a [530‴] contrary direction, the action which m exerts on M; so that in this relative motion, m is attracted towards M, by a force equal to the sum of the masses M and m, divided by the square of their distance; the body m will therefore, upon the same principle, [380^{iv}], describe a conic section about M. But the importance of this subject in the theory of the system of the world, requires that we should resume the investigation, in order to place it in a new point of view.

For this purpose, let us consider the equations (K) § 15 [525]. If we put

$$\mu = M + m, \qquad [530^{iv}]$$

it is evident by § 14 [499′], that if we notice only the reciprocal action of M and m, we shall have*

$$Q = \frac{\mu}{r} = \frac{\mu \, . \, u}{\sqrt{1+ss}}; \qquad [530^{v}]$$

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* (380) Putting m', m'', &c. equal to nothing, it makes λ [412] vanish, and Q, [499′], becomes simply $Q = \frac{M+m}{r} = \frac{\mu}{r}$. But $\frac{1}{r} = u \, . \cos.\theta$, [517′], and $\mathrm{tang.}\theta = s$, [517″], hence, [530a] $\cos.\theta = \frac{1}{\sec.\theta} = \frac{1}{\sqrt{1+\mathrm{tang.}^2\theta}} = \frac{1}{\sqrt{1+ss}}$, consequently $\frac{1}{r} = \frac{u}{\sqrt{1+ss}}$, and

[530^{vi}] the equations (K) [525] will then become, dv being constant,

Differential equations of the motion of one body about another considered as at rest.

$$dt=\frac{dv}{hu^2};$$

$$0=\frac{ddu}{dv^2}+u-\frac{\mu}{h^2.(1+ss)^{\frac{3}{2}}};$$

[531]
$$0=\frac{dds}{dv^2}+s;$$

The area described during the element of time dt, by the projection of the radius vector, being equal to* $\frac{1}{2}.\frac{dv}{u^2}$, the first of these equations shows that this area is proportional to that element, consequently in a finite time, the area is proportional to the time. The last equation gives by integration,† [531′]

[532]
$$s=\gamma.\sin.(v-\theta),$$

γ and θ being two arbitrary constant quantities. The second equation [531] gives by integration,‡

[533]
$$u=\frac{\mu}{h^2.(1+\gamma^2)}.\{\sqrt{1+ss}+e.\cos.(v-\varpi)\}=\frac{\sqrt{1+ss}}{r};$$

[530b] $Q=\frac{\mu}{r}=\frac{\mu u}{\sqrt{1+ss}}$. This gives $\left(\frac{dQ}{dv}\right)=0$; $\left(\frac{dQ}{du}\right)=\frac{\mu}{\sqrt{1+ss}}$; $\left(\frac{dQ}{ds}\right)=\frac{-\mu su}{(1+ss)^{\frac{3}{2}}}$. These being substituted in [525] they will become as in [531].

* (381) By [372a] this area is equal to the square of the projection of the radius vector, multiplied by half the differential of the arch, or by referring to the figure in page 306, [530c] $\frac{1}{2}.CB^2\times dv$, and since $CB=r.\cos.\theta=\frac{1}{u}$, [517′], it becomes $\frac{1}{2}.\frac{dv}{u^2}$, as in [531′].

† (382) This equation is obtained as in [864a], putting $y=s$, $t=v$, $a=1$, $b=\gamma$, $\varphi=-\theta$. And it is easily proved, for $s=\gamma.\sin.(v-\theta)$ gives $ds=\gamma dv.\cos.(v-\theta)$, and as dv is constant [530^{vi}], $dds=-\gamma dv^2.\sin.(v-\theta)$, hence $\frac{dds}{dv^2}+s=0$.

‡ (383) That the assumed value of u, [533] satisfies the second equation [531] is easily proved by substitution and reduction, and as it contains two arbitrary constant quantities, it must be the complete integral. For, by putting $\varepsilon=\frac{\mu e}{h^2.(1+\gamma^2)}$, the terms of [533], depending on the angle $v-\varpi$ will become $\varepsilon.\cos.(v-\varpi)$; substituting this for u in the second of the equations [531], it produces the terms $-\varepsilon.\cos.(v-\varpi)+\varepsilon.\cos.(v-\varpi)$,

e and ϖ being two new arbitrary constant quantities. Substitute in this expression of u the value of s in terms of v [532] and then this value of u in the expression $dt = \frac{dv}{h \,.\, u^2}$; the integral of this equation will give t in a function of v; we shall then have v, u, s, in functions of the time. [533′]

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which mutually destroy each other, so that it will be only necessary to notice the other term depending on $\sqrt{1+ss}$, and if for brevity we put $\frac{\mu}{h^2 \,.\, (1+\gamma^2)} = b$, this term will become $b \,.\, (1+ss)^{\frac{1}{2}}$, which, being substituted in the second of the equations [531], produces the following terms, observing that $\mu = b\,h^2 \,.\, (1+\gamma^2)$ in the last term,

$$b \,.\, \left\{ \frac{d^2 \,.\, (1+ss)^{\frac{1}{2}}}{dv^2} + (1+ss)^{\frac{1}{2}} - (1+\gamma^2) \,.\, (1+ss)^{-\frac{3}{2}} \right\}. \qquad [532a]$$

The first of these terms $b \,.\, \frac{d^2 \,.\, (1+ss)^{\frac{1}{2}}}{dv^2}$ being developed becomes

$$b \,.\, (1+ss)^{-\frac{3}{2}} \,.\, \left\{ \frac{ds^2}{dv^2} + s \,.\, \frac{dds}{dv^2} \,.\, (1+ss) \right\},$$

and since $\frac{dds}{dv^2} = -s$, [531], it changes into $b \,.\, (1+ss)^{-\frac{3}{2}} \left\{ \frac{ds^2}{dv^2} - s^2 - s^4 \right\}$,

connecting this with the second term of [532*a*], $b \,.\, (1+ss)^{\frac{1}{2}}$, which may be put under the form $b \,.\, (1+ss)^{-\frac{3}{2}} \,.\, \{1 + 2s^2 + s^4\}$, the sum becomes $b \,.\, (1+ss)^{-\frac{3}{2}} \,.\, \left\{1 + s^2 + \frac{ds^2}{dv^2}\right\}$, and if we substitute the value of s, [532], it becomes

$$b \,.\, (1+ss)^{-\frac{3}{2}} \,.\, \left\{ 1 + \gamma^2 \,.\, \sin.^2 (v-\theta) + \gamma^2 \,.\, \cos.^2 (v-\theta) \right\} = b \,.\, (1+s^2)^{-\frac{3}{2}} \,.\, (1+\gamma^2),$$

which being equal, and of an opposite sign to the third term of [532*a*] renders the whole equal to nothing, therefore the assumed value [533], satisfies the second of the equations [531]. Finally, by [517′] we have $u = \frac{1}{r \,.\, \cos.\, \theta} = \frac{1}{r} \,.\, \sqrt{1 + \text{tang.}^2\, \theta} = \frac{\sqrt{1+ss}}{r}$, as in [533]. [532b]

We shall now show how the same equation may be solved directly by the method given in [865*a*, *b*]. Putting in [865*a*], $y = u$, $t = v - \theta$, $a = 1$, $\alpha = \frac{\mu}{h^2}$, $Q = -\frac{1}{(1+ss)^{\frac{3}{2}}}$, also $b = \frac{\mu e}{h^2 \,.\, (1+\gamma^2)}$, $\varphi = \theta - \varpi$, it will become like the second of the equations [531], and the general solution [865*b*], will give the following value of u,

$$u = \frac{\mu e}{h^2 \,.\, (1+\gamma^2)} \,.\, \cos.\, (v - \varpi) + \alpha \,.\, \sin.\, t \,.\, \int \frac{dt \,.\, \cos.\, t}{(1+ss)^{\frac{3}{2}}} - \alpha \,.\, \cos.\, t \,.\, \int \frac{dt \,.\, \sin.\, t}{(1+ss)^{\frac{3}{2}}}.$$

The calculation may be considerably abridged by observing that the value of s [532] indicates that the orbit is wholly in a plane inclined to the fixed
[533″] plane by an angle whose tangent is γ, and the longitude of the node θ,

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In which, for brevity, t is retained instead of $v-\theta$, and this makes [532] become $s=\gamma\,.\,\sin.\,t$, hence $\sin.\,t=\frac{s}{\gamma}$; $\cos.\,t=\sqrt{1-\frac{s^2}{\gamma^2}}$, whose differentials are $dt\,.\,\cos.\,t=\frac{ds}{\gamma}$; $dt\,.\,\sin.\,t=\frac{s\,ds}{\gamma^2\,.\sqrt{1-\frac{s^2}{\gamma^2}}}$, these being substituted we shall get

$$u=\frac{\mu e}{h^2\,.\,(1+\gamma^2)}\,.\,\cos.(v-\varpi)+\frac{\alpha s}{\gamma^2}\,.\int\frac{ds}{(1+ss)^{\frac{3}{2}}}-\frac{\alpha\,.\sqrt{1-\frac{s^2}{\gamma^2}}}{\gamma^2}\,.\int\frac{s\,ds}{\sqrt{1-\frac{s^2}{\gamma^2}\,.\,(1+ss)^{\frac{3}{2}}}}.$$

Now $\int\frac{ds}{(1+ss)^{\frac{3}{2}}}=\frac{s}{\sqrt{1+ss}}$, as is easily proved by differentiation; also

$$\int\frac{s\,ds}{\sqrt{1-\frac{ss}{\gamma^2}\,.\,(1+ss)^{\frac{3}{2}}}}=-\frac{\gamma\,.\sqrt{\gamma^2-s^2}}{(1+\gamma^2)\,.\sqrt{1+ss}},$$

for the differential of the second member is

$$\frac{\gamma s\,ds}{(1+\gamma^2)\,.\sqrt{\gamma^2-s^2}\,.\sqrt{1+s^2}}+\frac{\gamma\,.\sqrt{\gamma^2-s^2}\,.\,s\,ds}{(1+\gamma^2)\,.\,(1+ss)^{\frac{3}{2}}}$$

$$=\frac{\gamma s\,ds}{(1+\gamma^2)\,.\sqrt{\gamma^2-s^2}\,.\,(1+ss)^{\frac{3}{2}}}\,.\left\{1+ss+\gamma^2-s^2\right\}$$

$$=\frac{\gamma s\,ds}{\sqrt{\gamma^2-s^2}\,.\,(1+ss)^{\frac{3}{2}}}=\frac{s\,ds}{\sqrt{1-\frac{s^2}{\gamma^2}\,.\,(1+ss)^{\frac{3}{2}}}}.$$

Hence,

$$u=\frac{\mu e}{h^2\,.\,(1+\gamma^2)}\,.\,\cos.\,(v-\varpi)+\frac{\alpha s^2}{\gamma^2\,.\sqrt{1+s^2}}+\frac{\alpha\,.\sqrt{1-\frac{s^2}{\gamma^2}}}{\gamma^2}\,.\,\frac{\gamma\,.\sqrt{\gamma^2-s^2}}{(1+\gamma^2)\,.\sqrt{1+s^2}}.$$

The two last terms of which may be thus written,

$$\frac{\alpha}{\gamma^2\,.\sqrt{1+s^2}}\,.\left\{s^2+\frac{(\gamma^2-s^2)}{1+\gamma^2}\right\}=\frac{\alpha}{\gamma^2\,.\sqrt{1+s^2}}\,.\,\frac{\gamma^2\,.\,(1+s^2)}{1+\gamma^2}=\frac{\alpha\,.\sqrt{1+ss}}{1+\gamma^2},$$

and by substituting the value of α, it becomes $\frac{\mu\,.\sqrt{1+s^2}}{h^2\,.\,(1+\gamma^2)}$. Hence

$$u=\frac{\mu}{h^2\,.\,(1+\gamma^2)}\,.\left\{e\,.\,\cos.\,(v-\varpi)+\sqrt{1+s^2}\right\},$$

as in [533].

counted from the origin of the angle v.* Referring therefore to this plane [533''']
the motion of m, we shall have $s=0$, and $\gamma=0$, which gives [533]

$$u=\frac{1}{r}=\frac{\mu}{h^2}\,.\,\{1+e\,.\,\cos.\,(v-\varpi)\}\,;$$ [534]

This equation corresponds to an ellipsis in which the origin of r is at the focus; $\frac{h^2}{\mu\,.\,(1-e^2)}$ is the semi-transverse axis,† and we shall put

$a=\frac{h^2}{\mu\,.\,(1-e^2)}=$ the semi-transverse axis,

$e=$ the ratio of the excentricity to the semi-transverse axis, [534']

$\varpi=$ the longitude of the perihelion.

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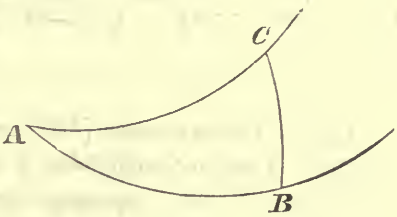
* (384) Suppose a spherical surface ABC to be drawn about the origin of r, with a radius equal to unity, to intersect the plane of x, y, in the great circle AB, and the plane of the orbit in its ascending node A. Through this point draw, on the spherical surface, a great circle AC, such that the tangent of the angle BAC may be equal to γ. Make the arch $AB=v-\theta$, and draw, perpendicular to it, the arch BC cutting AC in C. Then by spherics, $\text{tang.}\,BC=\text{tang.}\,BAC\,.\,\sin.\,AB=\gamma\,.\,\sin.\,(v-\theta)$, or, by [532], [533a]
tang. $BC=s$, and since s [517''], represents the tangent of the latitude, it follows that the planet in its motion must have the same latitude as if it moved in the plane of the great circle AC.



† (385) From [378] we get $\frac{1}{r}=\frac{1}{a.(1-e^2)}\,.\,\{1+e\,.\,\cos.\,(v-\varpi)\}$, putting this equal to $\frac{1}{r}$, [534] we shall get $\frac{1}{a.(1-e^2)}=\frac{\mu}{h^2}$, or $\frac{h^2}{\mu.(1-e^2)}=a$; a being the [534a] semi-transverse axis, [377''], $a\,e$ the excentricity, &c. This gives

$$h=\sqrt{\mu}\,.\,a^{\frac{1}{2}}\,.\,(1-e^2)^{\frac{1}{2}},$$ [534b]

which being substituted in the first of the equations [531], it becomes

$$dt=\frac{dv}{\sqrt{\mu}\,.\,a^{\frac{1}{2}}\,.\,(1-e^2)^{\frac{1}{2}}\,.\,u^2},$$ [534c]

and since $u=\frac{1}{r}=\frac{1+e\,.\,\cos.(v-\varpi)}{a\,.\,(1-e^2)}$, [534, 378], it will become, by substitution, as in [535].

The equation $dt = \frac{dv}{h . u^2}$ [531] will by this means become

[535] $$dt = \frac{a^{\frac{3}{2}} . (1-e^2)^{\frac{3}{2}} . dv}{\sqrt{\mu} . \{1 + e . \cos. (v - \varpi)\}^2}.$$

We shall develop the second member of this equation, in a series of cosines of the angle $v - \varpi$ and of its multiples. For this purpose, we shall commence with the development of the quantity $\frac{1}{1 + e . \cos. (v - \varpi)}$, in a similar series. If we put*

[536] $$\lambda = \frac{e}{1 + \sqrt{1 - e^2}};$$

we shall have†

[537] $$\frac{1}{1 + e . \cos. (v - \varpi)} = \frac{1}{\sqrt{1 - e^2}} . \left\{ \frac{1}{1 + \lambda . c^{(v - \varpi) . \sqrt{-1}}} - \frac{\lambda . c^{-(v - \varpi) . \sqrt{-1}}}{1 + \lambda . c^{-(v - \varpi) . \sqrt{-1}}} \right\};$$

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* (386) The expression [536], may be put under other forms which it will be useful to notice. First, by multiplying it by $1 + \sqrt{1 - e^2}$, and transposing λ, we get $\lambda . \sqrt{1 - ee} = e - \lambda$; squaring, rejecting λ^2 from each side of the equation, and dividing by e, we get $-\lambda^2 e = e - 2\lambda$, whence

[536a] $$e = \frac{2\lambda}{1 + \lambda\lambda}.$$

This gives

[536b] $$1 - e = \frac{(1 - \lambda)^2}{1 + \lambda\lambda}; \qquad 1 + e = \frac{(1 + \lambda)^2}{1 + \lambda\lambda}; \qquad \frac{1 - e}{1 + e} = \frac{(1 - \lambda)^2}{(1 + \lambda)^2}; \qquad \sqrt{\frac{1 - e}{1 + e}} = \frac{1 - \lambda}{1 + \lambda}.$$

Also,

[536c] $$1 - ee = 1 - \frac{4\lambda^2}{(1 + \lambda\lambda)^2} = \frac{1 - 2\lambda^2 + \lambda^4}{(1 + \lambda\lambda)^2} = \left(\frac{1 - \lambda\lambda}{1 + \lambda\lambda}\right)^2, \quad \text{and} \quad \sqrt{1 - e^2} = \frac{1 - \lambda\lambda}{1 + \lambda\lambda}.$$

Again, $\sqrt{\frac{1 + e}{1 - e}} - 1 = \frac{1 + \lambda}{1 - \lambda} - 1 = \frac{2\lambda}{1 - \lambda}$, and $\sqrt{\frac{1 + e}{1 - e}} + 1 = \frac{1 + \lambda}{1 - \lambda} + 1 = \frac{2}{1 - \lambda}$,

[536d] consequently $$\frac{\sqrt{\frac{1 + e}{1 - e}} - 1}{\sqrt{\frac{1 + e}{1 - e}} + 1} = \lambda.$$

[537a] † (387) By [12] Int. $\cos. (v - \varpi) = \frac{c^{(v - \varpi) . \sqrt{-1}} + c^{-(v - \varpi) . \sqrt{-1}}}{2}$. This, by

[537b] putting for brevity $x = c^{(v - \varpi) . \sqrt{-1}}$, becomes $\cos. (v - \varpi) = \frac{1}{2} x + \frac{1}{2} x^{-1}$, and since

c being the number whose hyperbolic logarithm is unity. By developing [537'] the second member of this expression in a series, of which the first part is arranged according to the powers of $c^{(v-\varpi).\sqrt{-1}}$, and the second according to the powers of $c^{-(v-\varpi).\sqrt{-1}}$, and then substituting, instead of the imaginary exponential quantities, their corresponding sines and cosines;* we shall find

$$\frac{1}{1+e.\cos.(v-\varpi)}=\frac{1}{\sqrt{1-e^2}} \qquad [538]$$
$$.\{1-2\lambda.\cos.(v-\varpi)+2\lambda^2.\cos.2(v-\varpi)-2\lambda^3.\cos.3(v-\varpi)+\&c.\}$$

Put the second member of this equation equal to φ, and $q=\frac{1}{e}$; we shall [538'] have in general†

$$\frac{1}{\{1+e.\cos.(v-\varpi)\}^{m+1}}=\pm\frac{e^{-m-1}.d^m.\left(\frac{\varphi}{q}\right)}{1.2.3....m.dq^m}; \qquad [539]$$

$e=\frac{2\lambda}{1+\lambda\lambda}$, [536a], we shall get

$$\frac{1}{1+e.\cos.(v-\varpi)}=\frac{1}{1+\frac{\lambda}{1+\lambda\lambda}.(x+x^{-1})}=\frac{1+\lambda^2}{1+\lambda^2+\lambda x+\lambda x^{-1}}$$
$$=\frac{1+\lambda^2}{(1+\lambda x).(1+\lambda x^{-1})}=\frac{1+\lambda^2}{1-\lambda^2}.\left\{\frac{1}{1+\lambda x}-\frac{\lambda x^{-1}}{1+\lambda x^{-1}}\right\},$$

substituting $\frac{1+\lambda^2}{1-\lambda^2}=\frac{1}{\sqrt{1-e^2}}$, [536c], it becomes as in [537].

* (388) Using the symbol x [537b], and developing the terms of [537], according to the powers of x we shall find $\frac{1}{1+\lambda x}=1-\lambda x+\lambda^2x^2-\lambda^3x^3+\&c.$ and in like manner $-\frac{\lambda x^{-1}}{1+\lambda x^{-1}}=-\lambda x^{-1}+\lambda^2x^{-2}-\lambda^3x^{-3}+\&c.$ The sum of these two series is $1-\lambda.(x+x^{-1})+\lambda^2.(x^2+x^{-2})-\lambda^3.(x^3+x^{-3})+\&c.$; and by [537a, b], $x+x^{-1}=2\cos.(v-\varpi)$; $x^2+x^{-2}=2\cos.2.(v-\varpi)$; $x^3+x^{-3}=2\cos.3.(v-\varpi)$, &c. therefore the preceding series is equal to

$$1-2\lambda.\cos.(v-\varpi)+2\lambda^2.\cos.2.(v-\varpi)-2\lambda^3.\cos.3.(v-\varpi)+\&c.$$

consequently, the formula [537] becomes as in [538].

† (389) This value of $\varphi=\frac{1}{1+e.\cos.(v-\varpi)}$ gives $\frac{1}{\frac{1}{e}+\cos.(v-\varpi)}=e\varphi$, or [538a]

$d\,q$ being supposed constant, and the sign + or — taking place, according as m is an even or odd number. Hence it is evident that if we suppose

[540]
$$\frac{1}{\{1+e\,.\cos.(v-\varpi)\}^2}=(1-e^2)^{-\frac{3}{2}}$$
$$.\{1+E^{(1)}.\cos.(v-\varpi)+E^{(2)}.\cos.2(v-\varpi)+E^{(3)}.\cos.3(v-\varpi)+\&c.\}\,;$$

we shall have, whatever i may be*

[541]
$$E^{(i)}=\pm\frac{2\,e^i\,.\,\{1+i.\sqrt{1-e^2}\}}{(1+\sqrt{1-e^2})^i}\;;$$

$\frac{1}{q+\cos.(v-\varpi)}=\frac{\varphi}{q}$, and by putting for brevity $q+\cos.(v-\varpi)=\mathcal{A}$, $\mathcal{A}^{-1}=\frac{\varphi}{q}$, now the assumed value of $\mathcal{A}$ gives $\left(\frac{d\mathcal{A}}{d\,q}\right)=1$, and if we take successively the differentials of the equation $\mathcal{A}^{-1}=\left(\frac{\varphi}{q}\right)$, considering q or e only as variable, and substitute $\left(\frac{d\mathcal{A}}{d\,q}\right)=1$, we shall have

$$-\mathcal{A}^{-2}=\frac{d\,.\left(\frac{\varphi}{q}\right)}{d\,q}\;;\qquad 2\,\mathcal{A}^{-3}=\frac{d^2\,.\left(\frac{\varphi}{q}\right)}{d\,q^2}\;;\qquad -2\,.\,3\,.\,\mathcal{A}^{-4}=\frac{d^3\,.\left(\frac{\varphi}{q}\right)}{d\,q^3}\;;$$

$$2\,.\,3\,.\,4\,.\,\mathcal{A}^{-5}=\frac{d^4\,.\left(\frac{\varphi}{q}\right)}{d\,q^4}\;;\ \&c.\qquad \ldots\ldots\pm1\,.\,2\,.\,3\,.\,4\ldots.\,m\,.\,\mathcal{A}^{-m-1}=\frac{d^m\,.\left(\frac{\varphi}{q}\right)}{d\,q^m}\,.$$

Hence $\mathcal{A}^{-m-1}=\pm\frac{d^m\,.\left(\frac{\varphi}{q}\right)}{1.2.3\ldots.m\,.\,d\,q^m}$, but

$$\mathcal{A}^{-m-1}=\frac{1}{\{q+\cos.(v-\varpi)\}^{m+1}}=\frac{e^{m+1}}{\{1+e\,.\cos.(v-\varpi)\}^{m+1}}\,;$$

putting these two expressions equal to each other, we shall get

$$\frac{1}{\{1+e\,.\cos.(v-\varpi)\}^{m+1}}=\pm\frac{e^{-m-1}\,.\,d^m\,.\left(\frac{\varphi}{q}\right)}{1.2.3\ldots.m\,.\,d\,q^m}\;;\ \ [539].$$

[541a] * [390) Putting $m=1$, in [539], it becomes $\frac{1}{\{1+e\,.\cos.(v-\varpi)\}^2}=\frac{-e^{-2}\,.\,d\,.\left(\frac{\varphi}{q}\right)}{d\,q}$. Now from the assumed values of φ, q, [538′], we shall get,

$$\frac{\varphi}{q}=\frac{1}{\sqrt{q\,q-1}}\,.\left\{1-2\,\lambda\,.\cos.(v-\varpi)+2\,\lambda^2\,.\cos.2\,(v-\varpi)-2\,\lambda^3\,.\cos.3\,(v-\varpi)+\&c.\right\}.$$

Hence

$$\frac{d\,.\left(\frac{\varphi}{q}\right)}{d\,q}=-\frac{q}{(q\,q-1)^{\frac{3}{2}}}\,.\left\{1-2\,\lambda\,.\cos.(v-\varpi)+2\,\lambda^2\,.\cos.2\,(v-\varpi)-\&c.\right\}$$
$$+\frac{1}{\sqrt{q^2-1}}\,.\left\{-2\cos.(v-\varpi)+4\,\lambda\,.\cos.2\,.\,(v-\varpi)-\&c.\right\}.\frac{d\,\lambda}{d\,q}\,;$$

the sign + taking place if i is even ; and the sign —, if i is odd ; supposing therefore

$$n = a^{-\frac{3}{2}} . \sqrt{\mu} , \qquad [541']$$

we shall have*

$$n\,dt = dv . \{1 + E^{(1)} . \cos . (v - \varpi) + E^{(2)} . \cos . 2(v - \varpi) + E^{(3)} . \cos . 3(v - \varpi) + \&c. \} ; \qquad [542]$$

and by integration

$$nt + \varepsilon = v + E^{(1)} . \sin . (v - \varpi) + \tfrac{1}{2} . E^{(2)} . \sin . 2(v - \varpi) + \tfrac{1}{3} . E^{(3)} . 3(v - \varpi) + \&c. ; \qquad [543]$$

ε being an arbitrary constant quantity. This expression of $n\,t + \varepsilon$ is very converging when the orbits are nearly circular, as is the case with the orbits [543'] of the planets and satellites ; and we may, by inverting the series, find the value of v in t ; we shall attend to this subject in the following articles.

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this being multiplied by $-e^{-2}$, or $-q^2$, gives $\frac{1}{\{1 + e . \cos . (v - \varpi)\}^2}$, [541a], equal to

$$\frac{q^3}{(qq-1)^{\frac{3}{2}}} . \left\{ 1 - 2\lambda . \cos . (v - \varpi) + 2\lambda^2 . \cos . 2\,(v - \varpi) - \&c. \right\}$$
$$- \frac{q^2}{\sqrt{q^2-1}} . \left\{ -2 \cos . (v - \varpi) + 4\lambda . \cos . 2 . (v - \varpi) - \&c. \right\} . \frac{d\lambda}{dq} ,$$

and if we put, as in [540], the term of this series corresponding to $\cos . i . (v - \varpi)$, equal to $(1 - e^2)^{-\frac{3}{2}} . E^{(i)}$, we shall have

$$(1 - e^2)^{-\frac{3}{2}} . E^{(i)} = \pm \frac{q^3}{(qq-1)^{\frac{3}{2}}} . 2\lambda^i \mp \frac{q^2}{\sqrt{q^2-1}} . 2i . \lambda^{i-1} . \frac{d\lambda}{dq} ,$$

or by substituting $q = \frac{1}{e}$, and multiplying by $(1 - ee)^{\frac{3}{2}}$;

$$E^{(i)} = \pm 2\lambda^i \mp \frac{1 - ee}{e} . 2i . \lambda^{i-1} . \frac{d\lambda}{dq} .$$

Now $\lambda = \frac{e}{1 + \sqrt{1 - e^2}} = \frac{1}{q + \sqrt{q^2 - 1}}$ gives

$$\frac{d\lambda}{dq} = \frac{-1}{\sqrt{q^2-1} . \{q + \sqrt{qq-1}\}} = \frac{-\lambda}{\sqrt{q^2-1}} = \frac{-\lambda e}{\sqrt{1-e^2}} ;$$

hence $E^{(i)} = \pm 2\lambda^i . \{1 + i . \sqrt{1 - e^2}\}$, and by substituting for λ its value $\frac{e}{1 + \sqrt{1 - ee}}$, [536], it becomes as in [541].

* (391) Multiply the first member of [535] by n, its second member by the value of n [541'], $a^{-\frac{3}{2}} . \sqrt{\mu}$, and substitute the expression [540], it will become as in [542].

When the planet returns to the same point of its orbit, v is increased by
[543′] the *circumference of the circle, which we shall always denote by* 2π; putting
T for the time of a revolution, we shall have,*

[544] $$T=\frac{2\pi}{n}=\frac{2\pi\,.\,a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

[544′] This expression of T may be deduced directly from the differential expression of dt without having recourse to series. For by resuming the equation [531] $dt=\frac{dv}{h\,.\,u^2}$, or $dt=\frac{r^2dv}{h}$, [534], it gives $T=\frac{\int r^2dv}{h}$. $\int r^2dv$ is double the surface of the ellipsis [372*a*], consequently it is equal to
[544″] $2\pi\,.\,a^2.\sqrt{1-e^2}$ [378*v*]; also h^2 is equal to $\mu a\,.\,(1-e^2)$ [534*a*]; hence we deduce the same expression of T as that above given [544].

If we neglect the mass of the planets with respect to that of the sun, we
[544‴] shall have $\sqrt{\mu}=\sqrt{M}$ [$530^{\rm iv}$]; the value of $\sqrt{\mu}$ will then be the same for all the planets; T is therefore proportional to $a^{\frac{3}{2}}$, consequently the squares of the times of revolution are as the cubes of the transverse axes of the
[$544^{\rm iv}$] orbits. We see also, that the same law takes place in the motion of the satellites about their primary planet, neglecting their masses in comparison with that of the planet.

The motion of one body about another, computed in a different manner.

17. We may also integrate the differential equations of the motions of two bodies M and m, which attract each other in the inverse duplicate ratio of the distances, in the following manner. Resuming the equations (1), (2) and (3), § 9 [416—418], they will become, by considering only the action
[$544^{\rm v}$] of two bodies† M and m, and putting $M+m=\mu$ [$530^{\rm iv}$],

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* (392) Since $nt+\varepsilon=v+E^{(1)}.\sin.(v-\varpi)+\&c.$ [543]. If we increase t by T, and v by 2π, we shall have $n.(t+T)+\varepsilon=(v+2\pi)+E^{(1)}.\sin.(v-\varpi)+\&c.$ Subtracting the former from the latter we get $nT=2\pi$, or $T=\frac{2\pi}{n}$; substituting n, [541′], it becomes $T=\frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$, as in [544].

† (393) In this case λ, [412], vanishes, and the equations [416—418] become as in [545].

$$\left.\begin{array}{l} 0=\dfrac{ddx}{dt^2}+\dfrac{\mu.x}{r^3} \\ 0=\dfrac{ddy}{dt^2}+\dfrac{\mu.y}{r^3} \\ 0=\dfrac{ddz}{dt^2}+\dfrac{\mu.z}{r^3} \end{array}\right\}. \qquad (O)$$ [545]

The integrals of these equations will give the three co-ordinates x, y, z, of the body m, referred to the centre of M, in functions of the time t; we shall then have, by § 9, the co-ordinates ξ, Π and γ of the body M, referred to a fixed point, by means of the equations [421, 422],

$$\xi=a+bt-\frac{mx}{M+m}; \qquad \Pi=a'+b't-\frac{my}{M+m}; \qquad \gamma=a''+b''t-\frac{mz}{M+m};$$ [546]

Lastly, we shall have the co-ordinates of m with respect to the same fixed point, by adding x to ξ, y to Π, and z to γ; we shall thus have the relative motions of M and m, and their absolute motions in space. All that is now required is to integrate the differential equations (O) [545].

For this purpose, we shall observe, that if we have between the n variable quantities $x^{(1)}$, $x^{(2)}$, $x^{(3)} \ldots . x^{(n)}$, and the variable quantity t, whose differential is supposed constant, a number of differential equations denoted by n, of the following form,

$$0=\frac{d^i x^{(s)}}{dt^i}+A.\frac{d^{i-1}x^{(s)}}{dt^{i-1}}+B.\frac{d^{i-2}x^{(s)}}{dt^{i-2}} \ldots\ldots + H.x^{(s)},$$ [547]

in which we suppose s to be successively equal to 1, 2, 3, ... n; [547′]
A, B, ... H, being functions of the variable quantities $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, &c., and t; A, B, ... H, being symmetrical* with respect to the variable quantities $x^{(1)}$, $x^{(2)}$, ... $x^{(n)}$; or, in other words, they will remain the same, when we change any one of these quantities $x^{(1)}$, $x^{(2)}$, ... $x^{(n)}$, into any other of them, and the contrary; we may suppose

$$\begin{array}{l} x^{(1)}=a^{(1)}.x^{(n-i+1)}+b^{(1)}.x^{(n-i+2)} \ldots\ldots + h^{(1)}.x^{(n)}; \\ x^{(2)}=a^{(2)}.x^{(n-i+1)}+b^{(2)}.x^{(n-i+2)} \ldots\ldots + h^{(2)}.x^{(n)}; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^{(n-i)}=a^{(n-i)}.x^{(n-i+1)}+b^{(n-i)}.x^{(n-i+2)} \ldots\ldots + h^{(n-i)}.x^{(n)}; \end{array}$$ [548]

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* (393a) The only condition necessary to be observed relative to the quantities $A, B \ldots . H$ is that they must be the same for all the differential equations of the form [547], it is not generally necessary that they should be symmetrical.

$a^{(1)}$, $b^{(1)}$, ... $h^{(1)}$; $a^{(2)}$, $b^{(2)}$, &c., being arbitrary constant quantities, the number of which is $i\,.\,(n-i)$. It is evident that these values satisfy the proposed system of differential equations;* moreover they reduce these equations to i differential equations, between the i variable quantities $x^{(n-i+1)}$, $x^{(n-i+2)}$, ... $x^{(n)}$. Their integrals will introduce i^2 new constant quantities, which being united with the $i\,.\,(n-i)$, just mentioned, will

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* (393) To illustrate this we shall take the case where $i=3$, and it will be easy to extend the demonstration to $i=4$, 5, &c. In this case the n differential equations of the order i are

$$0=\frac{d^3x^{(1)}}{dt^3}+A\,.\,\frac{d^2x^{(1)}}{dt^2}+B\,.\,\frac{dx^{(1)}}{dt}+H\,.\,x^{(1)};$$

$$0=\frac{d^3x^{(2)}}{dt^3}+A\,.\,\frac{d^2x^{(2)}}{dt^2}+B\,.\,\frac{dx^{(2)}}{dt}+H\,.\,x^{(2)};$$

$$0=\frac{d^3x^{(3)}}{dt^3}+A\,.\,\frac{d^2x^{(3)}}{dt^2}+B\,.\,\frac{dx^{(3)}}{dt}+H\,.\,x^{(3)};$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

[548a] $$0=\frac{d^3x^{(g)}}{dt^3}+A\,.\,\frac{d^2x^{(g)}}{dt^2}+B\,.\,\frac{dx^{(g)}}{dt}+H\,.\,x^{(g)};$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$0=\frac{d^3x^{(n-2)}}{dt^3}+A\,.\,\frac{d^2x^{(n-2)}}{dt^2}+B\,.\,\frac{dx^{(n-2)}}{dt}+H\,.\,x^{(n-2)};$$

$$0=\frac{d^3x^{(n-1)}}{dt^3}+A\,.\,\frac{d^2x^{(n-1)}}{dt^2}+B\,.\,\frac{dx^{(n-1)}}{dt}+H\,.\,x^{(n-1)};$$

$$0=\frac{d^3x^{(n)}}{dt^3}+A\,.\,\frac{d^2x^{(n)}}{dt^2}+B\,.\,\frac{dx^{(n)}}{dt}+H\,.\,x^{(n)};$$

In this case the expressions [548] will give the quantities $x^{(1)}$, $x^{(2)}$, $x^{(3)}$....$x^{(g)}$....$x^{(n-3)}$, in terms of $x^{(n-2)}$, $x^{(n-1)}$, $x^{(n)}$, any one of these quantities, as $x^{(g)}$, will be expressed in the
[548b] following manner, $x^{(g)}=a^{(g)}\,.\,x^{(n-2)}+b^{(g)}\,.\,x^{(n-1)}+c^{(g)}\,.\,x^{(n)}$; $a^{(g)}$, $b^{(g)}$, $c^{(g)}$, being
arbitrary constant quantities. For this value of $x^{(g)}$, gives

$$dx^{(g)}=a^{(g)}\,.\,dx^{(n-2)}+b^{(g)}\,.\,dx^{(n-1)}+c^{(g)}\,.\,dx^{(n)};$$

[548c] $$d^2x^{(g)}=a^{(g)}\,.\,d^2x^{(n-2)}+b^{(g)}\,.\,d^2x^{(n-1)}+c^{(g)}\,.\,d^2x^{(n)};$$

$$d^3x^{(g)}=a^{(g)}\,.\,d^3x^{(n-2)}+b^{(g)}\,.\,d^3x^{(n-1)}+c^{(g)}\,.\,d^3x^{(n)};$$

multiplying these values respectively by H, $\frac{B}{dt}$, $\frac{A}{dt^2}$, $\frac{1}{dt^3}$, and adding the products together, the sum will be equal to $\frac{d^3x^{(g)}}{dt^3}+A\,.\,\frac{d^2x^{(g)}}{dt^2}+B\,.\,\frac{dx^{(g)}}{dt}+H\,.\,x^{(g)}$, and this ought to be equal to nothing, if the assumed value of $x^{(g)}$ is correct. Now this sum is equal to the following expression, make the $i\,n$ constant quantities necessary to complete the integrals of the proposed differential equations.

If we apply this theorem to the equations (O) [545]; we shall find that

$$z = a\,x + b\,y, \qquad [548']$$

a and b being two arbitrary constant quantities.* This equation is that of a plane passing through the origin of the co-ordinates [19c]; therefore the [548''] orbit of m is wholly in the same plane.

The equations (O) [545] give†

$$a^{(g)} \cdot \frac{d^3 x^{(n-2)}}{d t^3} + A \cdot a^{(g)} \cdot \frac{d^2 x^{(n-2)}}{d t^2} + B \cdot a^{(g)} \cdot \frac{d x^{(n-2)}}{d t} + H \cdot a^{(g)} \cdot x^{(n-2)}$$
$$+ b^{(g)} \cdot \frac{d^3 x^{(n-1)}}{d t^3} + A \cdot b^{(g)} \cdot \frac{d^2 x^{(n-1)}}{d t^2} + B \cdot b^{(g)} \cdot \frac{d x^{(n-1)}}{d t} + H \cdot b^{(g)} \cdot x^{(n-1)} \qquad [548d]$$
$$+ c^{(g)} \cdot \frac{d^3 x^{(n)}}{d t^3} + A \cdot c^{(g)} \cdot \frac{d^2 x^{(n)}}{d t^2} + B \cdot c^{(g)} \cdot \frac{d x^{(n)}}{d t} + H \cdot c^{(g)} \cdot x^{(n)},$$

which is evidently equal to the sum of the three last of the differential equations [548a], multiplied respectively by $a^{(g)}$, $b^{(g)}$, $c^{(g)}$, and added together; therefore this sum is equal to nothing, and the assumed value of $x^{(g)}$, [548b], containing the three constant quantities $a^{(g)}$, $b^{(g)}$, $c^{(g)}$, is correct, and each of the n—3, or n—i, of the first of the proposed equations [548a], furnishes 3 or i constant quantities, making in all $(n-i) \cdot i$ quantities. Again, the values [548e] $x^{(1)}, x^{(2)} \ldots\ldots x^{(n-3)}$, [548] being substituted in the three, or i, last equations [548a], they will contain only the quantities $x^{(n-2)}$, $x^{(n-1)}$, $x^{(n)}$, and their differentials of the order 3 or i. These three, or i equations of the order 3, or i, being integrated will introduce 3×3, or i^2, new arbitrary constant quantities, adding these to the $i\,n - i^2$ quantities [548e], the sum becomes $i\,n$, which is the whole number required to complete the integrals of the n proposed equations of the order i.

* (395) The equations [545] being compared with the general form of the expression [547] give $n = 3$, $i = 2$, $A = 0$, $H = \frac{\mu}{r^3}$, r and H being symmetrical in x, y, z. In this case $n - i$ becomes 1, and the series of equations [548] will be reduced to the first $x^{(1)} = a^{(1)} \cdot x^{(2)} + b^{(1)} \cdot x^{(3)}$, and by putting $x^{(1)} = z$, $x^{(2)} = y$, $x^{(3)} = x$, $b^{(1)} = a$, $a^{(1)} = b$, it becomes as in [548'].

† (397) Multiplying the equations [545] by r^3 and taking their differentials we get [549]. The differential of $r^2 = x^2 + y^2 + z^2$, [411], gives $r\,d\,r = x\,d\,x + y\,dy + z\,d\,z$, [549']. [549$a$]

[549]
$$\left.\begin{aligned} 0 &= d\,.\left(r^3\,.\,\frac{d\,d\,x}{d\,t^2}\right)+\mu\,.\,d\,x \\ 0 &= d\,.\left(r^3\,.\,\frac{d\,d\,y}{d\,t^2}\right)+\mu\,.\,d\,y \\ 0 &= d\,.\left(r^3\,.\,\frac{d\,d\,z}{d\,t^2}\right)+\mu\,.\,d\,z \end{aligned}\right\}. \qquad (O')$$

Now by taking the differential of the equation [411]

[549']
$$r\,d\,r = x\,d\,x + y\,d\,y + z\,d\,z$$

twice in succession, we shall have

[550]
$$r\,.\,d^3r + 3\,d\,r\,.\,d\,d\,r = x\,.\,d^3x + y\,.\,d^3y + z\,.\,d^3z + 3\,.\left\{d\,x\,.\,d\,d\,x + d\,y\,.\,d\,d\,y + d\,z\,.\,d\,d\,z\right\};$$

consequently*

[551]
$$d\,.\left(r^3\,.\,\frac{d\,d\,r}{d\,t^2}\right) = r^2\,.\left\{x\,.\,\frac{d^3x}{d\,t^2} + y\,.\,\frac{d^3y}{d\,t^2} + z\,.\,\frac{d^3z}{d\,t^2}\right\} + 3\,r^2\,.\left\{d\,x\,.\,\frac{d\,d\,x}{d\,t^2} + d\,y\,.\,\frac{d\,d\,y}{d\,t^2} + d\,z\,.\,\frac{d\,d\,z}{d\,t^2}\right\}.$$

Substituting, in the second member of this equation, for d^3x, d^3y, d^3z, their values given by the equations (O') [549], and then, instead of $d\,d\,x$, $d\,d\,y$, $d\,d\,z$, their values deduced from the equations (O) [545], we shall find†

[552]
$$0 = d\,.\left(r^3\,.\,\frac{d\,d\,r}{d\,t^2}\right) + \mu\,d\,r.$$

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* (398) Multiplying the equation [550] by $\frac{r^2}{d\,t^2}$, the first member of the resulting equation becomes $r^3\,.\,\frac{d^3r}{d\,t^2} + 3\,r^2\,.\,d\,r\,.\,\frac{d\,d\,r}{d\,t^2}$, which is evidently equal to the differential of $r^3\,.\,\frac{d\,d\,r}{d\,t^2}$. The second member, without any reduction, is of the form [551].

† (399) The terms of the second member of the equation [551], depending explicitly on x, are $r^2\,.\,x\,.\,\frac{d^3x}{d\,t^2} + 3\,r^2\,.\,d\,x\,.\,\frac{d\,d\,x}{d\,t^2}$. The first of the equations [549] being developed and multiplied by $\frac{x}{r}$, gives $r^2\,.\,x\,.\,\frac{d^3x}{d\,t^2} = -3\,r\,d\,r\,.\,x\,.\,\frac{d\,d\,x}{d\,t^2} - \frac{\mu\,x\,d\,x}{r}$; substituting this in the preceding expression we get $-3\,r\,d\,r\,.\,x\,.\,\frac{d\,d\,x}{d\,t^2} - \frac{\mu\,x\,d\,x}{r} + 3\,r^2\,.\,d\,x\,.\,\frac{d\,d\,x}{d\,t^2}$, and since $\frac{d\,d\,x}{d\,t^2} = -\frac{\mu\,x}{r^3}$, [545], it becomes $3\,r\,d\,r\,.\,x\,.\,\frac{\mu\,x}{r^3} - \frac{\mu\,x\,d\,x}{r} - 3\,r^2\,.\,d\,x\,.\,\frac{\mu\,x}{r^3}$, or by reduction $\frac{-4\,\mu}{r}\,.\,x\,d\,x + \frac{3\,\mu\,d\,r}{r^2}\,.\,x^2$. In a similar manner the terms depending on y

If we compare this equation with the equations (O') [549]; we shall have, by means of the theorem above given [548], supposing $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, $\frac{dr}{dt}$, [552'] to be the variable quantities $x^{(3)}$, $x^{(4)}$, $x^{(2)}$, $x^{(1)}$; and r to be a function of the time t;*

$$dr = \lambda . dx + \gamma . dy ; \qquad [553]$$

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and z, found by changing successively x into y and z, are $\frac{-4\mu}{r} . y\,dy + \frac{3\mu dr}{r^2} . y^2$; $\frac{-4\mu}{r} . z\,dz + \frac{3\mu dr}{r^2} . z^2$. The sum of these three expressions, putting

$$x\,dx + y\,dy + z\,dz = r\,dr, \qquad x^2 + y^2 + z^2 = r^2, \quad [549a],$$

is $\frac{-4\mu}{r} . r\,dr + \frac{3\mu dr}{r^2} . r^2 = -\mu\,dr.$ This being substituted for the second member of [551] gives $d . \left(r^3 . \frac{ddr}{dt^2}\right) = -\mu\,dr$, as in [552].

* (400) Divide [552] by dt, and put $\frac{dr}{dt} = x^{(1)}$, it will become

$$0 = d . \left(r^3 . \frac{dx^{(1)}}{dt^2}\right) + \mu x^{(1)}, \qquad \text{or} \qquad 0 = r^3 . \frac{ddx^{(1)}}{dt^2} + 3r^2 . \frac{dr}{dt} . \frac{dx^{(1)}}{dt} + \mu x^{(1)}.$$

Dividing this by r^3 and putting $A = \frac{3dr}{r\,dt}$, $H = \frac{\mu}{r^3}$, we shall get

$$0 = \frac{ddx^{(1)}}{dt^2} + A . \frac{dx^{(1)}}{dt} + H . x^{(1)}.$$

Putting $\frac{dz}{dt} = x^{(2)}$, in the third of the equations (O'), divided by dt, we shall get $0 = d . \left(r^3 . \frac{dx^{(2)}}{dt^2}\right) + \mu x^{(2)}$, which being developed and divided by r^3 becomes $0 = \frac{ddx^{(2)}}{dt^2} + A . \frac{dx^{(2)}}{dt} + H . x^{(2)}$. The first of the equations (O'), developed in the same manner, putting $\frac{dx}{dt} = x^{(3)}$, gives $0 = \frac{ddx^{(3)}}{dt^2} + A . \frac{dx^{(3)}}{dt} + H . x^{(3)}$, and the second of the equations (O'), by putting $\frac{dy}{dt} = x^{(4)}$, becomes

$$0 = \frac{ddx^{(4)}}{dt^2} + A . \frac{dx^{(4)}}{dt} + H . x^{(4)} .$$

In these four equations in $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$, the terms A, H, may be considered as functions of t, being all similar to the equation [547], making $i=2$, $n=4$, and they will furnish two equations of the series [548],

$$x^{(1)} = a^{(1)} . x^{(3)} + b^{(1)} . x^{(4)}; \qquad x^{(2)} = a^{(2)} . x^{(3)} + b^{(2)} . x^{(4)}.$$

λ and γ being constant quantities; and by integration,

[554] $$r = \frac{h^2}{\mu} + \lambda x + \gamma y,$$

$\frac{h^2}{\mu}$ being a constant quantity. This equation, combined with the following [548′, 549*a*],

[555] $$z = ax + by; \qquad r^2 = x^2 + y^2 + z^2;$$

gives an equation of the second degree, in x and y, in x and z, or in y and z;* whence it follows that the three projections of the curve described by m about M, are of the second order; and as this curve is wholly in the same plane
[555′] [548″], it is itself a curve of the second order, or a conic section. It is easy to prove, from the properties of curves of this kind, that as the radius vector
[555″] r is expressed by a linear function of the co-ordinates x, y; the origin of these co-ordinates must be at the focus of the section.†

Substituting in the first, for $x^{(1)}$, $x^{(3)}$, $x^{(4)}$, their values $\frac{dr}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$, [552′], it becomes $dr = a^{(1)}.dx + b^{(1)}.dy$, which agrees with [553], putting $a^{(1)} = \lambda$, $b^{(1)} = \gamma$. The second becomes $dz = a^{(2)}.dx + b^{(2)}.dy$, and agrees with $z = ax + by$, [548′], putting $a^{(2)} = a$, $b^{(2)} = b$.

* (401) Substituting the value of r [554], in the second of the equations [555], it
[553*a*] becomes $\left(\frac{h^2}{\mu} + \lambda x + \gamma y\right)^2 = x^2 + y^2 + z^2$; and if in this we substitute either the value of z, x, or y, deduced from the first equation, [555], $z = ax + by$, we shall obtain an equation of the second degree. Thus if the value of z is substituted, the result will be an equation of the second degree in x and y, representing the projection of the curve upon the plane of x, y, as in [555′]. The projections of the curve being of the second degree, the curve itself must be of the same order. This result may also be obtained by observing that since the curve described by the body is on a plane, [548″], we may take, on that plane, the rectangular co-ordinates $x_{,,,}$, $y_{,,,}$, to denote the point corresponding to the co-ordinates x, y, z. Then by [172*a*],

[553*b*] $$x = A_0 x_{,,,} + B_0 y_{,,,}; \qquad y = A_1 x_{,,,} + B_1 y_{,,,}; \qquad z = A_2 x_{,,,} + B_2 y_{,,,};$$

because $z_{,,,} = 0$, the body being supposed to move on the plane of $x_{,,,}$, $y_{,,,}$. These values of x, y, z, being substituted in the equation [553*a*], will produce the equation of the curve described by m about M, expressed in terms of $x_{,,,}$, $y_{,,,}$, and this will evidently be of the second degree, or a conic section.

† (402) The equation [554] may be reduced to an expression of r in terms of the co-ordinates $x_{,,,}$, $y_{,,,}$, taken on the plane of the apparent path of the body m about M, by

Now the equation $r=\frac{h^2}{\mu}+\lambda x+\gamma y$ [554] gives, by means of the equations (O) [545],*

$$0=\frac{ddr}{dt^2}+\mu\,.\frac{\left(r-\frac{h^2}{\mu}\right)}{r^3}.$$ [556]

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substituting for x and y their values [553b], which give

$$r=\frac{h^2}{\mu}+(\lambda A_0+\gamma A_1)\,.\,x_{,,,}+(\lambda B_0+\gamma B_1)\,.\,y_{,,,}.$$ [554a]

In the plane of $x_{,,,}$, $y_{,,,}$, take two other rectangular co-ordinates, x'', y'', so that the axis of x'' may make, with the axis of $x_{,,,}$, an angle denoted by ε; then the co-ordinates x'', y'', and $x_{,,,}$, $y_{,,,}$, being supposed to correspond to the same point of the curve, we shall have, as in [252], by writing $x_{,,,}$, $y_{,,,}$, for x', y',

$$x_{,,,}=x''\,.\cos.\varepsilon+y''\,.\sin.\varepsilon;\qquad y_{,,,}=y''\,.\cos.\varepsilon-x''\,.\sin.\varepsilon.$$

These being substituted in the preceding expression of r, it will become

$$r=\frac{h^2}{\mu}+x''\,.\left\{(\lambda A_0+\gamma A_1)\,.\cos.\varepsilon-(\lambda B_0+\gamma B_1)\,.\sin.\varepsilon\right\}$$
$$+y''\,.\left\{(\lambda A_0+\gamma A_1)\,.\sin.\varepsilon+(\lambda B_0+\gamma B_1)\,.\cos.\varepsilon\right\}.$$

Now as ε is arbitrary, we can take it so that the coefficient of y'' may be nothing; this value of ε being substituted in the coefficient of x'', let its result be $-e$, and we shall have $r=\frac{h^2}{\mu}-ex''=e\,.\left\{\frac{h^2}{\mu e}-x''\right\}$. Which is the noted theorem used in page 243 to demonstrate the properties of the conic sections. For by referring to the figure in that page, and putting $SD=\frac{h^2}{\mu e}$, $SF=x''$, $SP=r$, the preceding equation will become [554b] $SP=e\,.(SD-SF)=e\,.PE$, being the same as in [378a], where the origin of the co-ordinates is taken at the focus S.

* (403) The second differential of [554] divided by dt^2, gives

$$\frac{ddr}{dt^2}=\lambda\,.\frac{ddx}{dt^2}+\gamma\,.\frac{ddy}{dt^2},$$

and from [545] we get $\frac{ddx}{dt^2}=-\frac{\mu x}{r^3}$; $\frac{ddy}{dt^2}=-\frac{\mu y}{r^3}$; hence $\frac{ddr}{dt^2}=-\frac{\mu}{r^3}\,.(\lambda x+\gamma y)$; now from [554] we get $(\lambda x+\gamma y)=r-\frac{h^2}{\mu}$, whence $\frac{ddr}{dt^2}=-\frac{\mu}{r^3}\,.\left(r-\frac{h^2}{\mu}\right)$, as in [556]. This being multiplied by $2\,dr$ becomes $0=2\,.\frac{dr\,.ddr}{dt^2}+2\,\mu\,.\frac{dr}{r^2}-\frac{2h^2dr}{r^3}$, whose integral is $\frac{dr^2}{dt^2}-\frac{2\mu}{r}+\frac{h^2}{r^2}+\frac{\mu}{a'}=0$, and this multiplied by r^2 gives

$$r^2\,.\frac{dr^2}{dt^2}-2\,\mu r+\frac{\mu r^2}{a'}+h^2=0,\ [557],$$

whence we easily obtain dt, [558].

Multiplying this equation by dr, and taking the integrals, we shall have

[557] $$r^2 \cdot \frac{dr^2}{dt^2} - 2\mu r + \frac{\mu r^2}{a'} + h^2 = 0\,;$$

a' being an arbitrary constant quantity. Hence we deduce

[558] $$dt = \frac{r\,dr}{\sqrt{2r - \frac{r^2}{a'} - \frac{h^2}{\mu}} \cdot \sqrt{\mu}}\,;$$

[558] this equation will give r in a function of t; and as x, y, z, are given, by what precedes [554, 555], in functions of r; we shall have the co-ordinates of m, in functions of the time.

18. We may obtain these equations by the following method, which has the advantage of giving the arbitrary constant quantities, in functions of the co-ordinates x, y, z, and of their first differentials; which will be useful in the course of this work.

Suppose that $V = \text{constant}$ is an integral of the first order of the equations

[558''] (O) [545], V being a function of $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; and if we put

[558'''] $$x' = \frac{dx}{dt}, \qquad y' = \frac{dy}{dt}, \qquad z' = \frac{dz}{dt},$$

the equation $V = \text{constant}$ will give, by taking its differential,

[559] $$0 = \left(\frac{dV}{dx}\right) \cdot \frac{dx}{dt} + \left(\frac{dV}{dy}\right) \cdot \frac{dy}{dt} + \left(\frac{dV}{dz}\right) \cdot \frac{dz}{dt} + \left(\frac{dV}{dx'}\right) \cdot \frac{dx'}{dt} + \left(\frac{dV}{dy'}\right) \cdot \frac{dy'}{dt} + \left(\frac{dV}{dz'}\right) \cdot \frac{dz'}{dt}\,;$$

but the equations (O) [545] give

[560] $$\frac{dx'}{dt} = -\frac{\mu x}{r^3}\,; \qquad \frac{dy'}{dt} = -\frac{\mu y}{r^3}\,; \qquad \frac{dz'}{dt} = -\frac{\mu z}{r^3}\,;$$

therefore we shall have this identical equation of partial differentials,*

[561] $$0 = x' \cdot \left(\frac{dV}{dx}\right) + y' \cdot \left(\frac{dV}{dy}\right) + z' \cdot \left(\frac{dV}{dz}\right) - \frac{\mu}{r^3} \cdot \left\{ x \cdot \left(\frac{dV}{dx'}\right) + y \cdot \left(\frac{dV}{dy'}\right) + z \cdot \left(\frac{dV}{dz'}\right) \right\}. \quad (I)$$

[560a] * (404) The differentials of [558'''], divided by dt, are $\frac{dx'}{dt} = \frac{ddx}{dt^2}$, $\frac{dy'}{dt} = \frac{ddy}{dt^2}$, $\frac{dz'}{dt} = \frac{ddz}{dt^2}$, hence the equations [545] become as in [560]. Substituting these in the differential equation [559], and putting for $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, their values x', y', z', [558'''], we shall obtain [561], which is a differential equation of the first order, without a constant quantity, and is evidently identical.

It is evident that every function of x, y, z, x', y', z', which substituted for V in this equation renders it identically nothing, becomes, by putting it equal [561'] to an arbitrary constant quantity, an integral of the first order of the equations (O) [545].

Suppose

$$V = U + U' + U'' + \&c. ;$$ [562]

U being a function of the three variable quantities x, y, z ; U' a function of the six variable quantities x, y, z, x', y', z', but of the first order* relative to [562'] x', y', z' ; U'' being a function of the same quantities, but of the second order [562''] with respect to x', y', z' ; and so on for the rest. Substitute this value in the equation (I) [561], and compare separately, *First*, the terms without x', y', z' ; *Second*, those containing the first power of these quantities ; *Third*, those containing their squares and products ; and so on for others ; we shall have

$$\left.\begin{array}{l} 0 = x.\left(\frac{dU'}{dx'}\right) + y.\left(\frac{dU'}{dy'}\right) + z.\left(\frac{dU'}{dz'}\right) \\ x'.\left(\frac{dU}{dx}\right) + y'.\left(\frac{dU}{dy}\right) + z'.\left(\frac{dU}{dz}\right) = \frac{\mu}{r^3}.\left\{x.\left(\frac{dU''}{dx'}\right) + y.\left(\frac{dU''}{dy'}\right) + z.\left(\frac{dU''}{dz'}\right)\right\} \\ x'.\left(\frac{dU'}{dx}\right) + y'.\left(\frac{dU'}{dy}\right) + z'.\left(\frac{dU'}{dz}\right) = \frac{\mu}{r^3}.\left\{x.\left(\frac{dU'''}{dx'}\right) + y.\left(\frac{dU'''}{dy'}\right) + z.\left(\frac{dU'''}{dz'}\right)\right\} \\ x'.\left(\frac{dU''}{dx}\right) + y'.\left(\frac{dU''}{dy}\right) + z'.\left(\frac{dU''}{dz}\right) = \frac{\mu}{r^3}.\left\{x.\left(\frac{dU^{\text{iv}}}{dx'}\right) + y.\left(\frac{dU^{\text{iv}}}{dy'}\right) + z.\left(\frac{dU^{\text{iv}}}{dz'}\right)\right\} \end{array}\right\} ; (I') \qquad [563]$$

&c.

The integral of the first of these equations is, by the theory of partial differentials,†

$$U' = \text{function}\,\{xy' - yx',\;\; xz' - zx',\;\; yz' - zy',\;\; x, y, z,\} ;$$ [564]

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* (405) By the first order is to be understood terms of the first degree in x', y', z', excluding their powers and products. By the second order, terms of the second degree in x', y', z', excluding their powers and products of the third degree, or above, &c. ; and since by substituting the value of $V = U + U' + \&c.$ in the equation [561], it ought to be identically nothing, the terms of the first, second, third, &c. orders, must be separately equal to nothing, which will give the equations [563].

† (406) In the equation [563]

$$0 = x.\left(\frac{dU'}{dx'}\right) + y.\left(\frac{dU'}{dy'}\right) + z.\left(\frac{dU'}{dz'}\right),$$ [564a]

as the value of U' ought to be linear in x', y', z', [562'], we shall suppose it to be of this form,

[565] $$U' = A\,.\,(xy' - yx') + B\,.\,(xz' - zx') + C\,.\,(yz' - zy')\,;$$

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the partial differentials relative to x', y', z', exist, but those relative to x, y, z, do not occur, so that it is in the same situation as if U' was a function of only the three *variable* quantities x', y', z'. In this supposition the general value of dU' will be

$$dU' = \left(\frac{dU'}{dx'}\right).dx' + \left(\frac{dU'}{dy'}\right).dy' + \left(\frac{dU'}{dz'}\right).dz'\,;$$

and if to this we add the preceding equation, multiplied by $\frac{-dx'}{x}$, we shall eliminate $\left(\frac{dU'}{dx'}\right)$ and shall find $dU' = \left(\frac{dU'}{dy'}\right).\left(\frac{x\,dy' - y\,dx'}{x}\right) + \left(\frac{dU'}{dz'}\right).\left(\frac{x\,dz' - z\,dx'}{x}\right)$.

[564b] If we now put $xy' - yx' = p$, $xz' - zx' = q$, $yz' - zy' = s$, we shall get, by taking their differentials, $x\,dy' - y\,dx' = dp$, $x\,dz' - z\,dx' = dq$; therefore $dU' = \frac{1}{x}.\left(\frac{dU'}{dy'}\right).dp + \frac{1}{x}.\left(\frac{dU'}{dz'}\right).dq$; in which the second member must, like the

[564c] first, be an exact differential. This condition is evidently satisfied by putting $U' = \varphi.(p, q)$, φ being a function of p, q, and we may include in it the terms x, y, z, considered as constant, putting $U' = \varphi.(p, q, x, y, z)$. But $\frac{y}{x}.q - \frac{z}{x}.p$, is a function of the five quantities p, q, x, y, z, included under the sign φ, we might therefore introduce this quantity also, or its equal, $\frac{y}{x}.(xz' - zx') - \frac{z}{x}.(xy' - yx') = yz' - zy' = s$, [564b]. Hence a more

[564d] symmetrical expression is $U' = \varphi.(p, q, s, x, y, z)$, which is the same as that assumed in [564]. To prove its correctness *a posteriori*, we may substitute it in the proposed equation [564a], writing for brevity, φ instead of $\varphi.(p, q, s, x, y, z)$, we shall have

$$x.\left(\frac{dU'}{dx'}\right) = x.\left\{\left(\frac{d\varphi}{dp}\right).\left(\frac{dp}{dx'}\right) + \left(\frac{d\varphi}{dq}\right).\left(\frac{dq}{dx'}\right) + \left(\frac{d\varphi}{ds}\right).\left(\frac{ds}{dx'}\right)\right\},$$

and since by [564b], $\left(\frac{dp}{dx'}\right) = -y$, $\left(\frac{dq}{dx'}\right) = -z$, $\left(\frac{ds}{dx'}\right) = 0$, it becomes $x.\left(\frac{dU'}{dx'}\right) = -xy.\left(\frac{d\varphi}{dp}\right) - xz.\left(\frac{d\varphi}{dq}\right)$. In a similar manner

$$y.\left(\frac{dU'}{dy'}\right) = xy.\left(\frac{d\varphi}{dp}\right) - yz.\left(\frac{d\varphi}{ds}\right), \qquad z.\left(\frac{dU'}{dz'}\right) = xz.\left(\frac{d\varphi}{dq}\right) + yz.\left(\frac{d\varphi}{ds}\right).$$

The sum of all these is

$$x.\left(\frac{dU'}{dx'}\right) + y.\left(\frac{dU'}{dy'}\right) + z.\left(\frac{dU'}{dz'}\right)$$

$$= \left(\frac{d\varphi}{dp}\right).\{-xy + xy\} + \left(\frac{d\varphi}{dq}\right).\{-xz + xz\} + \left(\frac{d\varphi}{ds}\right).\{-yz + yz\},$$

and as the terms of the second member mutually destroy each other, it becomes as in [564a]; therefore the assumed value of U' [564], must satisfy the proposed equation [564a].

A, *B* and *C* being constant quantities. Suppose the value of V to terminate at U'', making U''', U^{iv}, &c., nothing; the third of the equations (I') [563] will become

$$0=x'.\left(\frac{d\,U'}{d\,x}\right)+y'.\left(\frac{d\,U'}{d\,y}\right)+z'.\left(\frac{d\,U'}{d\,z}\right). \qquad [566]$$

The preceding value of U' satisfies also this equation.* The fourth of the equations (I') [563] will become

$$0=x'.\left(\frac{dU''}{d\,x}\right)+y'.\left(\frac{dU''}{d\,y}\right)+z'.\left(\frac{dU''}{d\,z}\right); \qquad [567]$$

the integral of which is†

$$U''=\text{function}\,\{x\,y'-y\,x',\quad x\,z'-z\,x',\quad y\,z'-z\,y',\quad x',\,y',\,z'\}. \qquad [568]$$

This function ought to satisfy the second of the equations (I') [563], and the first member of this equation multiplied by $d\,t$ is evidently equal to $d\,U$;‡ the second member ought therefore to be the exact differential of a

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* (407) From [565] we get $\left(\frac{d\,U'}{d\,x}\right)=A\,y'+B\,z'$, $\left(\frac{d\,U'}{d\,y}\right)=-A\,x'+C\,z'$, $\left(\frac{d\,U'}{d\,z}\right)=-B\,x'-C\,y'$. Hence

$$x'.\left(\frac{d\,U'}{d\,x}\right)+y'.\left(\frac{d\,U'}{d\,y}\right)+z'.\left(\frac{d\,U'}{d\,z}\right)=A.(x'y'-x'y')+B.(x'z'-x'z')+C.(y'z'-y'z'),$$

in which the terms of the second member mutually destroy each other, and it becomes as in [566].

† (408) The integral of the equation [567] may be easily deduced from that of [564*a*], since the former may be derived from the latter by changing U', x, y, z, x', y', z', into U'', x', y', z', x, y, z, respectively.

Now these changes being made in p, q, s, [564*b*], they would become respectively $-p$, $-q$, $-s$, and the expression of U', [564*d*] would be changed into

$$U''=\varphi.(-p,-q,-s,x',y',z').$$

or by neglecting the signs of p, q, s, $U''=\text{function}\,(p,q,s,x',y',z')$, as in [568].

‡ (409) The first member of this equation being multiplied by $d\,t$, gives

$$x'.d\,t.\left(\frac{dU}{d\,x}\right)+y'd.t.\left(\frac{dU}{dy}\right)+z'd\,t.\left(\frac{dU}{d\,z}\right),$$

and by substituting $x'd\,t=d\,x$, $y'd\,t=d\,y$, $z'd\,t=d\,z$, [558'''], it becomes $d\,x.\left(\frac{dU}{d\,x}\right)+d\,y.\left(\frac{dU}{d\,y}\right)+d\,z.\left(\frac{dU}{d\,z}\right)$, which is evidently equal to $d\,U$, because by function of x, y, z. Now it is easy to see that we may satisfy this condition, and at the same time conform to the nature of the function U'', and to the supposition that this function is of the second order in x', y', z'; by making*

[569] $$U''=(Dy'-Ex').(xy'-yx')+(Dz'-Fx').(xz'-zx')$$
$$+(Ez'-Fy').(yz'-zy')+G.(x'^2+y'^2+z'^2);$$

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hypothesis [562'], U does not contain x', y', z'. Therefore we shall have

[568a] $$dU=\frac{\mu dt}{r^3}.\left\{x.\left(\frac{dU''}{dx'}\right)+y.\left(\frac{dU''}{dy'}\right)+z.\left(\frac{dU''}{dz'}\right)\right\},$$

and the second member, like the first, must be an exact differential in x, y, z.

* (410) This assumed value of U'', [569], is of the form required by [568]. It is also of the second degree in x', y', z', as it ought to be by [562']. Moreover it is necessary that it should render the value of dU, [568a], integrable. Now as D, E, F, G, are supposed to be arbitrary constant quantities, wholly independent of each other, it will follow that the terms multiplied by each of these coefficients must be separately integrable. We shall therefore examine each of them in succession. The term of U'', [569], depending on D, is $D.\{xy'^2-yx'y'+xz'^2-zx'z'\}$. If this be substituted for U'', in [568a], and then reduced by means of [549a], it will become

$$dU=\frac{D\mu dt}{r^3}.\left\{x.(-yy'-zz')+y.(2xy'-yx')+z.(2xz'-zx')\right\}$$
$$=\frac{D\mu dt}{r^3}.\left\{x.(yy'+zz')-x'.(y^2+z^2)\right\}$$
[569a] $$=\frac{D\mu}{r^3}.\left\{x.(ydy+zdz)-dx.(y^2+z^2)\right\}$$
$$=\frac{D\mu}{r^3}.\left\{x.(rdr-xdx)-dx.(r^2-x^2)\right\}$$
$$=\frac{D\mu}{r^3}.\left\{xrdr-r^2dx\right\}=-D\mu.\frac{(rdx-xdr)}{r^2}=-D\mu.d.\left(\frac{x}{r}\right).$$

The terms depending on E, F, may be found in the same manner, or much more simply, by the consideration that the function U'', [569] is symmetrical, as it respects the *three* series of quantities $D, E, F,$ $x, y, z,$ x', y', z'. So that the expression [569] will not be altered by changing each of these quantities into the following one of the same series, commencing each series again when we arrive at the last terms F, z, and z'; this would not affect the value of $r=\sqrt{x^2+y^2+z^2}$, or the coefficient of G, [569], If we make these changes in the term $-D\mu.d.\left(\frac{x}{r}\right)$, [569a], we shall obtain the terms depending on E and F, which

[569b] will be respectively $-E\mu.d.\left(\frac{y}{r}\right)$, $-F\mu.d.\left(\frac{z}{r}\right)$. Lastly the term depending on

D, E, F, G, being arbitrary constant quantities; and then r being equal to $\sqrt{x^2+y^2+z^2}$, we shall find

$$U = -\frac{\mu}{r}.\{Dx+Ey+Fz+2G\}; \qquad [570]$$

we shall thus have the values of U, U', U''; and the equation $V=\text{constant}$ [558″, 562], will become [570, 565, 569]

$$\begin{aligned}\text{constant} = -\frac{\mu}{r}.\{Dx+Ey+Fz+2G\} &+ (A+Dy'-Ex').(xy'-yx') \\ &+ (B+Dz'-Fx').(xz'-zx') + (C+Ez'-Fy').(yz'-zy') \\ &+ G.(x'^2+y'^2+z'^2).\end{aligned} \qquad [571]$$

This equation satisfies the equation (I) [561], consequently also the differential equations (O) [545, 561′], whatever be the arbitrary quantities A, B, C, D, E, F, G. If we suppose, *First*, that all except A are nothing; *Second*, that all except B are nothing; *Third*, that all except C are nothing; &c., and then resubstitute $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, for x', y', z', we shall obtain these integrals: [571′]

G, [569], being taken for U'', and substituted in dU, [568a], will produce the quantity

$$\begin{aligned}G.\frac{\mu\,dt}{r^3}.\left\{2xx'+2yy'+2zz'\right\} &= G.\frac{\mu}{r^3}.\left\{2x\,dx+2y\,dy+2z\,dz\right\} \\ = G.\frac{\mu}{r^3}.2r\,dr = 2G\mu.\frac{dr}{r^2} &= -2G\mu.d.\left(\frac{1}{r}\right).\end{aligned} \qquad [569c]$$

Connecting together all these terms [569a, b, c], we shall get the complete value of dU,

$$dU = -D\mu.d.\left(\frac{x}{r}\right) - E\mu.d.\left(\frac{y}{r}\right) - F\mu.d.\left(\frac{z}{r}\right) - 2G\mu.d.\left(\frac{1}{r}\right),$$

the integral of which gives U, as in [570].

In order to abridge the demonstration, it has been supposed that the form of the function U'' is given as in [569]. If this form were unknown, it might be investigated, by the consideration that U'', [562′] is of the second degree in p, q, s, x', y', z'; and the most general form of a function of this kind, connected with constant coefficients a, a', a'', &c. is to be substituted for U'', in [568a], and the constant quantities a, a', &c. are to be taken, so as to make the second member of this equation to be, like the first, a complete differential. In this way we might obtain the function [569], connected with a few other terms, which were neglected, not being of any use in the subsequent calculations. It was not thought necessary to explain this calculation more fully; it may however be proper to remark that in making these substitutions, we may consider p, q, s, as constant, since the terms arising in [568a], from dp, dq, ds, mutually destroy each other.

Important integrals corresponding to the relative motion of one body about another in a conic section.

[572]

$$\left.\begin{aligned}
&c=\frac{x\,dy-y\,dx}{dt}; \quad c'=\frac{x\,dz-z\,dx}{dt}; \quad c''=\frac{y\,dz-z\,dy}{dt};\\
&0=f+x.\left\{\frac{\mu}{r}-\left(\frac{dy^2+dz^2}{dt^2}\right)\right\}+\frac{y\,dy.dx}{dt^2}+\frac{z\,dz.dx}{dt^2};\\
&0=f'+y.\left\{\frac{\mu}{r}-\left(\frac{dx^2+dz^2}{dt^2}\right)\right\}+\frac{x\,dx.dy}{dt^2}+\frac{z\,dz.dy}{dt^2};\\
&0=f''+z.\left\{\frac{\mu}{r}-\left(\frac{dx^2+dy^2}{dt^2}\right)\right\}+\frac{x\,dx.dz}{dt^2}+\frac{y\,dy.dz}{dt^2};\\
&0=\frac{\mu}{a}-\frac{2\mu}{r}+\frac{dx^2+dy^2+dz^2}{dt^2};
\end{aligned}\right\}\quad (P)$$

c, c', c'', f, f', f'', and a being arbitrary constant quantities.

The differential equations (*O*) [545], can have but six distinct integrals of the first order,* and if from these we eliminate the differentials dx, dy, dz, we shall obtain the three variable quantities x, y, z, in functions of the time t; therefore at least one of the seven preceding integrals is comprised in the
[572'] six others. In fact it is easy to perceive *a priori* that two of these integrals ought to be contained in the remaining five. For these integrals do not contain the time t explicitly, but merely its differential dt, therefore they cannot give the variable quantities x, y, z, in functions of the time,† consequently they are not sufficient to determine completely the motion of m about M. We shall now examine in what manner these integrals are equivalent only to five distinct integrals.

If we multiply the fourth of the equations (*P*) [572] by $\frac{z\,dy-y\,dz}{dt}$, and add to it the fifth multiplied by $\frac{x\,dz-z\,dx}{dt}$; we shall have*

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* (411) A differential equation of the second order, has generally in its complete finite integral only two distinct arbitrary constant quantities, and if between this integral and its first differential, we eliminate first the one, then the other of these constant quantities, we
[571a] shall obtain two equations of the first order, each containing a different constant quantity. In this way the three equations [545] may have six distinct integrals of the first order, containing x, y, z, dx, dy, dz, and by eliminating dx, dy, dz, there would remain three equations containing x, y, z, in functions of t, and of the six arbitrary constant quantities.

† (412) All the equations [572] contain dt, but none of them contain t explicitly, therefore t cannot be obtained from them without another integration.

$$0=f.\left(\frac{z\,dy-y\,dz}{dt}\right)+f'.\left(\frac{x\,dz-z\,dx}{dt}\right)+z.\left(\frac{x\,dy-y\,dx}{dt}\right).\left\{\frac{\mu}{r}-\left(\frac{dx^2+dy^2}{dt^2}\right)\right\}$$
$$+\left(\frac{x\,dy-y\,dx}{dt}\right).\left\{\frac{x\,dx\,.\,dz}{dt^2}+\frac{y\,dy\,.\,dz}{dt^2}\right\}. \qquad [573]$$

By substituting for $\frac{x\,dy-y\,dx}{dt}$, $\frac{x\,dz-z\,dx}{dt}$, $\frac{y\,dz-z\,dy}{dt}$, their values given by the three first of the equations (P) [572]; we shall have

$$0=\frac{f'c'-fc''}{c}+z.\left\{\frac{\mu}{r}-\left(\frac{dx^2+dy^2}{dt^2}\right)\right\}+\frac{x\,dx.dz}{dt}+\frac{y\,dy.dz}{dt^2}; \qquad [574]$$

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* (413) In performing these reductions it will be convenient to put the fourth, fifth and sixth of the equations [572], under a different form, by substituting $d\omega^2=dx^2+dy^2+dz^2$, $r\,dr=x\,dx+y\,dy+z\,dz$, [549']. For by this means we shall have in the fourth, $dy^2+dz^2=d\omega^2-dx^2$ and $y\,dy.dx+z\,dz.dx=dx.(y\,dy+z\,dz)=dx.(r\,dr-x\,dx)$, [571*b*] hence $0=f+x.\left\{\frac{\mu}{r}-\left(\frac{d\omega^2-dx^2}{dt^2}\right)\right\}+\frac{dx.(r\,dr-x\,dx)}{dt^2}$, or by reduction

$$0=f+x.\left\{\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right\}+\frac{r\,dr.dx}{dt^2}. \qquad [572a]$$

And by changing successively x into y, z, and f into f', f'', we shall obtain the fifth and sixth equations [572],

$$0=f'+y.\left\{\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right\}+\frac{r\,dr.dy}{dt^2}; \qquad 0=f''+z.\left\{\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right\}+\frac{r\,dr.dz}{dt^2}. \qquad [572b]$$

Multiplying the fourth by $\frac{z\,dy-y\,dz}{dt}$, the fifth by $\frac{x\,dz-z\,dx}{dt}$, and adding the products we shall get

$$0=f.\left(\frac{z\,dy-y\,dz}{dt}\right)+f'.\left(\frac{x\,dz-z\,dx}{dt}\right)+\left\{\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right\}.\left\{\frac{x.(z\,dy-y\,dz)+y.(x\,dz-z\,dx)}{dt}\right\}$$
$$+\frac{r\,dr}{dt^2}.\left\{dx.\left(\frac{z\,dy-y\,dz}{dt}\right)+dy.\left(\frac{x\,dz-z\,dx}{dt}\right)\right\};$$

but $x.(z\,dy-y\,dz)+y.(x\,dz-z\,dx)=z.(x\,dy-y\,dx)$, also $dx.(z\,dy-y\,dz)+dy.(x\,dz-z\,dx)=dz.(x\,dy-y\,dx)$, hence

$$0=f.\left(\frac{z\,dy-y\,dz}{dt}\right)+f'.\left(\frac{x\,dz-z\,dx}{dt}\right)+\left\{\left(\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right).z+\frac{r\,dr}{dt}.\frac{dz}{dt}\right\}.\left(\frac{x\,dy-y\,dx}{dt}\right),$$

resubstituting for $d\omega^2$ its value $dx^2+dy^2+dz^2$, and $r\,dr=x\,dx+y\,dy+z\,dz$, [571*b*], and neglecting the terms multiplied by $z\,dz^2$, which mutually destroy each other, we obtain the equation [573]. Substituting in this the values c, c', c'', [572], and dividing by c we get [574].

which becomes the same as the sixth of the equations (*P*) [572], by making $f''=\frac{f'c'-fc''}{c}$, or

[574′] $$0=fc''-f'c'+f''c.$$

Hence the sixth of the equations (*P*) [572], results from the first five, and the six arbitrary quantities c, c', c'', f, f', f'', are connected by the preceding equation [574′].

If we take the squares of the values of f, f', f'', given by the equations (*P*) [572], and add them together, putting for brevity

[574″] $$f^2+f'^2+f''^2=l^2;$$

we shall have*

[575] $$l^2-\mu^2=\left\{r^2.\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)-\left(\frac{rdr}{dt}\right)^2\right\}.\left\{\frac{dx^2+dy^2+dz^2}{dt^2}-\frac{2\mu}{r}\right\};$$

but if we add the squares of the values of c, c', c'', given by the same equations [572], putting

[575′] $$c^2+c'^2+c''^2=h^2;$$

we shall have†

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* (414) Putting for brevity $\frac{\mu}{r}-\frac{d\omega^2}{dt^2}=A$, $\frac{rdr}{dt^2}=B$, we shall have by [572*a*, *b*], $-f=Ax+Bdx$; $-f'=Ay+Bdy$; $-f''=Az+Bdz$. The squares of these added together, using l^2, [574″], make
$l^2=A^2.(x^2+y^2+z^2)+2AB.(xdx+ydy+zdz)+B^2.(dx^2+dy^2+dz^2)$, or by [571*b*], $l^2=A^2.r^2+2AB.rdr+B^2.d\omega^2$. Substituting the values of A, B, it becomes $l^2=r^2.\left(\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right)^2+2.\left(\frac{rdr}{dt}\right)^2.\left(\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right)+\left(\frac{rdr.d\omega}{dt^2}\right)^2$. The first term of the second member being developed becomes $\mu^2-2\mu r.\frac{d\omega^2}{dt^2}+r^2.\frac{d\omega^4}{dt^4}$ Substituting this, and connecting together the terms depending on the different powers of μ,

$$l^2-\mu^2=\left(\frac{r^2d\omega^2}{dt^2}-\frac{r^2dr^2}{dt^2}\right).\frac{d\omega^2}{dt^2}-\frac{2\mu}{r}.\left(r^2.\frac{d\omega^2}{dt^2}-\frac{r^2dr^2}{dt^2}\right)$$
$$=\left\{\frac{r^2d\omega^2}{dt^2}-\left(\frac{rdr}{dt}\right)^2\right\}.\left\{\frac{d\omega^2}{dt^2}-\frac{2\mu}{r}\right\},$$

and by resubstituting $d\omega^2=dx^2+dy^2+dz^2$, [571*b*], it becomes as in [575].

† (415) The values of c, c', c'', [572], multiplied by dt, and squared, give

$$c^2.dt^2=x^2.dy^2-2xy.dxdy+y^2.dx^2;\qquad c'^2.dt^2=x^2.dz^2-2xz.dxdz+z^2.dx^2;$$
$$c''^2.dt^2=y^2.dz^2-2yz.dydz+z^2.dy^2;$$

$$r^2.\left(\frac{d\,x^2+d\,y^2+d\,z^2}{d\,t^2}\right)-\left(\frac{r\,d\,r}{d\,t}\right)^2=h^2\,;$$ [576]

therefore the preceding equation will become

$$0=\frac{d\,x^2+d\,y^2+d\,z^2}{d\,t^2}-\frac{2\,\mu}{r}+\frac{\mu^2-l^2}{h^2}.$$ [577]

By comparing this with the last of the equations (P) [572], we shall obtain this equation of condition,

$$\frac{\mu^2-l^2}{h^2}=\frac{\mu}{a}.$$ [578]

The last of the equations (P) [572] is therefore included in the six others, which are equivalent to five distinct integrals only, the seven arbitrary quantities c, c', c'', f, f', f'', a, being connected by the two preceding equations of condition [574', 578]. Hence it follows, that we shall find the most general expression of V, which satisfies the equation (I) [561], by [578'] taking for this expression, an arbitrary function of the values of c, c', c'', f, and f', given by the five first of the equations (P) [572].*

19. Although these integrals are not sufficient to compute x, y, z, in functions of the time, they determine however the nature of the curve

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and if to the sum of these we add $(r\,d\,r)^2=(x\,d\,x+y\,d\,y+z\,d\,z)^2$, or by developing $(r\,d\,r)^2=x^2\,d\,x^2+y^2\,d\,y^2+z^2\,d\,z^2+2\,x\,y\,.\,d\,x\,d\,y+2\,x\,z\,.\,d\,x\,d\,z+2\,y\,z\,.\,d\,y\,d\,z,$ we shall have

$$(c^2+c'^2+c''^2)\,.\,d\,t^2+(r\,d\,r)^2=(x^2+y^2+z^2)\,.\,d\,x^2+(x^2+y^2+z^2)\,.\,d\,y^2+(x^2+y^2+z^2)\,.\,d\,z^2,$$

and by substituting $c^2+c'^2+c''^2=h^2$; $x^2+y^2+z^2=r^2$, it becomes

$$h^2\,.\,d\,t^2+(r\,d\,r)^2=r^2\,.\,(d\,x^2+d\,y^2+d\,z^2).$$

Dividing this by $d\,t^2$, we get [576]. Substituting this value of h^2 in [575], divided by h^2, we shall get [577].

* (416) Using for brevity, the letters c, c', c'', f, f', to represent the quantities to which they are respectively equal in the equations [572]; then it is stated, [578'], that the most general value of V will be expressed by $V=\varphi\,.\,(c,c',c'',f,f')$. To find whether this will satisfy the equation [561], we shall suppose

$$d\,V=\varphi'\,.\,d\,c+\varphi''\,.\,d\,c'+\varphi'''\,.\,d\,c''+\varphi^{\text{iv}}\,.\,d\,f+\varphi^{\text{v}}\,.\,d\,f',$$ [578a]

denoting, as usual, by φ', φ'', &c., the coefficients of $d\,c$, $d\,c'$, &c. in the general differential of V. If we put successively the values of c, c', c'', f, f', for V in the second member of described by m about M. For if we multiply the first of the equations (P) [572] by z, the second by $-y$, and the third by x; we shall have, by adding these products,

[579] $$0=cz-c'y+c''x\,;$$

which is the equation of a plane* whose position depends on the constant quantities c, c', c''.

If we multiply the fourth of the equations (P) [572] by x, the fifth by y, the sixth by z, we shall have, by adding these products,†

[580] $$0=fx+f'y+f''z+\mu r-r^2\,.\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)+\frac{r^2\,dr^2}{dt^2}\,;$$

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the equation [561], and represent the resulting quantities by C, C', C'', F, F', we shall have

[578b] $$C=x'\,.\left(\frac{dc}{dx}\right)+y'\,.\left(\frac{dc}{dy}\right)+z'\,.\left(\frac{dc}{dz}\right)-\frac{\mu}{r^3}\,.\left\{x\,.\left(\frac{dc}{dx'}\right)+y\,.\left(\frac{dc}{dy'}\right)+z\,.\left(\frac{dc}{dz'}\right)\right\};$$
$$C'=x'\,.\left(\frac{dc'}{dx}\right)+y'\,.\left(\frac{dc'}{dy}\right)+z'\,.\left(\frac{dc'}{dz}\right)-\frac{\mu}{r^3}\,.\left\{x\,.\left(\frac{dc'}{dx'}\right)+y\,.\left(\frac{dc'}{dy'}\right)+z\,.\left(\frac{dc'}{dz'}\right)\right\};$$
&c.

[578c] Then we shall have [571', 572], $0=C$, $0=C'$, $0=C''$, $0=F$, $0=F'$. Now if we substitute the above assumed general value of $V=\varphi\,.\,(c,c',c'',f,f')$, in the second member of [561] it will be $\varphi'\,.\,C+\varphi''\,.\,C'+\varphi'''\,.\,C''+\varphi^{\text{iv}}\,.\,F+\varphi^{\text{v}}\,.\,F'$, and this becomes nothing, by means of the equations [578c]; therefore $V=\varphi\,.\,(c,c',c'',f,f')$, satisfies the equation [561], and as it contains an arbitrary function φ, it will be the complete integral.

* (417) The first member of this sum is $cz-c'y+c''x$, its second member, omitting for brevity the divisor dt, is

$$z\,.\,(x\,dy-y\,dx)+y\,.\,(z\,dx-x\,dz)+x\,.\,(y\,dz-z\,dy),$$

the terms of which mutually destroy each other, therefore we shall have $0=cz-c'y+c''x$, as in [579]. If we put $c'=Bc$, $c''=-Ac$, and divide by c, we shall get [579a] $z=Ax+By$, which is the equation of a plane, [19c] passing through the origin of the co-ordinates.

† (417a) The equations being put under the same forms as in [572a, b], and multiplied respectively by x, y, z, the sum of the products will be

$$0=fx+f'y+f''z+\left(\frac{\mu}{r}-\frac{d\omega^2}{dt^2}\right).\,(x^2+y^2+z^2)+\frac{r\,dr}{dt^2}\,.\,(x\,dx+y\,dy+z\,dz).$$

but by the preceding article [576]

$$h^2 = r^2 \cdot \left(\frac{d x^2 + d y^2 + d z^2}{d t^2}\right) - \frac{r^2 d r^2}{d t^2} \, ; \qquad [581]$$

therefore

$$0 = \mu\, r - h^2 + f x + f' y + f'' z. \qquad [582]$$

This equation being combined with the following [579, 555],

$$0 = c'' x - c' y + c z \, ; \qquad r^2 = x^2 + y^2 + z^2 \, ; \qquad [582']$$

gives the equation of the conic sections,* in which the origin of r is in the focus. The planets and comets describe therefore, about the sun, nearly conic sections, the sun being placed in one of the foci; and the motion of any planet is such that the areas described by the radius vector are proportional to the times of description. For if we put $d v$ for the angle included between the infinitely near radii r and $r + d r$, we shall have† [582'']

$$d x^2 + d y^2 + d z^2 = r^2 d v^2 + d r^2 \, ; \qquad [583]$$

Which by substituting [549*a*], $rr = x x + y y + z z$, and $r d r = x d x + y d y + z d z$, becomes $0 = f x + f' y + f'' z + \mu\, r - r^2 \cdot \frac{d \omega^2}{d t^2} + \left(\frac{r d r}{d t}\right)^2$, as in [580]. This being added to [581] gives [582].

* (418) The equation [582] gives $r = \frac{h^2}{\mu} - \frac{f}{\mu} . x - \frac{f'}{\mu} . y - \frac{f''}{\mu} . z$, this being [582*a*] substituted in $r^2 = x^2 + y^2 + z^2$, becomes $\left(\frac{h^2}{\mu} - \frac{f}{\mu} . x - \frac{f'}{\mu} . y - \frac{f''}{\mu} . z\right)^2 = x^2 + y^2 + z^2$, which is of the second degree in x, y, z. The equation [579], $0 = c z - c' y + c'' x$, may, as in [579*a*], be put under the form $z = A x + B y$. From this and the preceding equation, we find the equation of the conic section, as in note 401, page 336. Again, if we substitute in the above value of r, [582*a*], the expressions of x, y, z, [553*b*], it will become of the form $r = \frac{h^2}{\mu} + D x_{\prime\prime\prime} + E y_{\prime\prime\prime}$, which is similar to that in [554*a*], from which we have proved in [554*b*], that the origin of r is at the focus.

† (419) By [372], $r^2 d v^2 + d r^2$ expresses the square of the space passed over by the body in the time $d t$, being limited by the two radii r, $r + d r$, and the included angle $d v$, and by [40*a*], the square of the same space is also expressed by $d x^2 + d y^2 + d z^2$, according to the common principles of orthographic projection, with three rectangular co-ordinates, x, y, z. Putting these two expressions equal to each other, we get, [583].

and the equation [576]

[584] $$r^2 \cdot \left(\frac{dx^2+dy^2+dz^2}{dt^2}\right) - \frac{r^2 dr^2}{dt^2} = h^2,$$

becomes $r^4 dv^2 = h^2 dt^2$; therefore

[585] $$dv = \frac{h\,dt}{r^2}.$$

Hence we find that the elementary area $\tfrac{1}{2} r^2 dv$, described by the radius vector r, is proportional to the element of the time dt; the area described [585′] in a finite time is therefore proportional to the time. We also see that the angular motion of m about M is, at each point of the orbit, inversely as the [585″] square of the radius vector; and as we may, without any sensible error, take very small intervals of time, instead of an infinitely small instant; we shall have, by means of the preceding equation, the horary motions of the planets and comets in different parts of their orbits.

The elements of the conic section described by m, are the arbitrary constant quantities of its motion; consequently these elements are functions [585‴] of the quantities $c, c', c'', f, f', f'', \frac{\mu}{a}$. To determine these functions, let θ be the angle which the axis of x makes with the line of intersection of the plane of the orbit with the plane of x, y, which line we shall call the *line of* [585iv] *nodes;* let φ be the inclination of the planes to each other. If we call x' and y' the co-ordinates of m referred to the line of nodes as the axis of the abscisses; we shall find*

[586] $$x' = x \cdot \cos.\theta + y \cdot \sin.\theta;$$
$$y' = y \cdot \cos.\theta - x \cdot \sin.\theta;$$

we shall have also

[587] $$z = y' \cdot \text{tang.}\,\varphi;$$

Substituting this in [584] we find $r^2 \cdot \left(\frac{r^2 dv^2 + dr^2}{dt^2}\right) - \frac{r^2 dr^2}{dt^2} = h^2$. Multiplying by dt^2, and reducing we obtain $r^4 dv^2 = h^2 dt^2$, whose square root is $r^2 dv = h\,dt$, or $dv = \frac{h\,dt}{r^2}$, as in [585]. Now by [372*a*], $r^2 dv$ is double the area included by the radii r and $r + dr$, therefore this area is proportional to the time.

* (420) In the annexed figure let CX, CY, be the rectangular axes of x, y; CX', CY', those of x', y'; the line CX' being the line of intersection of the

therefore we shall have

$$z = y \,.\, \cos.\, \theta \,.\, \text{tang.}\, \varphi - x \,.\, \sin.\, \theta \,.\, \text{tang.}\, \varphi.$$ [588]

By comparing this with the equation [579],

$$0 = c''x - c'y + cz\,;$$ [589]

we shall get*

$$c' = c \,.\, \cos.\, \theta \,.\, \text{tang.}\, \varphi\,;$$
$$c'' = c \,.\, \sin.\, \theta \,.\, \text{tang.}\, \varphi\,;$$ [590]

whence we deduce

$$\text{tang.}\, \theta = \frac{c''}{c'}\,;$$
$$\text{tang.}\, \varphi = \frac{\sqrt{c'^2 + c''^2}}{c}\,;$$ [591]

---

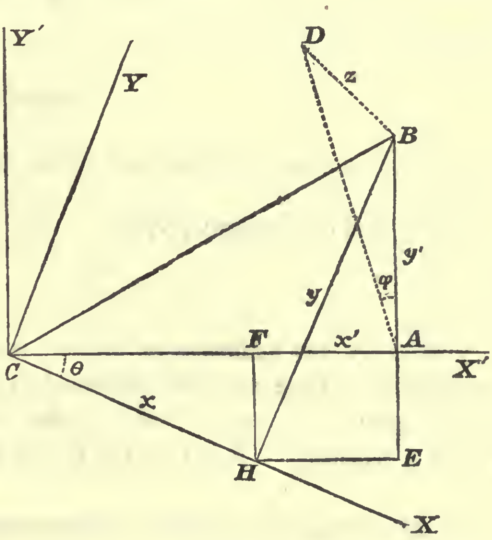
plane of the orbit with the plane of x, y; and the axis of z being perpendicular to the plane of the figure; so that the coordinates of any point D, may be expressed either by $CH = x$, $HB = y$, $BD = z$, or by $CA = x'$, $AB = y'$, $BD = z$. The angle $HCF = HBE = \theta$, angle $BAD = \varphi$, FH, AE, being parallel to the axis of y', and HE parallel to the axis of x'. Then we evidently have

$$CA = CF + HE = x \,.\, \cos.\, \theta + y \,.\, \sin.\, \theta.$$
$$AB = BE - FH = y \,.\, \cos.\, \theta - x \,.\, \sin.\, \theta.$$

which agree with [586]. Again, in the rectangular triangle DBA, we have

$$BD = AB \,.\, \text{tang.}\, BAD = y' \,.\, \text{tang.}\, \varphi,$$

as in [587]. Substituting in this the value of y', [586], it becomes as in [588].



* (421) Multiplying [588] by $-c$, we get

$$0 = (c \,.\, \sin.\, \theta \,.\, \text{tang.}\, \varphi) \,.\, x - (c \,.\, \cos.\, \theta \,.\, \text{tang.}\, \varphi) \,.\, y + c \,.\, z,$$

which being compared with [589] gives the values of c', c'', [590]. The latter divided by the former gives tang. θ, [591]. The sum of the squares of [590] is $c'^2 + c''^2 = c^2 \,.\, \text{tang.}^2 \varphi$, whence we deduce tang. φ, [591].

If the plane of the orbit be taken for the plane of x, y, we shall have $\text{tang.}\, \varphi = 0$, and [591a]
the last equation [591] will give $c'^2 + c''^2 = 0$, which requires that $c' = 0$, $c'' = 0$,
and then the expression of h [575'], will become $h = \sqrt{c^2 + c'^2 + c''^2} = c$.

Determination of the nodes, and inclination of the orbit. Thus the positions of the nodes and the inclination of the orbit will be determined in functions of the arbitrary constant quantities c, c', c''.

At the perihelion, we have*

[592] $$r\,d\,r=0\,;\qquad\text{or}\qquad x\,d\,x+y\,d\,y+z\,d\,z=0\,;$$

[592′] let X, Y, Z, be the co-ordinates of the planet at this point; the fourth and fifth of the equations (P) [572] of the preceding article, will give†

[593] $$\frac{Y}{X}=\frac{f'}{f}.$$

[593′] But if we put I for the longitude of the projection of the perihelion upon the plane of x and y, this longitude being counted from the axis of x, we shall have‡

[594] $$\frac{Y}{X}=\text{tang.}\,I\,;$$

therefore

[594′] $$\text{tang.}\,I=\frac{f'}{f}\,;$$

Determination of the place of the perihelion. which determines the position of the transverse axis of the conic section.

If from the equation [576], $r^2.\left(\frac{d\,x^2+d\,y^2+d\,z^2}{d\,t^2}\right)-\frac{r^2\,d\,r^2}{d\,t^2}=h^2$, we eliminate $\frac{dx^2+dy^2+dz^2}{dt^2}$, by means of the last of the equations (P) [572]; we shall find

---

* (422) At the perihelion the radius r is a minimum, at the aphelion it is a maximum, consequently at those points the differential $d\,r$ must be nothing, hence $r\,d\,r=0$, or [549′], $x\,d\,x+y\,d\,y+z\,d\,z=0$, and by using the capital letters, in conformity to the notation [592a] [592′], it becomes $X\,d\,X+Y\,d\,Y+Z\,d\,Z=0$.

† (423) Putting $r\,d\,r=0$, in the values of f, f', deduced from [572a, b], we shall find, $f=X.\left(\frac{d\,\omega^2}{d\,t^2}-\frac{\mu}{r}\right)$; $f'=Y.\left(\frac{d\,\omega^2}{d\,t^2}-\frac{\mu}{r}\right)$. The latter divided by the former gives $\frac{Y}{X}=\frac{f'}{f}$, [593].

‡ (424) Suppose, in the preceding figure, D to be the place of the perihelion, so that $C\,H=X$, $H\,B=Y$, $B\,D=Z$, the angle $H\,C\,B=I$, we shall evidently have $\text{tang.}\,H\,C\,B=\frac{H\,B}{C\,H}$, or $\text{tang.}\,I=\frac{Y}{X}$, as in [594], and this, by means of [593], becomes $\text{tang.}\,I=\frac{f'}{f}$, as in [594′].

$$2\mu r - \frac{\mu r^2}{a} - \frac{r^2 dr^2}{dt^2} = h^2\,; \qquad [595]$$

but dr is nothing at the extremities of the transverse axis; therefore we shall have at those points

$$0 = r^2 - 2ar + \frac{a \cdot h^2}{\mu}. \qquad [596]$$

The sum of the values of r in this equation is the transverse axis of the conic section, and their difference is the double of the excentricity;* therefore a is the semi-transverse axis of the orbit, or the mean distance of m from M; and $\sqrt{1-\frac{h^2}{\mu a}}$ is the ratio of the excentricity to the semi-transverse axis. Let e be this ratio; we shall have by the preceding article†

Excentricity and transverse axis.

[596′]

[596″]

$$\frac{\mu}{a} = \frac{\mu^2 - l^2}{h^2}, \qquad [597]$$

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* (425) The equation [596], gives $r = a \pm a \cdot \sqrt{1-\frac{h^2}{\mu a}}$. The greatest value being $a + a \cdot \sqrt{1-\frac{h^2}{\mu a}}$, the least $a - a \cdot \sqrt{1-\frac{h^2}{\mu a}}$, whose sum is the transverse diameter $2a$, and difference $2a \cdot \sqrt{1-\frac{h^2}{\mu a}}$ is double the excentricity. Hence $\sqrt{1-\frac{h^2}{\mu a}}$ expresses the ratio of the excentricity to the semi-transverse axis, [596a] and as this ratio is put equal to e [596″], the excentricity will be represented by ae.

† (426) Since $e = \sqrt{1-\frac{h^2}{\mu a}}$, [596a], we get, by squaring and reducing $h^2 = a\mu \cdot (1-e^2)$. Multiplying by $\frac{\mu}{h^2 a}$, we find $\frac{\mu}{a} = \frac{\mu^2 \cdot (1-e^2)}{h^2} = \frac{\mu^2 - l^2}{h^2}$, [578]. [596b] Hence we have $\mu^2 \cdot (1-e^2) = \mu^2 - l^2$, therefore $\mu^2 e^2 = l^2$, and $\mu e = l$, [597′]. The preceding value of h^2, by substituting $e = \frac{l}{\mu}$, [597′], becomes $h^2 = \frac{a}{\mu} \cdot (\mu^2 - l^2)$, [599]. When the plane of the orbit is taken for the plane of x, y, we shall have as in [591a], $h = c$, hence, in this case $h = c = \sqrt{\mu a \cdot (1-e^2)}$, [599]. [596c]

[578]; therefore we shall have

[597′] $$\mu\, e = l.$$

We shall thus know all the elements which determine the nature of the conic section, and its position in space.

20. The three finite equations, found in the preceding article [582, 582′], between x, y, z, and r, give x, y, z, in functions of r; therefore to obtain the co-ordinates in functions of the time, it will be sufficient to ascertain the value of r, by a similar function, which requires another integration. To obtain this, let us resume the equation [595]

[598] $$2\,\mu\, r - \frac{\mu\, r^2}{a} - \frac{r^2 \,.\, d\,r^2}{d\,t^2} = h^2\,;$$

we have by the preceding article [597, 597′],

[599] $$h^2 = \frac{a}{\mu} \,.\, (\mu^2 - l^2) = a\,\mu \,.\, (1 - e^2)\,;$$

hence we shall have

[600] $$d\,t = \frac{r\,d\,r}{\sqrt{\mu} \,.\, \sqrt{2\,r - \frac{r^2}{a} - a\,.\,(1 - e^2)}}\,.$$

[600′] To obtain the integral of this equation, put $r = a\,.\,(1 - e\,.\,\cos.\,u)$; we shall have*

[601] $$d\,t = \frac{a^{\frac{3}{2}} \,.\, d\,u}{\sqrt{\mu}} \,.\, (1 - e\,.\,\cos.\,u)\,;$$

hence by integration,

[602] $$t + T = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \,.\, (u - e\,.\,\sin.\,u)\,; \qquad (S)$$

---

* (427) This value of r gives

$$2\,r = a\,.\,(2 - 2\,e\,.\,\cos.\,u), \qquad \frac{-r^2}{a} = a\,.\,(-1 + 2\,e\,.\,\cos.\,u - e^2\,.\,\cos.^2 u).$$

Hence the term of the denominator,

$$\sqrt{2\,r - \frac{r\,r}{a} - a\,.\,(1 - e^2)} = \sqrt{a\,e^2\,.\,(1 - \cos.^2 .\,u)} = \sqrt{a\,e^2\,.\,\sin.^2 u} = a^{\frac{1}{2}}\,e\,.\,\sin.\,u.$$

The same value of r gives $d\,r = a\,e\,.\,d\,u\,.\,\sin.\,u$, hence

$$r\,d\,r = a^2 e\,.\,d\,u\,.\,\sin.\,u\,.\,(1 - e\,.\,\cos.\,u).$$

These being substituted in [600], we shall get $d\,t = \frac{a^{\frac{3}{2}}\,.\,d\,u}{\sqrt{\mu}}\,.\,(1 - e\,.\,\cos.\,u)$, as in [601].

T being an arbitrary constant quantity. This equation gives u, and thence r, in functions of t; and as x, y, z, are given in functions of r, [558′]; we [602′] shall have the values of these co-ordinates, at any instant.

Thus we have completely integrated the differential equations (O) § 17 [545], which has introduced the six arbitrary quantities a, e, I, θ, φ, and T: [602″] the two first depend on the nature of the orbit; the three following, on its position in space; and the last, on the position of the body at a given epoch, or, which is the same thing, it depends on the time of passing the perihelion.

We shall refer the co-ordinates of the body m to others more convenient for astronomical uses; for this purpose, let us put v for the angle which the [602‴] radius vector r makes with the transverse axis, counted from the perihelion; Polar equation of an ellipsis. the equation of the ellipsis [378] will be*

$$r = \frac{a.(1-e^2)}{1+e.\cos. v}. \qquad [603]$$

The equation $r = a.(1-e.\cos. u)$, of the preceding article [600′], indicates that u is nothing at the perihelion,† consequently this point is the [603′] origin of both the angles u and v; and it is easy to perceive that the angle u is formed by the transverse axis of the orbit, and the radius drawn from its centre to the point where the circumference of the circle described about [603″] the transverse axis as a diameter, is intersected by the ordinates drawn from the body perpendicular to the transverse axis.‡ This angle is what is called Excentric anomaly.

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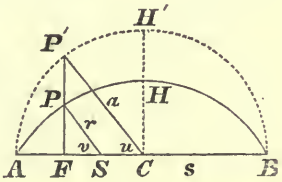
* (428) This is the same as [378], writing v for $v - \varpi$, or counting the angle v from the perihelion.

† (429) At the perihelion $r = a.(1-e)$, putting this $= a.(1-e.\cos. u)$, [600′], we get $\cos. u = 1$, hence $u = 0$.



‡ (430) In the annexed figure, which is similar to that in page 243, APB is a semi-ellipsis, whose transverse axis is AB, foci S, s, centre C; P a point of the orbit corresponding to $SP = r$, $ASP = v$, the angle ϖ, [377″], being nothing. Draw the ordinate FP perpendicular to AB, and continue it to meet the semi-circle $AP'B$, in the point P' join CP'. Then $CS = ae$, [378e], $SF = SP.\cos. ASP = r.\cos. v$, consequently $CF = CS + SF = ae + r.\cos. v$; in the triangle CFP' we have

True anomaly. the *excentric anomaly*, and the angle v is called the *true anomaly*. Comparing these two expressions of r [600′, 603], we find

[604] $$1 - e\,.\,\cos.\,u = \frac{1-e^2}{1+e\,.\,\cos.\,v}\,;$$

hence we deduce*

[605] $$\mathrm{tang.}\,\tfrac{1}{2}\,v = \sqrt{\frac{1+e}{1-e}}\,.\,\mathrm{tang.}\,\tfrac{1}{2}\,u.$$

If we fix the origin of the time t at the moment the body m passes the perihelion, T will be nothing,† and putting for brevity [530$^{\mathrm{iv}}$]

[605′] $$n = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}} = \frac{\sqrt{M+m}}{a^{\frac{3}{2}}},$$

[605″] we shall have [602] $n\,t = u - e\,.\,\sin.\,u$. Collecting these equations of the motion of m about M [605′, 600′, 605], we shall have

Equations of the elliptical motion.

[606] $$\left.\begin{array}{l} n\,t = u - e\,.\,\sin.\,u \\ r = a\,.\,(1 - e\,.\,\cos.\,u) \\ \mathrm{tang.}\,\tfrac{1}{2}\,v = \sqrt{\dfrac{1+e}{1-e}}\,.\,\mathrm{tang.}\,\tfrac{1}{2}\,u \end{array}\right\}; \qquad (f)$$

[603a] $CF = CP'\,.\,\cos.\,ACP' = a\,.\,\cos.\,ACP'$, hence $a\,.\,\cos.\,ACP' = a\,e + r\,.\,\cos.\,v$, the angle ACP' being the excentric anomaly. Now the expressions of r, [600′, 603], being put equal to each other, we get $a\,e\,.\,\cos.\,u = a - \frac{a\,.\,(1-e^2)}{1+e\,.\,\cos.\,v} = \frac{a\,.\,(e\,.\,\cos.\,v+e^2)}{1+e\,.\,\cos.\,v}$.

[603b] Dividing this by e we find $a\,.\,\cos.\,u = \frac{a\,.\,(\cos.\,v+e)}{1+e\,.\,\cos.\,v} = a\,e + \frac{a\,.\,(1-e^2)\,.\,\cos.\,v}{1+e\,.\,\cos.\,v}$, and by using the value of r, [603] it becomes $a\,.\,\cos.\,u = ae + r\,.\,\cos.\,v = a\,.\,\cos.\,ACP'$, [603a], hence $\cos.\,u = \cos.\,ACP'$, and $u = ACP' =$ the excentric anomaly.

[*] (431) Dividing $a\,.\,\cos.\,u$, [603b], by a, we get $\cos.\,u = \frac{\cos.\,v+e}{1+e\,.\,\cos.\,v}$, hence $1 - \cos.\,u = \frac{(1-e)\,.\,(1-\cos.\,v)}{1+e\,.\,\cos.\,v}$, and $1 + \cos.\,u = \frac{(1+e)\,.\,(1+\cos.\,v)}{1+e\,.\,\cos.\,v}$. Dividing the former by the latter gives $\frac{1-\cos.\,u}{1+\cos.\,u} = \frac{1-\cos.\,v}{1+\cos.\,v}\,.\,\frac{1-e}{1+e}$. Now $\frac{1-\cos.\,u}{1+\cos.\,u} = \mathrm{tang.}^2\,\tfrac{1}{2}\,u$, [40] Int. and $\frac{1-\cos.\,v}{1+\cos.\,v} = \mathrm{tang.}^2\,\tfrac{1}{2}\,v$. Substituting these and extracting the square root we shall get $\mathrm{tang.}\,\tfrac{1}{2}\,v$, [605].

[†] (431a) If $t=0$, when $u=0$, we shall get also $T=0$, [602], and this by means of [605′], gives $n\,t$, [606].

the angle $n t$ being what is called the *mean anomaly*. The first of these **Mean anomaly.** equations gives u in a function of the time t, the two others give r and v, after u is ascertained. [606′] The equation between u and t is transcendental, and can only be solved by approximation. Fortunately the circumstances of the celestial motions render these approximations very rapid. For the orbits of the heavenly bodies are either nearly circular or very excentric, and in both cases, we can determine u in terms of t, by very converging formulas, which we shall now investigate. [606″] For this purpose, we shall give some general theorems on the reduction of functions into series, which will be useful in the rest of this work.

21. Let u be any function of α, which is to be developed according to the [606‴] powers of α. By representing this series in the following manner,

$$u = \mathrm{u} + \alpha \,.\, q_1 + \alpha^2 \,.\, q_2 + \alpha^3 \,.\, q_3 \ldots\ldots + \alpha^n \,.\, q_n + \alpha^{n+1} \,.\, q_{n+1} + \&c. \,; \qquad [607]$$

u, q_1, q_2, &c., being quantities independent of α; it is evident that u is the value of u, when α is nothing, and that whatever be n, we shall have*

$$\left(\frac{d^n u}{d\alpha^n}\right) = 1\,.\,2\,.\,3\ldots.\,n\,.\,q_n + 2\,.\,3\ldots.\,(n+1)\,.\,\alpha\,.\,q_{n+1} + \&c. \,; \qquad [608]$$

---

* (432) Taking the differential of u [607] relative to α, n times, we shall get the expression [608]; then putting $\alpha = 0$, and dividing by $1.2.3\ldots.n$, we shall obtain the value of q_n, [609]. Therefore the values of q_1, q_2, q_3, &c. may be found from the first, second, third, &c. differentials of u, by putting $\alpha = 0$ after taking the differentials, and dividing by the factors 1, 1.2, 1.2.3, &c. respectively.

Hence we shall obtain the development of the function $u = f(x)$, according to the powers **Maclaurin's Theorem.** of x, by changing α into x in [607, 609], therefore

$$u = f(x) = \mathrm{u} + \left(\frac{d\,\mathrm{u}}{d x}\right) . x + \left(\frac{d^2 \mathrm{u}}{d x^2}\right) . \frac{x^2}{1.2} + \left(\frac{d^3 \mathrm{u}}{d x^3}\right) . \frac{x^3}{1.2.3} + \&c. \qquad [607a]$$

observing to put $x = 0$, in u and its differential coefficients. This formula is generally known by the name of Maclaurin's Theorem. The use of it may be shown by a few examples.

Thus if $u = a^x$, we shall have, by using hyperbolic logarithms, $\frac{d u}{d x} = a^x . \log. a$, $\frac{d d u}{d x^2} = a^x . (\log. a)^2$, &c. and when $x = 0$, we get $\mathrm{u} = 1$, $\left(\frac{d\,\mathrm{u}}{d x}\right) = \log. a$, $\left(\frac{d d \mathrm{u}}{d x^2}\right) = (\log. a)^2$, &c. thus the expression [607a] becomes

$$a^x = 1 + x \,.\, \log. a + \frac{x^2}{1.2} . (\log. a)^2 + \frac{x^3}{1.2.3} . (\log. a)^3 + \&c. \qquad [607b]$$

the differentials $\left(\frac{d^n u}{d\,\alpha^n}\right)$ being taken upon the supposition that every thing which varies with α must vary in u. Therefore if we suppose, after taking the differentials, that $\alpha = 0$, in the expression of $\left(\frac{d^n u}{d\,\alpha^n}\right)$, we shall have

[609] $$q_n = \frac{\left(\frac{d^n u}{d\,\alpha^n}\right)}{1.2.3\ldots.n}.$$

Let u be a function of two quantities α and α', which is proposed to be developed in a series according to the powers and products of α and α'. If we represent this series in the following manner,

[610] $$\begin{aligned} u = \mathrm{u} &+ \alpha\,.\,q_{1,0} + \alpha^2\,.\,q_{2,0} + \text{\&c.} \\ &+ \alpha'\,.\,q_{0,1} + \alpha\alpha'.\,q_{1,1} + \text{\&c.} \\ &+ \alpha'^2.\,q_{0,2} + \text{\&c.}\,; \end{aligned}$$

the coefficient $q_{n,n'}$ of the product $\alpha^n\,.\,\alpha'^{n'}$, will in like manner be

[611] $$q_{n,n'} = \frac{\left(\frac{d^{n+n'} u}{d\,\alpha^n\,.\,d\,\alpha'^{n'}}\right)}{1.2.3\ldots.n\,.\,1.2.3\ldots.n'}\,;$$

---

and if $a = c =$ number whose hyp. log. is 1, we shall find

[607c] $$c^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \text{\&c.}$$

If $u = \sin.\,x$, we shall have $\frac{d\,u}{d\,x} = \cos.\,x$, $\frac{d^2 u}{d\,x^2} = -\sin.\,x$, $\frac{d^3 u}{d\,x^3} = -\cos.\,x$, $\frac{d^4 u}{d\,x^4} = \sin.\,x$, &c. and when $x = 0$, we get $\mathrm{u} = 0$, $\left(\frac{d\,\mathrm{u}}{d\,x}\right) = 1$, $\left(\frac{d^2 \mathrm{u}}{d\,x^2}\right) = 0$, $\left(\frac{d^3 \mathrm{u}}{d\,x^3}\right) = -1$, &c. then the expression [607a] becomes

[607d] $$\sin.\,x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{\&c.}$$

If $u = \cos.\,x$, we shall have $\frac{d\,u}{d\,x} = -\sin.\,x$, $\frac{d^2 u}{d\,x^2} = -\cos.\,x$, $\frac{d^3 u}{d\,x^3} = \sin.\,x$, &c. and when $x = 0$, $\mathrm{u} = 1$, $\left(\frac{d^2 \mathrm{u}}{d\,x^2}\right) = -1$, $\left(\frac{d^4 \mathrm{u}}{d\,x^4}\right) = 1$, &c., the other coefficients depending on $d\,u$, $d^3 u$, $d^5 u$, &c. being nothing, hence [607a], becomes

[607e] $$\cos.\,x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{\&c.}$$

α and α' being supposed nothing after taking the differentials.*

In general, if u is a function of α, α', α'', &c., which is to be developed in a series arranged according to the powers and products of α, α', α'', &c., the coefficient of $\alpha^n . \alpha'^{n'} . \alpha''^{n''} . \&c.$, being represented by $q_{n,n',n'',\&c.}$, we shall have

$$q_{n,n',n'',\&c.} = \frac{\left(\dfrac{d^{n+n'+n''+\&c.}\,u}{d\alpha^n . d\alpha'^{n'} . d\alpha''^{n''} . \&c.}\right)}{1.2.3....n.1.2.3....n'.1.2.3....n''.\&c.}; \qquad [612]$$

provided that we suppose α, α', α'', &c., to be nothing after taking the differentials.

Supposing now u to be a function of α, α', α'', &c., and of the variable quantities t, t', t'', &c.; and that by the nature of this function, or by an equation of partial differentials which represents it, we can obtain

$$\left(\frac{d^{n+n'+n''+\&c.}\,u}{d\alpha^n . d\alpha'^{n'} . \&c.}\right), \qquad [613]$$

in a function of u, and of its differentials taken with respect to t, t', &c. Then if we call this function F, after u is changed into u, u being the value of u, when α, α', α'', &c., are supposed equal to nothing; it is evident that we shall obtain $q_{n,n',\&c.}$, by dividing F by the product [613']

$$1.2.3....n.1.2.3....n'.\&c.;$$

we shall therefore have the law of the series in which u is developed.

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* (433) Taking the differential of [610] n times and dividing by $d\alpha^n$, considering $d\alpha$ as constant, all the terms of u, depending on α^{n-1}, and lower powers of α will vanish in $d^n u$; the terms depending on α^n, will obtain the factor $1.2.3....n$, multiplied by α^0 or 1; the terms multiplied by α^{n+1}, α^{n+2}, &c. will produce terms multiplied by α, α^2, &c. If we now take in the same manner the differential of $d^n u$, n' times relative to α', and divide by $d\alpha'^{n'}$, all the terms, which in u are multiplied by $\alpha'^{n'-1}$, and lower powers of α' will vanish, and the term of u multiplied by $\alpha^n . \alpha'^{n'}$, will produce in the differential the quantity $1.2.3....n.1.2.3....n'.q_{n,n'}$, the other terms of the differential being multiplied by α, or α' and their powers and products will become nothing, when $\alpha = 0$, $\alpha' = 0$, and then $\left(\frac{d^{n+n'}.u}{d\alpha^n . d\alpha'^{n'}}\right) = 1.2.3....n.1.2.3....n'.q_{n,n'}$, which gives $q_{n,n'}$, as in [611]. In the same manner we shall obtain the general expression of $q_{n,n',n'',\&c.}$, [612].

Suppose in the first place that u is equal to any function of $t+\alpha$, $t'+\alpha'$, $t''+\alpha''$, &c.; which we shall denote by

[613″] $$u=\varphi\,.\,(t+\alpha,\ \ t'+\alpha',\ \ t''+\alpha'',\ \ \&c.)\,;$$

in this case the differential of u, relative to α, being taken a number of times denoted by i, and then divided by $d\,\alpha^i$, will be evidently equal to the like differentials taken with respect to t, and divided by $d\,t^i$.* The same equality exists between the differentials taken with respect to α' and t', or with respect to α'' and t'', &c.; hence it follows that we shall have in general

[614] $$\left(\frac{d^{n+n'+n''+\&c.}\,.\,u}{d\,\alpha^n\,.\,d\,\alpha'^{n'}\,.\,d\,\alpha''^{n''}\,.\,\&c.}\right)=\left(\frac{d^{n+n'+n''+\&c.}\,.\,u}{d\,t^n\,.\,d\,t'^{n'}\,.\,d\,t''^{n''}\,.\,\&c.}\right).$$

By changing in the second member of this equation, u into u, that is into $\varphi\,(t,t',t'',\&c.)$; we shall have, by what precedes, [612, 614].

[615] $$q_{n,\,n',\,n'',\,\&c.}=\frac{\left(\dfrac{d^{n+n'+n''+\&c.}\,.\,\varphi\,(t,\,t',\,t'',\,\&c.)}{d\,t^n\,.\,d\,t'^{n'}\,.\,d\,t''^{n''}\,.\,\&c.}\right)}{1\,.\,2\,.\,3\,\ldots\,.\,n\,.\,1\,.\,2\,.\,3\,\ldots\,.\,n'\,.\,1\,.\,2\,.\,3\,\ldots\,.\,n''\,.\,\&c.}\,;$$

If u is a function of $t+\alpha$ only, we shall have [615]

[616] $$q_n=\frac{d^n\,.\,\varphi\,(t)}{1\,.\,2\,.\,3\,\ldots\,.\,n\,.\,d\,t^n}\,;$$

Taylor's Theorem. therefore†

[617] $$\varphi(t+\alpha)=\varphi(t)+\alpha\,.\,\frac{d\,.\,\varphi\,(t)}{d\,t}+\frac{\alpha^2}{1\,.\,2}\,.\,\frac{d^2\,.\,\varphi\,(t)}{d\,t^2}+\frac{\alpha^3}{1\,.\,2\,.\,3}\,.\,\frac{d^3\,.\,\varphi\,(t)}{d\,t^3}+\&c.\qquad(i)$$

[617′] Suppose now that u, instead of being given explicitly in α and t, as in the preceding case, is a function of x; x being given by the equation of partial differentials‡

[617″] $$\left(\frac{d\,x}{d\,\alpha}\right)=z\,.\left(\frac{d\,x}{d\,t}\right),$$

---

* (433a) Because t and α occur in φ, only under the form of $t+\alpha$, and its functions, and we have $\left(\frac{d\,.\,(t+\alpha)}{d\,t}\right)=\left(\frac{d\,.\,(t+\alpha)}{d\,\alpha}\right)$.

† (434) Substituting in [607] the values deduced from [616], we shall get the formula [617].

‡ (434a) It will be proved in [632′] that the integral of this equation is $x=\varphi\,(t+a\,z)$, φ denoting an arbitrary function.

in which z is any function of x. To reduce u to a series arranged according [617‴] to the powers of α, it is necessary to determine $\left(\frac{d^n u}{d\alpha^n}\right)$ [609, 607], in the case of $\alpha = 0$; now we have, by means of the proposed equation of partial differentials,

$$\left(\frac{du}{d\alpha}\right) = \left(\frac{du}{dx}\right) . \left(\frac{dx}{d\alpha}\right) = z . \left(\frac{du}{dx}\right) . \left(\frac{dx}{dt}\right); \qquad [618]$$

therefore we shall have*

$$\left(\frac{du}{d\alpha}\right) = \left(\frac{d . \int z\,du}{dt}\right); \qquad (k) \qquad [619]$$

taking the differential of this equation with respect to α, we shall have

$$\left(\frac{ddu}{d\alpha^2}\right) = \left(\frac{dd . \int z\,du}{d\alpha\,dt}\right); \qquad [620]$$

now the equation (k) [619] gives, by changing u into $\int z\,du$,†

$$\left(\frac{d . \int z\,du}{d\alpha}\right) = \left(\frac{d . \int z^2\,du}{dt}\right); \qquad [621]$$

therefore

$$\left(\frac{ddu}{d\alpha^2}\right) = \left(\frac{dd . \int z^2\,du}{dt^2}\right). \qquad [622]$$

---

* (435) u being any function of x, [617′], and x a function of t, α, we evidently have $\left(\frac{du}{d\alpha}\right) = \left(\frac{du}{dx}\right) . \left(\frac{dx}{d\alpha}\right)$, and this, by means of [617″], becomes $\left(\frac{du}{d\alpha}\right) = z . \left(\frac{du}{dx}\right) . \left(\frac{dx}{dt}\right)$, as in [618]; and as z, u, are both functions of x [617′, 617‴], we may find the integral $\int z\,du = z'$, z' being also a function of x; hence $z . \left(\frac{du}{dx}\right) = \left(\frac{dz'}{dx}\right)$, and the preceding [617a] expression will become $\left(\frac{du}{d\alpha}\right) = \left(\frac{dz'}{dx}\right) . \left(\frac{dx}{dt}\right)$, the second member of which is evidently equal to $\left(\frac{dz'}{dt}\right)$, because z' is a function of x, and x a function of t, α, hence $\left(\frac{du}{d\alpha}\right) = \left(\frac{dz'}{dt}\right)$, [618a] which becomes the same as in [619] by substituting $d . \int z\,du$ for dz', [617a].

† (436) The equation [619] was derived from [617″], and in it we may take for u any function of x whatever, as $\int z\,du$, $\int z^2\,du$, $\int z^3\,du$, &c. and the equation will still exist, or in other words, we may, instead of du, put $z\,du$, $z^2\,du$, $z^3\,du$, &c. Thus by changing du into $z\,du$, in the equation [619] we shall obtain $\left(\frac{d . \int z du}{d\alpha}\right) = \left(\frac{d . \int z^2 du}{dt}\right)$, as in [621], the differential of this relative to t is $\left(\frac{dd . \int z\,du}{d\alpha\,dt}\right) = \left(\frac{dd . \int z^2\,du}{dt^2}\right)$, which being substituted in [620] gives [622].

Taking the differential again with respect to α, we shall have

[623] $$\left(\frac{d^3 u}{d\alpha^3}\right)=\left(\frac{d^3.\int z^2\,du}{d\alpha\,dt^2}\right);$$

now the equation (k) [619] gives, by changing u into $\int z^2\,du$,*

[624] $$\left(\frac{d.\int z^2\,du}{d\alpha}\right)=\left(\frac{d.\int z^3\,du}{dt}\right);$$

therefore

[625] $$\left(\frac{d^3 u}{d\alpha^3}\right)=\left(\frac{d^3.\int z^3\,du}{dt^3}\right).$$

Proceeding in this manner, it will be easy to conclude that in general

[626] $$\left(\frac{d^n u}{d\alpha^n}\right)=\left(\frac{d^n.\int z^n\,du}{dt^n}\right)=\left(\frac{d^{n-1}.z^n.\left(\frac{du}{dt}\right)}{dt^{n-1}}\right).$$

[626′] Suppose now that by making $\alpha=0$, we should have $x=T$, T being a function of t; and that this value of x, being substituted in z and u, makes
[626″] those quantities become Z and u; we shall have, by supposing $\alpha=0$,

[627] $$\left(\frac{d^n u}{d\alpha^n}\right)=\frac{d^{n-1}.\left(Z^n.\frac{d\mathrm{u}}{dt}\right)}{dt^{n-1}};$$

therefore by what precedes

[628] $$q_n=\frac{d^{n-1}.\left(Z^n.\frac{d\mathrm{u}}{dt}\right)}{1.2.3\ldots.n.dt^{n-1}};$$

which gives [607, 628]†

[629] $$u=\mathrm{u}+\alpha Z.\frac{d\mathrm{u}}{dt}+\frac{\alpha^2}{1.2}.\frac{d.\left(Z^2.\frac{d\mathrm{u}}{dt}\right)}{dt}+\frac{\alpha^3}{1.2.3}.\frac{d^2.\left(Z^3.\frac{d\mathrm{u}}{dt}\right)}{dt^2}+\&c. \qquad (p)$$

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[*] (437) This change of u into $\int z^2\,du$, or du into $z^2\,du$, may be made for the reasons stated in the preceding note, and [619] will be changed into [624]; taking its differential twice relative to t we get $\left(\frac{d^3.\int z^2\,du}{d\alpha.dt^2}\right)=\left(\frac{d^3.\int z^3\,du}{dt^3}\right)$, which being substituted in [623], it becomes as in [625]. Proceeding in this manner we shall evidently obtain the general expression [626].

[†] (438) The formula [629], given by La Place, is of great use in the inversion of series and is frequently referred to; it may not therefore be amiss to collect in one point of view

It now remains to determine the function of t and α, which is represented by x. This is done by taking the integral of the equation of partial differentials [617″] $\left(\frac{dx}{d\alpha}\right) = z \,.\, \left(\frac{dx}{dt}\right)$. For this purpose we shall observe that*

$$dx = \left(\frac{dx}{dt}\right) . dt + \left(\frac{dx}{d\alpha}\right) . d\alpha \,; \qquad [630]$$

---

the forms of the different functions used. Supposing therefore φ, ψ, F, to denote the characteristic of functions, we shall have [632′, 632″, 617‴, 626″, 617‴],

$$\begin{array}{ll} x = \varphi\,(t + \alpha z)\,; & T = \varphi\,(t)\,; \\ z = F\,(x)\,; & Z = F\,(T) = F.\,\{\varphi\,(t)\}\,; \\ u = \psi\,(x) = \psi.\,\{\varphi\,(t + \alpha z)\}\,; & \mathrm{u} = \psi\,(T) = \psi.\,\{\varphi\,(t)\}\,; \end{array} \qquad [629a]$$

T, Z, u, being the values of x, z, u, respectively when $\alpha = 0$.

If we take, as a simple case of the formula [629], $x = t + \alpha z = t + \alpha . F(x)$, we shall have $u = \psi\,(t + \alpha z)$, and that formula will become, by putting $\frac{d.\psi(t)}{dt} = \psi'(t)$,

$$u = \psi\,(t) + \alpha . F\,(t) . \psi'\,(t) + \frac{\alpha^2}{1.2} . \frac{d.\{F(t)^2.\psi'(t)\}}{dt} + \frac{\alpha^3}{1.2.3} . \frac{d^2.\{F(t)^3.\psi'(t)\}}{dt^2} + \&c. \qquad [629b]$$

If in this we put $\alpha = 1$, we shall obtain the celebrated Theorem of La Grange, which has been of such great use in analysis. In this theorem we have

Theorem of La Grange.

$$x = t + F\,(x),$$

$$\psi\,(x) = \psi\,(t) + F\,(t) . \psi'\,(t) + \frac{1}{1.2} . \frac{d.\{F(t)^2.\psi'(t)\}}{dt} + \frac{1}{1.2.3} . \frac{d^2.\{F(t)^3.\psi'(t)\}}{dt^2} + \&c. \qquad [629c]$$

For an example of the use of this last formula we shall suppose $x = t + \alpha x^n$, and that it is required to find x in a series arranged according to the powers of t. This value being compared with $x = t + F\,(x)$, [629c], gives $F\,(x) = \alpha x^n$, $\psi\,(x) = x$, hence $F\,(t) = \alpha t^n$, $\psi\,(t) = t$, and $\psi'\,(t) = 1$, and the formula [629c] becomes

$$x = t + \alpha t^n + \frac{\alpha^2}{1.2} . \frac{d.t^{2n}}{dt} + \frac{\alpha^3}{1.2.3} . \frac{d^2.t^{3n}}{dt^2} + \&c.$$

or by development

$$x = t + \alpha t^n + \frac{\alpha^2}{1.2} . 2n . t^{2n-1} + \frac{\alpha^3}{1.2.3} . 3n . (3n - 1) . t^{3n-2} + \&c.$$

in which the law of continuation is very manifest, and it is one of the great advantages of this beautiful formula, which is much used in the course of this work, as in [652, 657, 658, 666], which may be referred to, as striking examples of the importance of this method of development.

* (438a) In the equation [617″] the *partial* differentials of x relative to α and t only occur, or in other words x is considered as a function of t and α only, and then its complete differential dx will be as in [630].

by substituting for $\left(\frac{dx}{d\alpha}\right)$, its value $z.\left(\frac{dx}{dt}\right)$, we shall have*

[631] $$dx=\left(\frac{dx}{dt}\right).\left\{dt+z\,d\alpha\right\}=\left(\frac{dx}{dt}\right).\left\{d.(t+\alpha z)-\alpha.\left(\frac{dz}{dx}\right).dx\right\};$$

therefore we shall have

[632] $$dx=\frac{\left(\frac{dx}{dt}\right).d.(t+\alpha z)}{1+\alpha.\left(\frac{dz}{dx}\right).\left(\frac{dx}{dt}\right)};$$

which gives by integration

[632′] $$x=\varphi\,(t+\alpha z)\,;$$

$\varphi\,(t+\alpha z)$ being an arbitrary function of $(t+\alpha z)$; hence it follows that the
[632″] quantity we have denoted by T [626′] is equal to $\varphi\,(t)$. Therefore whenever there is given between x and α an equation which may be reduced to the form $x=\varphi(t+\alpha z)$; the value of u will be given by the formula (p) [629], in a series arranged according to the powers of α.

Suppose now that u is a function of two variable quantities x and x', these
[632‴] quantities being given by the equations of partial differentials,

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* (439) The last member of [631] is easily deduced from the preceding, by substituting for $dt+z\,d\alpha$, its value $d.(t+\alpha z)-\alpha.dz$, and as z is a function of x, [617‴], we may write $\left(\frac{dz}{dx}\right).dx$ for dz, we shall thus obtain

$$dx=\left(\frac{dx}{dt}\right).\left\{d.(t+\alpha z)-\alpha.\left(\frac{dz}{dx}\right).dx\right\}.$$

Transposing the last term and dividing by $1+\alpha.\left(\frac{dz}{dt}\right).\left(\frac{dx}{dt}\right)$, we shall obtain

$$dx=\frac{\left(\frac{dx}{dt}\right).d.(t+\alpha z)}{1+\alpha.\left(\frac{dz}{dx}\right).\left(\frac{dx}{dt}\right)}.$$

Putting now for brevity $t+\alpha z=\omega$, $\dfrac{\left(\frac{dx}{dt}\right)}{1+\alpha.\left(\frac{dz}{dx}\right).\left(\frac{dx}{dt}\right)}=V$, it becomes a common differential equation $dx=V\,d\omega$, and as the first member is a complete differential of x, the second member must also be complete, which cannot be, in general, unless we take V such that x may be a function of ω or $t+\alpha z$, which may be represented by $x=\varphi\,(t+\alpha z)$, as in [632′].

$$\left(\frac{dx}{d\alpha}\right)=z\cdot\left(\frac{dx}{dt}\right);\qquad\qquad \left(\frac{dx'}{d\alpha'}\right)=z'\cdot\left(\frac{dx'}{dt'}\right);$$ [633]

in which z and z' are any functions whatever of x and x'. It is easy to [633'] prove* that the integrals of these equations are

$$x=\varphi\,(t+\alpha z);\qquad\qquad x'=\psi\,(t'+\alpha' z');$$ [634]

$\varphi\,(t+\alpha z)$ and $\psi\,(t'+\alpha' z')$ being arbitrary functions, the one of $t+\alpha z$, and the other of $t'+\alpha' z'$. We shall also have†

$$\left(\frac{du}{d\alpha}\right)=z\,.\left(\frac{du}{dt}\right);\qquad\qquad \left(\frac{du}{d\alpha'}\right)=z'\,.\left(\frac{du}{dt'}\right);$$ [635]

This being premised, if we suppose x' to be eliminated from u and z, by means of the equation $x'=\psi(t'+\alpha'z')$ u and z will become functions of x, α', t', without α or t;‡ therefore we shall have, by what precedes [627]

$$\left(\frac{d^n u}{d\alpha^n}\right)=\left(\frac{d^{n-1}\,.\,z^n\,.\left(\frac{du}{dt}\right)}{dt^{n-1}}\right).$$ [636]

If we suppose $\alpha=0$, after taking the differentials, and substitute also in the second member of the equation [636] $x=\varphi\,(t+\alpha z^n)$, consequently§ $\left(\frac{du}{d\alpha}\right)=z^n\,.\left(\frac{du}{dt}\right)$; we shall have, by means of these suppositions, [636']

$$\left(\frac{d^n u}{d\alpha^n}\right)=\left(\frac{d^{n-1}\,.\left(\frac{du}{d\alpha}\right)}{dt^{n-1}}\right);$$ [637]

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* (440) This demonstration may be made as in the preceding note.

† (441) Put $t+\alpha z=\omega$, $t'+\alpha' z'=\omega'$, then u may be considered as a function of ω, ω', [632''', 634], and we shall have $\left(\frac{du}{dt}\right)=\left(\frac{du}{d\omega}\right)\cdot\left(\frac{d\omega}{dt}\right)$; $\left(\frac{du}{d\alpha}\right)=\left(\frac{du}{d\omega}\right)\cdot\left(\frac{d\omega}{d\alpha}\right)$, but from $\omega=t+\alpha z$, we get $\left(\frac{d\omega}{dt}\right)=1$, $\left(\frac{d\omega}{d\alpha}\right)=z$, therefore $\left(\frac{du}{dt}\right)=\left(\frac{du}{d\omega}\right)$; $\left(\frac{du}{d\alpha}\right)=\left(\frac{du}{d\omega}\right).z=\left(\frac{du}{dt}\right).z$, as in [635]. In like manner we find $\left(\frac{du}{d\alpha'}\right)=z'.\left(\frac{du}{dt'}\right)$.

‡ (442) z, u, do not contain α, t, explicitly, but implicitly only, by means of x, and they then come under the form supposed in [617', 617'''], and the result found in [627], from the suppositions made in [617—617'''], may be applied to this case, as is done in [636].

§ (443) This calculation is made in the same manner as in note 441, merely changing z into z^n, so that $\omega=t+\alpha z^n$, by which means we shall find that $\left(\frac{du}{d\alpha}\right)=z^n\,.\left(\frac{du}{dt}\right)$. Substituting this in [636] it becomes as in [637].

consequently*

[638] $$\left(\frac{d^{n+n'}.u}{d\alpha^{n}.d\alpha'^{n'}}\right)=\left(\frac{d^{n-1}.\left(\frac{d.\left(\frac{d^{n'}u}{d\alpha'^{n}}\right)}{d\alpha}\right)}{dt^{n-1}}\right).$$

in like manner we shall have†

[639] $$\left(\frac{d^{n'}u}{d\alpha'^{n'}}\right)=\left(\frac{d^{n'-1}.\left(\frac{du}{d\alpha'}\right)}{dt'^{n'-1}}\right);$$

supposing $\alpha'=0$, after taking the differentials, and putting also in the second member of this equation $x'=\psi(t'+\alpha' z'^{n'})$; therefore we shall have‡

[640] $$\left(\frac{d^{n+n'}u}{d\alpha^{n}.d\alpha'^{n'}}\right)=\left(\frac{d^{n+n'-2}.\left(\frac{ddu}{d\alpha d\alpha'}\right)}{dt^{n-1}.dt'^{n'-1}}\right);$$

provided we put α and α' equal to nothing after taking the differentials, and also, in the second member of this equation, make

[641] $$x=\varphi(t+\alpha z^{n})\,; \qquad x'=\varphi(t'+\alpha' z'^{n'})\,;$$

which amounts to the same thing as to suppose in both members

[642] $$x=\varphi(t+\alpha z)\,; \qquad x'=\varphi(t'+\alpha' z'),$$

changing at the same time in the partial differential $\left(\frac{ddu}{d\alpha d\alpha'}\right)$ of the second

[642′] member, z into z^{n}, and z' into $z'^{n'}$. These suppositions being made, and withal changing z into Z, z' into Z', u into u [626″], we shall find§

[643] $$q_{n,n'}=\left(\frac{d^{n+n'-2}.\left(\frac{ddu}{d\alpha d\alpha'}\right)}{1.2.3\ldots.n.1.2.3\ldots.n'.dt^{n-1}.dt'^{n'-1}}\right).$$

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* (444) Taking the differential of [637] n' times relative to α', we shall get [638].

† (445) The expression [639] was found in the same manner as [637] and it may be deduced from it by changing n, α, t, into n', α', t', respectively.

‡ (446) Substituting the value of $\left(\frac{d^{n'}u}{d\alpha'^{n'}}\right)$, [639] in [638] it changes into [640].

§ (447) Substituting in $q_{n,n'}$, [611] the value of $\left(\frac{d^{n+n'}u}{d\alpha^{n}.d\alpha'^{n'}}\right)$, [640], we shall get [643].

By following this method of reasoning, it is easy to perceive that if we have r equations

$$\begin{aligned} x &= \varphi\,(t+\alpha z)\;; \\ x' &= \psi\,(t'+\alpha' z')\;; \\ x'' &= \Pi\,(t''+\alpha'' z'')\;; \\ &\&c.\;; \end{aligned} \qquad [644]$$

z, z', z'', &c., being any functions whatever of x, x', x'', &c.; and if we suppose u to be a function of the same variable quantities, we shall have in general

$$q_{n,n',n'',\&c.} = \left(\frac{d^{n+n'+n''+\&c.-r}\,.\left(\dfrac{d^r u}{d\alpha\,.\,d\alpha'\,.\,d\alpha''\,.\,\&c.}\right)}{1.2.3\ldots n\,.\,1.2.3\ldots n'\,.\,1.2.3\ldots n''.\&c.\,dt^{n-1}\,.\,dt'^{\,n'-1}\,.\,dt''^{\,n''-1}\,.\,\&c.} \right); \qquad [645]$$

[645′] provided that we change in the partial differential $\left(\frac{d^r u}{d\alpha\,.\,d\alpha'\,.\,d\alpha''\,.\,\&c.}\right)$, z into z^n, z' into $z'^{\,n'}$, &c.; afterwards z into Z, z' into Z', z'' into Z'', &c.; and then u into u.

If there is but one variable quantity x, we shall have [635]

$$\left(\frac{du}{d\alpha}\right) = z\,.\left(\frac{du}{dt}\right); \qquad [646]$$

therefore*

$$q_n = \frac{d^{n-1}\,.\left\{Z^n\,.\left(\dfrac{d\mathrm{u}}{dt}\right)\right\}}{1.2.3\ldots.n\,.\,dt^{n-1}}. \qquad [647]$$

If there are two variable quantities x and x', we shall have [635]

$$\left(\frac{du}{d\alpha}\right) = z\,.\left(\frac{du}{dt}\right); \qquad [648]$$

taking the differential of this with respect to α', we shall find

$$\left(\frac{ddu}{d\alpha\,d\alpha'}\right) = \left(\frac{dz}{d\alpha'}\right).\left(\frac{du}{dt}\right) + z\,.\left(\frac{ddu}{d\alpha'\,dt}\right); \qquad [649]$$

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* (448) Putting $n'=0$, $n''=0$, &c. and $r=1$, the general expression [645] will become $q_n = \frac{d^{n-1}\,.\left(\frac{du}{d\alpha}\right)}{1.2.3\ldots.n.dt^{n-1}}$. Substituting the value of $\left(\frac{du}{d\alpha}\right)$, [646], and changing z into z^n, as in [645′] we shall get [647].

[649′] now we have* $\left(\frac{du}{d\alpha'}\right) = z' . \left(\frac{du}{dt'}\right)$ [635] ; and by changing in this equation

[649″] u into z, we shall get $\left(\frac{dz}{d\alpha'}\right) = z' . \left(\frac{dz}{dt'}\right)$; therefore†

[650]
$$\left(\frac{ddu}{d\alpha\, d\alpha'}\right) = z . \left(\frac{d . z'\left(\frac{du}{dt'}\right)}{dt}\right) + z' . \left(\frac{dz}{dt'}\right) . \left(\frac{du}{dt}\right).$$

Supposing α and α' to be nothing in the second member of this equation, and changing z into Z^n, z' into $Z'^{n'}$, and u into u [642′], we shall find the value of $\left(\frac{ddu}{d\alpha\, d\alpha'}\right)$ corresponding to these conditions ; hence we get‡

[651]
$$q_{n,n'} = \frac{d^{n+n'-2} . \left\{ \begin{array}{r} Z^n . Z'^{n'} . \left(\frac{dd\mathrm{u}}{dt\,dt'}\right) + Z'^{n'} . \left(\frac{d . Z^n}{dt'}\right) . \left(\frac{d\mathrm{u}}{dt}\right) \\ + Z^n . \left(\frac{d . Z'^{n'}}{dt}\right) . \left(\frac{d\mathrm{u}}{dt'}\right) \end{array} \right\}}{1 . 2 . 3 \ldots . n . dt^{n-1} . 1 . 2 . 3 \ldots . n . dt'^{n'-1}}.$$

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* (449) By [634] we have x' equal to a function of $t' + \alpha' z'$, and from this we have deduced in [635], $\left(\frac{du}{d\alpha'}\right) = z' . \left(\frac{du}{dt'}\right)$; u being any function of x, x', [632‴]; and as z is also a function of x, x', [633′], we may also in this equation change u into z, and we shall obtain $\left(\frac{dz}{d\alpha'}\right) = z' . \left(\frac{dz}{dt'}\right)$, [649″]. In this calculation we have neglected the consideration that z depends also on x or $t + \alpha z$, because the partial differentials relative to α, t, do not occur in the equations treated of in this note.

† (450) Taking the differential of the equation $\left(\frac{du}{d\alpha'}\right) = z' . \left(\frac{du}{dt'}\right)$, [649′], relative to t, we shall get $\left(\frac{ddu}{d\alpha' dt}\right) = \left(\frac{dz'}{dt}\right) . \left(\frac{du}{dt'}\right) + z' . \left(\frac{ddu'}{dt\,dt'}\right)$. Substituting this, and the value of $\left(\frac{dz}{d\alpha'}\right) = z' . \left(\frac{dz}{dt'}\right)$, [649″], in [649], it becomes

[650a]
$$\left(\frac{ddu}{d\alpha\, d\alpha'}\right) = z' . \left(\frac{dz}{dt'}\right) . \left(\frac{du}{dt}\right) + z . \left\{\left(\frac{dz'}{dt}\right) . \left(\frac{du}{dt'}\right) + z' . \left(\frac{ddu}{dt\,dt'}\right)\right\},$$

in which the terms between the braces, or the factor of z, may be put under this form, $\left(\frac{d . z' . \left(\frac{du}{dt'}\right)}{dt}\right)$, by which means it becomes as in [650].

‡ (451) Substituting $\left(\frac{ddu}{d\alpha\, d\alpha'}\right)$, [650a], in $q_{n,n'} = \frac{d^{n+n'-2} . \left(\frac{d^2u}{d\alpha\, d\alpha'}\right)}{1.2.3 \ldots . n . dt^{n-1} . 1.2.3 \ldots . n . dt'^{n'-1}}$, [645] ; changing also z into Z^n, z' into Z'^n, [645′], it becomes as in [651].

By continuing in this manner, we shall obtain the values of $q_{n,n',n'',\&c.}$, for any number of variable quantities.

Although we have supposed u, z, z', z'', &c., to be functions of x, x', x'', &c., without t, t', t'', &c., we may however suppose them to contain these last variable quantities, but we must then denote these quantities by $t_{,}, t'_{,}, t''_{,}$, [651‴] &c., and we must suppose $t_{,}, t'_{,}, t''_{,}$, &c., to be constant in the differentiations, and after taking these differentials we must resubstitute t, t', &c., for $t_{,}, t'_{,}$, &c.

22. Let us apply these results to the elliptical motion of the planets. For this purpose we shall resume the equations (f) § 20 [606]. If we compare the equation $nt = u - e\,.\sin.u$, or $u = nt + e\,.\sin.u$, with [651''''] $x = \varphi(t + \alpha z)$ [632'], x will change into u; t into nt; α into e; z into $\sin.u$; and $\varphi(t + \alpha z)$ into $nt + e\,.\sin.u$; hence the formula (p) [629] of the preceding article will become*

$$\psi(u) = \psi(nt) + e\,.\psi'(nt)\,.\sin nt + \frac{e^2}{1\,.\,2}\,.\,\frac{d\,.\{\psi'(nt)\,.\sin.^2 nt\}}{n\,dt} + \frac{e^3}{1\,.\,2\,.\,3}\,.\,\frac{d^2\,.\{\psi'(nt)\,.\sin.^3 nt\}}{n^2\,dt^2} + \&c.; \qquad (q) \quad [652]$$

$\psi'(nt)$ being equal to $\frac{d\,.\psi(nt)}{n\,dt}$. To develop this formula we shall observe that c being the number whose hyperbolic logarithm is unity, we shall have†

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* (452) The symbol u is used in a different sense in [651''], from what it is [629]. To prevent any confusion, it was thought best to accent the letters u, t, in this last formula, which will then become,

$$u' = \mathrm{u}' + \alpha Z\,.\,\frac{d\mathrm{u}'}{dt'} + \frac{\alpha^2}{1.2}\,.\,\frac{d\,.\left(Z^2\,.\,\frac{d\mathrm{u}'}{dt'}\right)}{dt'} + \frac{\alpha^3}{1.2.3}\,.\,\frac{d^2\,.\left(Z^3\,.\,\frac{d\mathrm{u}'}{dt'}\right)}{dt'^2} + \&c. \qquad [652a]$$

In which as in [629a], $x = \varphi(t' + \alpha z)$, $T = \varphi(t')$, $u' = \psi(x) = \psi\,.\{\varphi(t' + \alpha z)\}$; [652b] $\mathrm{u}' = \psi\,.\{\varphi(t')\}$. And by comparing $x = \varphi(t' + \alpha z)$ with $u = nt + e\,.\sin.u$, we shall have as above $x = u$, $\alpha = e$, $z = \sin.u$, $t' = nt$, $\varphi(t') = nt$, $\mathrm{u}' = \psi(nt)$, $\frac{d\mathrm{u}'}{dt'} = \frac{d\,.\psi(nt)}{n\,dt} = \psi'(nt)$; $Z = \sin.nt$. Substituting these in [652a], we shall get [652].

† (453) Putting $z = nt$, in [11, 12] Int. and involving to the power i, we shall get [653]. The formulas [653'] are derived from [15, 16] Int. by putting $z = rt$.

[653] $$\sin.^{i} nt = \left(\frac{c^{nt.\sqrt{-1}} - c^{-nt.\sqrt{-1}}}{2.\sqrt{-1}} \right)^{i}; \qquad \cos.^{i} nt = \left(\frac{c^{nt.\sqrt{-1}} + c^{-nt.\sqrt{-1}}}{2} \right)^{i};$$

i being any number whatever. By developing the second members of these equations, and substituting instead of $c^{rnt.\sqrt{-1}}$, and $c^{-rnt.\sqrt{-1}}$, their
[653′] values $\cos. r n t + \sqrt{-1} . \sin. r n t$, and $\cos. rnt - \sqrt{-1} . \sin. r n t$; r being any quantity whatever; we shall have the powers i of $\sin. nt$ and $\cos. nt$, developed in a series of sines and cosines of the angle $n t$ and its multiples; this being supposed, we shall find, that if the following function be put equal to P

[653″] $$\sin. n t + \frac{e}{1.2} . \sin.^{2} n t + \frac{e^{2}}{1.2.3} . \sin.^{3} n t + \frac{e^{3}}{1.2.3.4} . \sin.^{4} n t + \&c.;$$

we shall have [1—5, Int.]

[654] $$\begin{aligned} P = \sin. nt & - \frac{e}{1.2.2} . \left\{ \cos. 2 n t - 1 \right\} \\ & - \frac{e^{2}}{1.2.3.2^{2}} . \left\{ \sin. 3 n t - 3 . \sin. n t \right\} \\ & + \frac{e^{3}}{1.2.3.4.2^{3}} . \left\{ \cos. 4 n t - 4 . \cos. 2 n t + \tfrac{1}{2} . \frac{4.3}{1.2} \right\} \\ & + \frac{e^{4}}{1.2.3.4.5.2^{4}} . \left\{ \sin. 5nt - 5 . \sin. 3 n t + \frac{5.4}{1.2} . \sin. nt \right\} \\ & - \frac{e^{5}}{1.2.3.4.5.6.2^{5}} . \left\{ \cos. 6nt - 6 . \cos. 4nt + \frac{6.5}{1.2} . \cos. 2nt - \tfrac{1}{2} . \frac{6.5.4}{1.2.3} \right\} \\ & - \&c. \end{aligned}$$

[654′] Multiply this value of P by $\psi'(n t)$, and then take the differential of each of its terms, relative to t, a number of times equal to the exponent of the power of e by which it is multiplied, dt being supposed constant; afterwards
[654″] divide these differentials by the corresponding power of $n d t$. Let P' be the sum of these differentials thus divided; the formula (q) [652] will become

[655] $$\psi(u) = \psi(n t) + e P'.$$

It will be easy to obtain, by this method, the values of the angle u, and of the sines and cosines of its multiples. Supposing, for example, $\psi(u) = \sin. iu$,
[655′] we shall have $\psi'(nt) = i . \cos. i n t$. We must multiply the preceding value

of P by $i\,.\cos.\,i\,n\,t$, and develop the product,* in sines and cosines of the angle $n\,t$ and its multiples. The terms multiplied by the even powers of e will be sines, and those multiplied by the odd powers of e will be cosines. [655'']
Then we must change any term of the form $K\,.\,e^{2r}\,.\sin.\,s\,n\,t$ into [655'']
$\pm K\,.\,e^{2r}\,.\,s^{2r}\,.\sin.\,s\,n\,t$; the sign $+$ is to be used if r is even, and the sign $-$, if r is odd.† We must likewise change any term of the form
$K\,.\,e^{2r+1}\,.\cos.\,s\,n\,t$, into $\mp K\,.\,e^{2r+1}\,.\,s^{2r+1}\,.\sin.\,s\,n\,t$; the sign $-$ is to [655^iv]
be used if r is even, and the sign $+$, if r is odd. The sum of all these terms will be the value of P', and we shall have‡

$$\sin.\,i\,u = \sin.\,i\,n\,t + e\,P'. \qquad [656]$$

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* (454) If we multiply the above expression of P, [654] by $i\,.\cos.\,i\,n\,t$, and change the products like $i\,.\sin\,m\,t\,.\cos.\,i\,n\,t$, into $\tfrac{1}{2}i\,.\,\{\sin.\,(m+i)\,.\,n\,t+\sin.\,(m-i)\,.\,n\,t\}$, [18] Int., or as it may be written $\tfrac{1}{2}i\,.\sin.(m\pm i)\,.\,nt$; also products like $i\,.\cos.\,mt.\cos.\,int$, into $\tfrac{1}{2}i\,.\,\{\cos.\,(m+i)\,.\,n\,t+\cos.\,(m-i)\,.\,n\,t\}$, or $\tfrac{1}{2}i\,.\cos.\,(m\pm i)\,.\,n\,t$, [20] Int., it will become

$$\begin{aligned}
i\,.\,P\,.\cos.\,i\,n\,t &= \frac{i}{1.2}\,.\sin.\,(1\pm i)\,.\,n\,t - \frac{e\,i}{1.2.2^2}\,.\,\{\cos.\,(2\pm i)\,.\,n\,t - 2\cos.\,i\,n\,t\}\\
&- \frac{e^2\,i}{1.2.3.2^3}\,.\,\{\sin.\,(3\pm i)\,.\,n\,t - 3\sin.\,(1\pm i)\,.\,n\,t\}\\
&+ \frac{e^3\,i}{1.2.3.4.2^4}\,.\,\{\cos.\,(4\pm i)\,.\,n\,t - 4\,.\cos.\,(2\pm i)\,.\,n\,t + \frac{4.3}{1.2}\,.\cos.\,i\,n\,t\}\\
&+ \frac{e^4\,i}{1.2.3.4.5.2^5}\,.\,\{\sin.\,(5\pm i)\,.\,n\,t - 5\,.\sin.\,(3\pm i)\,.\,n\,t + \frac{5.4}{1.2}\,.\sin.\,(1\pm i)\,.\,n\,t\}\\
&- \frac{e^5\,i}{1.2.3.4.5.6.2^6}\,.\{\cos.(6\pm i).nt - 6.\cos.(4\pm i).nt + \frac{6.5}{1.2}.\cos.(2\pm i).nt - \frac{6.5.4}{1.2.3}.\cos.int\} + \&c
\end{aligned} \qquad [654a]$$

which is like the form mentioned above.

† (455) These values for the signs, and the factor s^{2r}, necessarily follow from taking the differential a number of times denoted by the exponent of e; that is $2\,r$ times. For if r is even as 0, 2, 4, &c. $d^{2r}\,.\sin.\,s\,n\,t$ will evidently have a positive sign; but if r is odd, as 1, 3, 5, &c. it will be negative. It being easy to perceive that the sign of the coefficient of $K\,e^{2r}$ is the same as that of $K\,e^{2r+4}$. Similar remarks may be made on the term $K\,e^{2r+1}\,.\cos.\,s\,n\,t$.

‡ (456) Taking the several terms of the expression of $i\,P\,.\cos.\,i\,n\,t$, [654a], for $K\,e^{2r}\,.\sin.\,s\,n\,t$, or $K\,e^{2r+1}\,.\cos.\,s\,n\,t$, and deducing from them $\pm K\,e^{2r}\,.\,s^{2r}\,.\sin.\,s\,n\,t$, $\mp K\,e^{2r+1}\,.\,s^{2r+1}\,.\sin.\,s\,n\,t$, by the above rules, we shall obtain the value of P', which being substituted in [656], it becomes

[656′] If we suppose $\psi(u)=u$, we shall have $\psi'(nt)=1$, and we shall find*

[656a]

$$\begin{aligned}\sin.\,iu=\;&\sin.\,int+\frac{ie}{1.2}\,.\,\sin.\,(1\pm i)\,.\,nt+\frac{ie^2}{1.2.2^2}\,.\,\{(2\pm i)\,.\,\sin.\,(2\pm i)\,.\,nt-2i\,.\,\sin.\,int\}\\&+\frac{ie^3}{1.2.3.2^3}\,.\,\{(3\pm i)^2\,.\,\sin.\,(3\pm i)\,.\,nt-3\,.\,(1\pm i)^2\,.\,\sin.\,(1\pm i)\,.\,nt\}\\&+\frac{ie^4}{1.2.3.4.2^4}\,.\,\left\{(4\pm i)^3\,.\,\sin.\,(4\pm i)\,.\,nt-4\,.\,(2\pm i)^3\,.\,\sin.\,(2\pm i)\,.\,nt+\frac{4.3}{1.2}\,.\,i^3\,.\,\sin.\,int\right\}\\&+\frac{ie^5}{1.2.3.4.5.2^5}\,.\,\left\{(5\pm i)^4.\sin.(5\pm i).nt-5.(3\pm i)^4.\sin.(3\pm i).nt+\frac{5.4}{1.2}\,.\,(1\pm i)^4.\sin.(1\pm i).nt\right\}\\&+\frac{ie^6}{1.2.3.4.5.6.2^6}\,.\,\left\{\begin{array}{l}(6\pm i)^5\,.\,\sin.\,(6\pm i)\,.\,nt-6\,.\,(4\pm i)^5\,.\,\sin.\,(4\pm i)\,.\,nt\\+\dfrac{6.5}{1.2}\,.\,(2\pm i)^5\,.\,\sin.\,(2\pm i)\,.\,nt-\dfrac{6.5.4}{1.2.3}\,.\,i^5\,.\,\sin.\,int\end{array}\right\}.\end{aligned}$$

By putting successively $i=1, 2, 3, 4, 5, 6$, &c. and making the deductions arising from $\sin.\,(-mnt)=-\sin.\,mnt$, we shall get

[656b]

$$\begin{aligned}\sin.u=\sin.nt+\frac{e}{1.2}\,.\,\sin.2nt+\frac{e^2}{1.2.2^2}\,.\,\{\sin.\,3nt-\sin.nt\}+\frac{e^3}{1.2.3.2^3}\,.\,\{4^2.\sin.\,4nt-8\sin.\,2nt\}\\+\frac{e^4}{1.2.3.4.2^4}\,.\,\{5^3\,.\,\sin.\,5nt-3\,.\,3^3\,.\,\sin.\,3nt+2\,.\,\sin.\,nt\}\\+\frac{e^5}{1.2.3.4.5.2^5}\,.\,\{6^4\,.\,\sin.\,6nt-4\,.\,4^4\,.\,\sin.\,4nt+5\,.\,2^4\,.\,\sin.\,2nt\}+\&c.\end{aligned}$$

[656c]

$$\begin{aligned}\sin.\,2u=\sin.\,2nt+\frac{2e}{1.2}\,.\,\{\sin.\,3nt-\sin.\,nt\}+\frac{2e^2}{1.2.2^2}\,.\,\{4\,.\,\sin.\,4nt-4\,.\,\sin.\,2nt\}\\+\frac{2e^3}{1.2.3.2^3}\,.\,\{5^2\,.\,\sin.\,5nt-3\,.\,3^2\,.\,\sin.\,3nt+4\,.\,\sin.\,nt\}\\+\frac{2e^4}{1.2.3.4.2^4}\,.\,\{6^3\,.\,\sin.\,6nt-4\,.\,4^3\,.\,\sin.\,4nt+7\,.\,2^3\,.\,\sin.\,2nt\}+\&c.\end{aligned}$$

[656d]

$$\begin{aligned}\sin.\,3u=\sin.\,3nt+\frac{3.e}{1.2}\,.\,\{\sin.\,4nt-\sin.\,2nt\}+\frac{3e^2}{1.2.2^2}\,.\,\{5\,.\,\sin.\,5nt-6\,.\,\sin.3nt+\sin.\,nt\}\\+\frac{3e^3}{1.2.3.2^3}\,.\,\{6^2\,.\,\sin.\,6nt-3\,.\,4^2\,.\,\sin.\,4nt+3\,.\,2^2\,.\,\sin.\,2nt\}+\&c.\end{aligned}$$

[656e]

$$\begin{aligned}\sin.\,4u=\sin.\,4nt+\frac{4e}{1.2}\,.\,\{\sin.\,5nt-\sin.\,3nt\}\\+\frac{4e^2}{1.2.2^2}\,.\,\{6\,.\,\sin.\,6nt-8\,.\,\sin.\,4nt+2\,.\,\sin.\,2nt\}+\&c.\end{aligned}$$

[656f] $\sin.\,5u=\sin.\,5nt+\frac{5e}{1.2}\,.\,\{\sin.\,6nt-\sin.\,4nt\}+\&c.$

[656g] $\sin.\,6u=\sin.\,6nt+\&c.$

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* (457) $\psi(u)=u$, [656′], hence $\psi(nt)=nt$, and $\psi'(nt)=\frac{d.\psi(nt)}{ndt}=1$, consequently the factor of P, [654′], is equal to unity. In order therefore to find P', [654″],

$$u=nt+e.\sin.nt+\frac{e^2}{1.2.2}.2.\sin.2nt+\frac{e^3}{1.2.3.2^2}.\left\{3^2.\sin.3nt-3.\sin.nt\right\}$$
$$+\frac{e^4}{1.2.3.4.2^3}.\left\{4^3.\sin.4nt-4.2^3.\sin.2nt\right\}$$ [657]
$$+\frac{e^5}{1.2.3.4.5.2^4}.\left\{5^4.\sin.5nt-5.3^4.\sin.3nt+\frac{5.4}{1.2}.\sin.nt\right\}$$
$$+\ \&c.$$

This series is very converging for the planets. Having thus determined u for any instant of time, we may deduce from it, by means of the equations [606], the corresponding values of r and v. We may also compute r and v by converging series in the following direct manner.

For this purpose, we shall observe that by § 20 [606] we have

$$r=a.(1-e.\cos.u);$$ [657′]

now if in the formula (q) [652], we suppose $\psi(u)=1-e.\cos.u$, we shall get* $\psi'(nt)=e.\sin.nt$; consequently

$$1-e.\cos.u=1-e.\cos.nt+e^2.\sin.^2nt+\frac{e^3}{1.2}.\frac{d.\sin.^3nt}{ndt}+\frac{e^4}{1.2.3}.\frac{d^2.\sin.^4nt}{n^2dt^2}+\&c.;$$ [658]

we shall therefore find by the preceding analysis†

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we must take the differential of each of the terms of P, [654], a number of times denoted by the exponent of e, in that term, and must divide by the corresponding powers of ndt. The general expression [655], $\psi(u)=\psi(nt)+eP'$, becomes, in this particular case, $u=nt+eP'$, and by substituting the value of P', found in the preceding manner, we shall obtain the value of u, [657].

* (458) Having $\psi u=1-e.\cos.u$, it gives $\psi(nt)=1-e.\cos.nt$, hence $\psi'(nt)=\frac{d.\psi(nt)}{ndt}=e.\sin.nt$, these being substituted in [652], give the value of $1-e.\cos.u$, [658].

† (459) Substituting for $\sin.^2nt$, $\sin.^3nt$, &c. their values [1—5] Int., then taking the differentials as in [658] we shall obtain the expression of $1-e.\cos.u=\frac{r}{a}$, [606], as in [659].

[659]

$$\begin{aligned}\frac{r}{a} = 1 + \frac{e^2}{2} - e\,.\cos.\,n\,t - \frac{e^2}{2}\,.\cos.\,2\,n\,t \\ - \frac{e^3}{1.2.2^2}\,.\left\{3.\cos.\,3\,n\,t - 3\,.\cos.\,n\,t\right\} \\ - \frac{e^4}{1.2.3.2^3}\,.\left\{4^2.\cos.\,4\,n\,t - 4.2^2.\cos.\,2\,n\,t\right\} \\ - \frac{e^5}{1.2.3.4.2^4}\,.\left\{5^3.\cos.\,5nt - 5.3^3.\cos.\,3nt + \frac{5.4}{1.2}.\cos.\,nt\right\} \\ - \frac{e^6}{1.2.3.4.5.2^5}\,.\left\{6^4.\cos.\,6nt - 6.4^4.\cos.\,4nt + \frac{6.5}{1.2}.2^4.\cos.\,2nt\right\} \\ - \&c.\end{aligned}$$

We shall now consider the third equation (*f*) § 20 [606]; it gives

[660]

$$\frac{\sin.\,\frac{1}{2}v}{\cos.\,\frac{1}{2}v} = \sqrt{\frac{1+e}{1-e}}\,.\,\frac{\sin.\,\frac{1}{2}u}{\cos.\,\frac{1}{2}u}.$$

Substituting in this equation, the exponential values of the sines and cosines, we shall find*

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* (460) Substituting the exponential values of $\sin.\,\frac{1}{2}v$, $\cos.\,\frac{1}{2}v$, $\sin.\,\frac{1}{2}u$, $\cos.\,\frac{1}{2}u$, [11, 12] Int. in $\frac{2.\sqrt{-1}.\sin.\,\frac{1}{2}v}{2.\cos.\,\frac{1}{2}v} = \sqrt{\frac{1+e}{1-e}}\,.\,\frac{2.\sqrt{-1}.\sin.\,\frac{1}{2}u}{2.\cos.\,\frac{1}{2}u}$, deduced from [660], it becomes $\frac{c^{\frac{1}{2}v.\sqrt{-1}} - c^{-\frac{1}{2}v.\sqrt{-1}}}{c^{\frac{1}{2}v.\sqrt{-1}} + c^{-\frac{1}{2}v.\sqrt{-1}}} = \sqrt{\frac{1+e}{1-e}}\,.\,\frac{c^{\frac{1}{2}u.\sqrt{-1}} - c^{-\frac{1}{2}u.\sqrt{-1}}}{c^{\frac{1}{2}u.\sqrt{-1}} + c^{-\frac{1}{2}u.\sqrt{-1}}}$. Multiplying the numerator and denominator of the first member by $c^{\frac{1}{2}v.\sqrt{-1}}$, those of the second by $c^{\frac{1}{2}u.\sqrt{-1}}$, we shall obtain [661]. Now the value of λ, [662], being the same as in [536] we get by [536*b*], $\sqrt{\frac{1+e}{1-e}} = \frac{1+\lambda}{1-\lambda}$, which being substituted in [661] we get $\frac{c^{v.\sqrt{-1}} - 1}{c^{v.\sqrt{-1}} + 1} = \frac{c^{u.\sqrt{-1}} - \lambda - 1 + \lambda c^{u.\sqrt{-1}}}{c^{u.\sqrt{-1}} - \lambda + 1 - \lambda c^{u.\sqrt{-1}}}$. From each member of this equation, we may deduce new fractional expressions, by adding the numerator and denominator for a new numerator, and subtracting the numerator from the denominator, for a new denominator; by this means the expression becomes $\frac{2c^{v.\sqrt{-1}}}{2} = \frac{2c^{u.\sqrt{-1}} - 2\lambda}{2 - 2\lambda.c^{u.\sqrt{-1}}}$, or $c^{v.\sqrt{-1}} = \frac{c^{u.\sqrt{-1}} - \lambda}{1 - \lambda.c^{u.\sqrt{-1}}} = c^{u.\sqrt{-1}}.\left\{\frac{1 - \lambda.c^{-u.\sqrt{-1}}}{1 - \lambda.c^{u.\sqrt{-1}}}\right\}$, as in [663]. Taking the logarithms of both sides, and dividing by $\sqrt{-1}$, we shall obtain [664].

$$\frac{c^{v.\sqrt{-1}}-1}{c^{v.\sqrt{-1}}+1}=\sqrt{\frac{1+e}{1-e}}.\left\{\frac{c^{u.\sqrt{-1}}-1}{c^{u.\sqrt{-1}}+1}\right\};$$ [661]

supposing therefore

$$\lambda=\frac{e}{1+\sqrt{1-e^2}};$$ [662]

we shall have

$$c^{v.\sqrt{-1}}=c^{u.\sqrt{-1}}.\left\{\frac{1-\lambda.c^{-u.\sqrt{-1}}}{1-\lambda.c^{u.\sqrt{-1}}}\right\};$$ [663]

consequently

$$v=u+\frac{\log.\left(1-\lambda.c^{-u.\sqrt{-1}}\right)-\log.\left(1-\lambda.c^{u.\sqrt{-1}}\right)}{\sqrt{-1}},$$ [664]

hence, by reducing the logarithms into series, we shall obtain[*]

$$v=u+2\lambda.\sin.u+\frac{2\lambda^2}{2}.\sin.2u+\frac{2\lambda^3}{3}.\sin.3u+\frac{2\lambda^4}{4}.\sin.4u+\&c.$$ [665]

We shall have, by what precedes, [657, 656a—g], u, $\sin.u$, $\sin.2u$, &c., in series arranged according to the powers of e, and developed in sines and cosines of the angle nt and its multiples; all that is now required to obtain v by a similar series, is to arrange the successive powers of λ, in a series proceeding according to the powers of e.

The equation[†] $u=2-\frac{e^2}{u}$, will give, by the formula (p) [629] of the [665']

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[*] (461) By [58] Int. $\log.(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$, &c. hence
$\log.\left(1-\lambda.c^{-u.\sqrt{-1}}\right)=-\lambda.c^{-u.\sqrt{-1}}-\frac{1}{2}.\lambda^2.c^{-2u.\sqrt{-1}}-\frac{1}{3}.\lambda^3.c^{-3u.\sqrt{-1}}-\&c.$
$-\log.\left(1-\lambda.c^{u.\sqrt{-1}}\right)=\lambda.c^{u.\sqrt{-1}}+\frac{1}{2}.\lambda^2.c^{2u.\sqrt{-1}}+\frac{1}{3}.\lambda^3.c^{3u.\sqrt{-1}}+\&c.$
Their sum is

$$\lambda.\left(c^{u.\sqrt{-1}}-c^{-u.\sqrt{-1}}\right)+\tfrac{1}{2}.\lambda^2.\left(c^{2u.\sqrt{-1}}-c^{-2u.\sqrt{-1}}\right)$$
$$+\tfrac{1}{3}.\lambda^3.\left(c^{3u.\sqrt{-1}}-c^{-3u.\sqrt{-1}}\right)+\&c.$$

and by [11] Int. this becomes $2.\sqrt{-1}.(\lambda.\sin.u+\frac{1}{2}.\lambda^2.\sin.2u+\frac{1}{3}.\lambda^3.\sin.3u+\&c.)$ hence [664] changes into [665].

[†] (462) This equation multiplied by u, is $u^2=2u-ee$, hence $u=1+\sqrt{1-e^2}$, [665a]
as in [666']. Substituting this in [662] we get $\lambda=\frac{e}{u}$, hence $\lambda^i=\frac{e^i}{u^i}$. Therefore to [666$a$]

preceding article,

[666] $$\frac{1}{u^i}=\frac{1}{2^i}+\frac{i\,.\,e^2}{2^{i+2}}+\frac{i.(i+3)}{1.2}\cdot\frac{e^4}{2^{i+4}}+\frac{i.(i+4).(i+5)}{1.2.3}\cdot\frac{e^6}{2^{i+6}}+\&c.\,;$$

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obtain λ^i in a series arranged according to the powers of e, it is necessary to find $\frac{1}{u^i}$, arranged according to the same powers of e, which may be done by means of the formula [652a]. For by comparing $u=2-\frac{e^2}{u}$, [665′], with $x=\varphi(t'+\alpha z)$, [652b], we find $x=u$, $t'=2$, $\alpha=e^2$, $z=-\frac{1}{u}$, $\mathrm{u}=2$, and $\varphi(t'+\alpha z)$, becomes $t'+\alpha z=2-\frac{e^2}{u}=u$. Now putting in [652$a$] $u'=u^{-i}$, it will become

$$u^{-i}=\mathrm{u}^{-i}-e^2.\frac{1}{\mathrm{u}}\cdot\frac{d.\mathrm{u}^{-i}}{dt'}+\frac{e^4}{1.2}\cdot\frac{d.\left(\mathrm{u}^{-2}.\frac{d.\mathrm{u}^{-i}}{dt'}\right)}{dt'}-\frac{e^6}{1.2.3}\cdot\frac{d^2.\left(\mathrm{u}^{-3}.\frac{d.\mathrm{u}^{-i}}{dt'}\right)}{dt'^2}+\&c.$$

and since $u=t'+\alpha z$, we shall have $\frac{d\mathrm{u}}{dt'}=1$, therefore $\frac{d.\mathrm{u}^{-i}}{dt'}=-i.\mathrm{u}^{-i-1}$,

consequently $u^{-i}=\mathrm{u}^{-i}+i\,e^2.\mathrm{u}^{-i-2}-\frac{i\,e^4}{1.2}\cdot\frac{d.(\mathrm{u}^{-i-3})}{dt'}+\frac{i\,e^6}{1.2.3}\cdot\frac{d^2.(\mathrm{u}^{-i-4})}{dt'^2}-\&c.$

Developing the differentials indicated in the formula, it changes into

$$u^{-i}=\mathrm{u}^{-i}+i\,e^2.\mathrm{u}^{-i-2}+\frac{i.(i+3)}{1.2}.e^4.\mathrm{u}^{-i-4}+\frac{i.(i+4).(i+5)}{1.2.3}.e^6.\mathrm{u}^{-i-6}+\&c.$$

Which, by putting $\mathrm{u}=2$, becomes as in [666]. This value of u^{-i} being substituted in [666a], $\lambda^i=\frac{e^i}{u^i}$, it becomes as in [667]. It may be observed that the factor $(i+4)$, in [666, 667], was by a small mistake in the original written $(i+3)$. The correctness of the present form is easily verified by examining a simple case, as for example, when $i=1$, corresponding to $\lambda=\frac{e}{1+\sqrt{1-ee}}$. Developing the denominator by the usual rule for extracting the square root, it becomes $2-\frac{1}{2}e^2-\frac{1}{8}e^4-\frac{1}{16}e^6$. Dividing e by this, by the usual method of division, we get $\lambda=\frac{1}{2}e+\frac{1}{8}e^3+\frac{1}{16}e^5+\frac{5}{128}e^7$, in which the coefficient of e^7 is $\frac{5}{128}$. This agrees with the corresponding term of the formula [667], $\frac{e^i}{2^i}\cdot\frac{i.(i+4).(i+5)}{1.2.3}\cdot\left(\frac{e}{2}\right)^6$, by putting $i=1$, whereas $\frac{e^i}{2^i}\cdot\frac{i.(i+3).(i+5)}{1.2.3}\cdot\left(\frac{e}{2}\right)^6$, would be $\frac{4}{128}e^7$. If in the expression of λ^i, [667], we put successively $i=1, 2, 3, 4, 5, 6$, we shall get

[667a]

$$\lambda=\tfrac{1}{2}e+\frac{e^3}{8}+\frac{e^5}{16}+\&c.;\qquad \lambda^2=\frac{e^2}{4}+\frac{e^4}{8}+\frac{5e^6}{64}+\&c.;\qquad \lambda^3=\frac{e^3}{8}+\frac{3e^5}{32}+\&c.;$$

$$\lambda^4=\frac{e^4}{16}+\frac{e^6}{16}+\&c.;\qquad \lambda^5=\frac{e^5}{32}+\&c.;\qquad \lambda^6=\frac{e^6}{64}+\&c.$$

and as $u=1+\sqrt{1-e^2}$ [665a], we shall have [666′]

$$\lambda^i=\frac{e^i}{2^i}\cdot\left\{1+i\cdot\left(\frac{e}{2}\right)^2+\frac{i.(i+3)}{1.2}\cdot\left(\frac{e}{2}\right)^4+\frac{i.(i+4).(i+5)}{1.2.3}\cdot\left(\frac{e}{2}\right)^6+\&c.\right\} \qquad [667]$$

This being premised, we shall find, by continuing the approximation to terms of the order e^6 inclusively,*

$$\begin{aligned}v=nt&+\left\{2e-\frac{1}{4}.e^3+\frac{5}{96}.e^5\right\}.\sin.nt+\left\{\frac{5}{4}.e^2-\frac{11}{24}.e^4+\frac{17}{192}.e^6\right\}.\sin.2nt\\&+\left\{\frac{13}{12}.e^3-\frac{43}{64}.e^5\right\}.\sin.3nt+\left\{\frac{103}{96}.e^4-\frac{451}{480}.e^6\right\}.\sin.4nt\\&+\frac{1097}{960}.e^5.\sin.5nt+\frac{1223}{960}.e^6.\sin.6nt.\end{aligned} \qquad [668]$$

The angles v and $n\,t$ are here counted from the perihelion; but if we wish to count them from the aphelion, we must evidently make e negative in [668′] the preceding expressions of r and v. The same result might also be obtained by increasing the angle $n\,t$ by two right angles, which would render the sines and cosines of the odd multiples of $n\,t$ negative; now since the [668″] results of both methods ought to be identical in the values of r and v, it is necessary that the sines and cosines of the odd multiples of $n\,t$, should be [668‴] multiplied by odd powers of e, and that the sines and cosines of the even multiples of the same angle, should be multiplied by even powers of that quantity. Which is confirmed by calculation *a posteriori.*

Suppose that instead of counting the angle v from the perihelion, we fix its origin at any other point whatever; it is evident that this angle would be increased by a constant quantity, which we shall denote by ϖ, and this will [668^{iv}] express the longitude of the perihelion. Instead of fixing the origin of t, at the instant of passing the perihelion, if we fix it at any other instant, the angle $n\,t$ will be increased by a constant quantity, which we shall denote by $\varepsilon-\varpi$; the preceding expressions of $\frac{r}{a}$, and v, will thus become [668^{v}]

$$\begin{aligned}\frac{r}{a}&=1+\tfrac{1}{2}e^2-(e-\tfrac{3}{8}e^3).\cos.(nt+\varepsilon-\varpi)-(\tfrac{1}{2}e^2-\tfrac{1}{3}e^4).\cos.2(nt+\varepsilon-\varpi)-\&c.;\\v&=nt+\varepsilon+(2e-\tfrac{1}{4}e^3).\sin.(nt+\varepsilon-\varpi)+(\tfrac{5}{4}e^2-\tfrac{11}{24}e^4).\sin.2(nt+\varepsilon-\varpi)+\&c.;\end{aligned} \qquad [669]$$

---

* (463) If in the expression of v, [665], we substitute the value of u, [657], those of sin. u, sin. 2 u, &c. [656b—g], and λ, λ^2, λ^3, &c. [667a], it becomes, by placing the terms in the same order as they occur in these formulas,

[669′] v is the true longitude of the planet, and $nt+\varepsilon$ its mean longitude, these longitudes being counted upon the plane of the orbit.

[688a]

$$
\begin{array}{l}
v=\left(\begin{array}{l}
nt+e\,.\,\sin.nt+\dfrac{e^2}{1.2.2}\,.\,\sin.2nt+\dfrac{e^3}{1.2.3.2^2}\,.\,\{3^2\,.\,\sin.3nt-3\sin.nt\} \\
\quad+\dfrac{e^4}{1.2.3.4.2^3}\,.\,\{4^3\,.\,\sin.4nt-4\,.\,2^3\,.\,\sin.2nt\} \\
\quad+\dfrac{e^5}{1.2.3.4.5.2^4}\,.\,\{5^4\,.\,\sin.5nt-5\,.\,3^4\,.\,\sin.3nt+\dfrac{5.4}{1.2}\,.\,\sin.nt\} \\
\quad+\dfrac{e^6}{1.2.3.4.5.6.2^5}\,.\,\{6^5\,.\,\sin.6nt-6\,.\,4^5\,.\,\sin.4nt+\dfrac{6.5}{1.2}\,.\,2^5\,.\,\sin.2nt\}
\end{array}\right) \\[2ex]
+2\,.\,\{\tfrac{1}{2}e+\tfrac{1}{8}e^3+\tfrac{1}{16}e^5\}\,.\left(\begin{array}{l}
\sin.nt+\dfrac{e}{1.2}\,.\,\sin.2nt+\dfrac{e^2}{1.2.2^2}\,.\,\{3\sin.3nt-\sin.nt\} \\
\qquad+\dfrac{e^3}{1.2.3.2^3}\,.\,\{4^2\,.\,\sin.4nt-8\sin.2nt\} \\
\qquad+\dfrac{e^4}{1.2.3.4.2^4}\,.\,\{5^3.\sin.5nt-3.3^3.\sin.3nt+2\sin.nt\} \\
\qquad+\dfrac{e^5}{1.2.3.4.5.2^5}\,.\,\{6^4.\sin.6nt-4.4^4.\sin.4nt+5.2^4.\sin.2nt\}
\end{array}\right) \\[2ex]
+\tfrac{2}{2}\,.\,\{\tfrac{1}{4}e^2+\tfrac{1}{8}e^4+\tfrac{5}{64}e^6\}\,.\left(\begin{array}{l}
\sin.2nt+\dfrac{2e}{1.2}\,.\,\{\sin.3nt-\sin.nt\} \\
\qquad+\dfrac{2e^2}{1.2.2^2}\,.\,\{4\,.\,\sin.4nt-4\,.\,\sin.2nt\} \\
\qquad+\dfrac{2e^3}{1.2.3.2^3}\,.\,\{5^2.\sin.5nt-3\,.\,3^2.\sin.3nt+4\,.\,\sin.nt\} \\
\qquad+\dfrac{2e^4}{1.2.3.4.2^4}\,.\,\{6^3.\sin.6nt-4.4^3.\sin.4nt+7.2^3.\sin.2nt\}
\end{array}\right) \\[2ex]
+\tfrac{2}{3}\,.\,\{\tfrac{1}{8}e^3+\tfrac{3}{32}e^5\}\,.\left(\begin{array}{l}
\sin.3nt+\dfrac{3e}{1.2}\,.\,\{\sin.4nt-\sin.2nt\} \\
\qquad+\dfrac{3e^2}{1.2.2^2}\,.\,\{5\,.\,\sin.5nt-6\,.\,\sin.3nt+\sin.nt\} \\
\qquad+\dfrac{3e^3}{1.2.3.2^3}\,.\,\{6^2\,.\,\sin.6nt-3\,.\,4^2\,.\,\sin.4nt+3\,.\,2^2\,.\,\sin.2nt\}
\end{array}\right) \\[2ex]
+\tfrac{2}{4}\,.\,\{\tfrac{1}{16}e^4+\tfrac{1}{16}e^6\}\,.\left\{\begin{array}{l}
\sin.4nt+\dfrac{4e}{1.2}\,.\,\{\sin.5nt-\sin.3nt\} \\
\qquad+\dfrac{4e^2}{1.2.2^2}\,.\,\{\sin.6nt-8\,.\,\sin.4nt+2\,.\,\sin.2nt\}
\end{array}\right\} \\[2ex]
+\tfrac{2}{5}\,.\,\{\tfrac{1}{32}e^5\}\,.\left\{\sin.5nt+\dfrac{5e}{1.2}\,.\,\{\sin.6nt-\sin.4nt\}\right\} \\[2ex]
+\tfrac{2}{6}\,.\,\{\tfrac{1}{64}e^6\}\,.\,\{\sin.6nt\}-\&c.
\end{array}
$$

We shall now refer the motion of the planet to a fixed plane, which is inclined by a small angle to the plane of the orbit. Put*

$\varphi =$ the angle of inclination of the two planes ;
$\theta =$ the longitude of the ascending node of the orbit, counted upon the fixed plane ; [669″]
$\beta =$ the longitude of the ascending node of the orbit, counted upon the plane of the orbit, so that θ may be the projection of β ;
$v_{,} =$ the projection of v upon the fixed plane ;

then we shall have

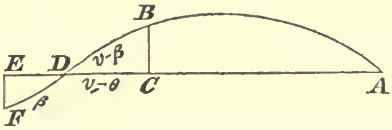
$$\text{tang.}\,(v_{,} - \theta) = \cos.\,\varphi\,.\,\text{tang.}\,(v - \beta). \qquad [670]$$

This equation will give $v_{,}$ in v and the contrary ; we may also obtain these angles by very converging series, as follows.

We have before deduced the series [665]†

---

connecting together the terms depending on $\sin. n\,t$, $\sin. 2\,n\,t$, &c. and making the necessary reductions we shall obtain the expression of v, [668]. This method appears long, but it is incomparably more simple and easy than the method formerly used by astronomers, as explained in La Lande's Astronomy. Several astronomers have calculated these series as far as e^{11} or e^{12}, and Schubert has shown how to calculate any one term of the series, independent of the rest.



* (464) Suppose a spherical surface to be described with the radius 1, about the focus, which is the origin of r, as its centre, and let the intersection of this surface by the plane of the orbit, and the fixed plane, be represented respectively by the great circles $FDBA$, $EDCA$. B being the place of the planet, C its projection on the fixed plane, D the place of the node, F the point from which the longitudes are counted, and E its projection on the fixed plane ; the angle $BDC = EDF = \varphi$; $FD = \beta$; $ED = \theta$; $FB = v$; $EC = v_{,}$; $DB = v - \beta$; $DC = v_{,} - \theta$. Then in the right-angled spherical triangles DEF, BCD, we shall have $\text{tang.}\,DC = \cos.\,BDC\,.\,\text{tang.}\,DB$; and $\text{tang.}\,DE = \cos.\,EDF\,.\,\text{tang.}\,FD$, which in symbols become

$$\text{tang.}\,(v_{,} - \theta) = \cos.\,\varphi\,.\,\text{tang.}\,(v - \beta), \qquad \text{and} \qquad \text{tang.}\,\theta = \cos.\,\varphi\,.\,\text{tang.}\,\beta, \qquad [670a]$$

as above.

† (465) This is the equation [665] divided by 2.

[671] $$\tfrac{1}{2}v=\tfrac{1}{2}u+\lambda\,.\,\sin.\,u+\frac{\lambda^2}{2}\,.\,\sin.\,2u+\frac{\lambda^3}{3}\,.\,\sin.\,3u+\&c.,$$

from the equation [606]

[672] $$\text{tang.}\,\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\,.\,\text{tang.}\,\tfrac{1}{2}u,$$

by putting*

[673] $$\lambda=\frac{\sqrt{\frac{1+e}{1-e}}-1}{\sqrt{\frac{1+e}{1-e}}+1}.$$

[673′] If we change $\tfrac{1}{2}v$ into $v_{,}-\theta$; $\tfrac{1}{2}u$ into $v-\beta$; and $\sqrt{\frac{1+e}{1-e}}$ into $\cos.\varphi$; we shall have†

[674] $$\lambda=\frac{\cos.\varphi-1}{\cos.\varphi+1}=-\,\text{tang.}^2\tfrac{1}{2}\varphi\,;$$

the equation between $\tfrac{1}{2}v$ and $\tfrac{1}{2}u$, will change into an equation between $v_{,}-\theta$ and $v-\beta$, and the preceding series [671] will become

[675] $$v_{,}-\theta=v-\beta-\text{tang.}^2\tfrac{1}{2}\varphi\,.\,\sin.\,2\,(v-\beta)+\tfrac{1}{2}\,.\,\text{tang.}^4\tfrac{1}{2}\varphi\,.\,\sin.\,4\,(v-\beta)$$
$$-\tfrac{1}{3}\,.\,\text{tang.}^6\tfrac{1}{2}\varphi\,.\,\sin.\,6\,(v-\beta)+\&c.$$

If in the equation between $\tfrac{1}{2}v$ and $\tfrac{1}{2}u$, we change $\tfrac{1}{2}v$ into $v-\beta$, $\tfrac{1}{2}u$ into

[675′] $v_{,}-\theta$, and $\sqrt{\frac{1+e}{1-e}}$ into $\frac{1}{\cos.\varphi}$,‡ we shall find

[676] $$\lambda=\text{tang.}^2\tfrac{1}{2}\varphi,$$

---

* (466) The value of λ, [536, 662], is in [536*d*] reduced to the form [673].

† (467) These changes of $\tfrac{1}{2}v$ into $v_{,}-\theta$, $\tfrac{1}{2}u$ into $v-\beta$, and $\sqrt{\frac{1+e}{1-e}}$ into $\cos.\varphi$, being made in [672], it will become as in [670]; the same changes being made in λ [673], will produce the first value of λ, [674]; its second value being deduced from the first by means of [40] Int. By the same process [671] will produce [675].

‡ (468) Dividing [670] by $\cos.\varphi$, we get $\text{tang.}\,(v-\beta)=\frac{\text{tang.}\,(v_{,}-\theta)}{\cos.\varphi}$, which might be derived from [672] by changing in this last expression $\tfrac{1}{2}v$ into $v-\beta$, $\tfrac{1}{2}u$ into $v_{,}-\theta$, and $\sqrt{\frac{1+e}{1-e}}$ into $\frac{1}{\cos.\varphi}$; and then [673] would become

and

$$v-\beta=v_{,}-\theta+\text{tang.}^2\tfrac{1}{2}\varphi\,.\,\text{sin.}\,2\,(v_{,}-\theta)+\tfrac{1}{2}\,\text{tang.}^4\tfrac{1}{2}\varphi\,.\,\text{sin.}^4\,(v_{,}-\theta)$$
$$+\tfrac{1}{3}\,.\,\text{tang.}^6\,\tfrac{1}{2}\varphi\,.\,\text{sin.}\,6\,(v_{,}-\theta)+\&\text{c}.$$ [676′]

We thus see that the two preceding series [675, 676′], mutually change into each other, by altering the sign of $\text{tang.}^2\,\tfrac{1}{2}\varphi$, and writing $v_{,}-\theta$, for $v-\beta$, and the contrary. We shall have $v_{,}-\theta$, in a function of sines and cosines [676″] of $n\,t$ and its multiples, observing that by what precedes*

$$v=n\,t+\varepsilon+e\,Q,$$ [677]

Q being a function of sines of the angle $n\,t+\varepsilon-\varpi$ and its multiples; and the formula (i) § 21 [617] gives, for any value of i,†

$$\text{sin.}\,i\,(v-\beta)=\text{sin.}\,i\,(n\,t+\varepsilon-\beta+e\,Q)$$
$$=\left\{1-\frac{i^2e^2.Q^2}{1.2}+\frac{i^4e^4.Q^4}{1.2.3.4}-\&\text{c.}\right\}.\,\text{sin.}\,i\,(n\,t+\varepsilon-\beta)$$ [678]
$$+\left\{i\,e\,.\,Q-\frac{i^3e^3\,.\,Q^3}{1.2.3}+\frac{i^5e^5\,.\,Q^5}{1.2.3.4.5}-\&\text{c.}\right\}.\,\text{cos.}\,i\,(n\,t+\varepsilon-\beta).$$

Lastly, s being the tangent of the latitude of the planet, above the fixed [678′]

$$\lambda=\frac{\frac{1}{\text{cos.}\,\varphi}-1}{\frac{1}{\text{cos.}\,\varphi}+1}=\frac{1-\text{cos.}\,\varphi}{1+\text{cos.}\,\varphi}=\text{tang.}^2\,\tfrac{1}{2}\varphi,$$

[40] Int., as in [676]. These changes being made in [671] it becomes as in [676′].

* (469) This is evident from the series [668] altered as in [669] and substituted in [675].

† (470) Writing τ for t in [617], to distinguish it from the time t, used in this article, and for $\varphi\,(\tau+\alpha)$ putting $\text{sin.}\,(\tau+\alpha)$, we shall get

$$\text{sin.}\,(\tau+\alpha)=\text{sin.}\,\tau+\alpha\,.\,\frac{d\,.\,\text{sin.}\,\tau}{d\tau}+\frac{\alpha^2}{1.2}\,.\,\frac{d^2\,.\,\text{sin.}\,\tau}{d\,\tau^2}+\frac{\alpha^3}{1.2.3}\,.\,\frac{d^3\,.\,\text{sin.}\,\tau}{d\,\tau^3}+\&\text{c.}$$
$$=\left\{1-\frac{\alpha^2}{1.2}+\frac{\alpha^4}{1.2.3.4}-\&\text{c.}\right\}.\,\text{sin.}\,\tau+\left\{\alpha-\frac{\alpha^3}{1.2.3}+\frac{\alpha^5}{1.2.3.4.5}-\&\text{c.}\right\}.\,\text{cos.}\,\tau,$$

which by substituting $\tau=i\,(n\,t+\varepsilon-\beta)$, $\alpha=i\,e\,Q$, becomes as in [678]. This may [678a] also be obtained from [21] Int. $\text{sin.}\,(\tau+\alpha)=\text{cos.}\,\alpha\,.\,\text{sin.}\,\tau+\text{sin.}\,\alpha\,.\,\text{cos.}\,\tau$, for by substituting the values of $\text{sin.}\,\alpha$, $\text{cos.}\,\alpha$, deduced from [43, 44] Int. it becomes as in [678].

plane, we shall have*

[679] $$s = \text{tang.}\,\varphi\,.\,\text{sin.}\,(v_{,} - \theta)\,;$$

[679′] and if we put $r_{,}$ for the projection of the radius vector r upon the fixed plane, we shall have†

[680] $$r_{,} = r\,.\,(1+s^2)^{-\frac{1}{2}} = r\,.\,\{1 - \tfrac{1}{2}\,.\,s^2 + \tfrac{3}{8}\,.\,s^4 - \&c.\}\,;$$

we may thus determine $v_{,}$, s, and $r_{,}$, by converging series of sines and cosines of the angle $n\,t$ and its multiples.

23. Let us now consider very excentrical orbits, like those of the comets. For this purpose we shall resume the equations of § 20 [603, 606].

$$r = \frac{a\,.\,(1-e^2)}{1+e\,.\,\text{cos.}\,v}\,;$$

[681] $$n\,t = u - e\,.\,\text{sin.}\,u\,;$$

$$\text{tang.}\,\tfrac{1}{2}\,v = \sqrt{\frac{1+e}{1-e}}\,.\,\text{tang.}\,\tfrac{1}{2}\,u.$$

In the case of very excentrical orbits, e differs but very little from unity; we shall therefore suppose

[681′] $$1 - e = \alpha,$$

α being very small. If we put D for the perihelion distance of the comet, we shall have

[681″] $$D = a\,.\,(1-e) = \alpha\,a\,;$$

the expression of r will therefore become‡

[682] $$r = \frac{(2-\alpha)\,.\,D}{2\,.\,\text{cos.}^2\,\frac{1}{2}\,v - \alpha\,.\,\text{cos.}\,v} = \frac{D}{\text{cos.}^2\frac{1}{2}\,v\,.\left\{1 + \frac{\alpha}{2-\alpha}\,.\,\text{tang.}^2\,\frac{1}{2}\,v\right\}}\,;$$

---

* (471) Referring to the figure in page 379 we shall have in the right angled spherical triangle $B\,C\,D$, $\text{tang.}\,B\,C = \text{tang.}\,B\,D\,C\,.\,\text{sin.}\,D\,C$, or by the symbols used in that [679a] note $s = \text{tang.}\,\varphi\,.\,\text{sin.}\,(v_{,} - \theta)$.

† (472) In the figure page 351, we have $C\,D = r$, $C\,B = r_{,}$, $\text{tang.}\,B\,C\,D = s$, or $\text{cos.}\,B\,C\,D = \frac{1}{\sqrt{1+s^2}} = (1+s^2)^{-\frac{1}{2}}$, and it is evident that $C\,B = C\,D.\text{cos.}\,B\,C\,D$, hence $r_{,} = r\,.\,(1+s^2)^{-\frac{1}{2}}$, as in [680]. Developing $(1+s^2)^{-\frac{1}{2}}$, by the binomial theorem we shall obtain the second formula [680].

‡ (473) Substitute $e = 1 - \alpha$, [681′] in r [681] and it becomes $r = \frac{a\,\alpha\,.\,(2-\alpha)}{1+\text{cos.}\,v - \alpha\,.\,\text{cos.}\,v}$,

which reduced to a series, gives

$$r=\frac{D}{\cos.^2\tfrac{1}{2}v}\cdot\left\{1-\frac{\alpha}{2-\alpha}\cdot\text{tang.}^2\tfrac{1}{2}v+\left(\frac{\alpha}{2-\alpha}\right)^2\cdot\text{tang.}^4\tfrac{1}{2}v-\&\text{c.}\right\}.$$ [683]

To obtain the ratio of v to the time t, we shall observe, that the expression of an arch by its tangent gives*

$$u=2\,.\,\text{tang.}\tfrac{1}{2}u\,.\{1-\tfrac{1}{3}\,.\,\text{tang.}^2\tfrac{1}{2}u+\tfrac{1}{5}\text{tang.}^4\tfrac{1}{2}u-\&\text{c.}\}\,;$$ [684]

now we have

$$\text{tang.}\tfrac{1}{2}u=\sqrt{\frac{\alpha}{2-\alpha}}\,.\,\text{tang.}\tfrac{1}{2}v\,;$$ [685]

therefore we shall have

$$u=2.\sqrt{\frac{\alpha}{2-\alpha}}.\text{tang.}\tfrac{1}{2}v.\left\{1-\tfrac{1}{3}.\left(\frac{\alpha}{2-\alpha}\right).\text{tang.}^2\tfrac{1}{2}v+\tfrac{1}{5}.\left(\frac{\alpha}{2-\alpha}\right)^2.\text{tang.}^4\tfrac{1}{2}v-\&\text{c.}\right\};$$ [686]

moreover†

$$\sin.u=\frac{2\,.\,\text{tang.}\tfrac{1}{2}u}{1+\text{tang.}^2\tfrac{1}{2}u}=2.\text{tang.}\tfrac{1}{2}u.\{1-\text{tang.}^2\tfrac{1}{2}u+\text{tang.}^4\tfrac{1}{2}u-\&\text{c.}\}\,;$$ [687]

hence we deduce‡

$$e.\sin.u=2.(1-\alpha).\sqrt{\frac{\alpha}{2-\alpha}}.\text{tang.}\tfrac{1}{2}v.\left\{1-\left(\frac{\alpha}{2-\alpha}\right).\text{tang.}^2\tfrac{1}{2}v+\left(\frac{\alpha}{2-\alpha}\right)^2.\text{tang.}^4\tfrac{1}{2}v-\&\text{c.}\right\}$$ [688]

---

but $a\alpha=D$, [681″], $\cos.v=\cos.^2\tfrac{1}{2}v-\sin.^2\tfrac{1}{2}v$, [32] Int. and $1+\cos.v=2\,.\cos.^2\tfrac{1}{2}v$, [6] Int.; hence by substitution

$$r=\frac{D.(2-\alpha)}{2\cos.^2\tfrac{1}{2}v-\alpha.(\cos.^2\tfrac{1}{2}v-\sin.^2\tfrac{1}{2}v)}=\frac{D.(2-\alpha)}{(2-\alpha).\cos.^2\tfrac{1}{2}v+\alpha.\sin.^2\tfrac{1}{2}v}$$

$$=\frac{D}{\cos.^2\tfrac{1}{2}v+\frac{\alpha}{2-\alpha}.\sin.^2\tfrac{1}{2}v}=\frac{D}{\cos.^2\tfrac{1}{2}v.\left\{1+\frac{\alpha}{2-\alpha}.\text{tang.}^2\tfrac{1}{2}v\right\}},$$

as in [682].

* (474) Put $z=\tfrac{1}{2}u$, and $t=\text{tang.}\tfrac{1}{2}u$, in [48] Int. and we shall get [684]. Again $\text{tang.}\tfrac{1}{2}u=\sqrt{\frac{1-e}{1+e}}.\text{tang.}\tfrac{1}{2}v$, [681], and $\frac{1-e}{1+e}=\frac{\alpha}{2-\alpha}$, hence we obtain [685]. Substitute this in [684], it becomes as in [686].

† (475) The first expression of $\sin.u$, [687], is easily deduced from [30″] Int. This being developed in series becomes like the second formula [687].

‡ (476) Multiplying the first member of the expression [687] by e, and the last member by its equal $1-\alpha$, then substituting for $\text{tang.}\tfrac{1}{2}u$, its value [685] we shall get [688].

Substituting these values of u and $e . \sin. u$, in the equation [681] $n t = u - e . \sin. u$; we shall have the time t, in a function of the anomaly v, by a very converging series; but before making these substitutions, we shall observe that we have, by § 20 [605′], $n = a^{-\frac{3}{2}} . \sqrt{\mu}$, and as $D = \alpha a$ [681″], we shall have

[689] $$\frac{1}{n} = \frac{D^{\frac{3}{2}}}{\alpha^{\frac{3}{2}} . \sqrt{\mu}}.$$

This being premised, we shall find[*]

[690] $$t = \frac{2 . D^{\frac{3}{2}}}{\sqrt{(2-\alpha) . \mu}} . \text{tang.} \tfrac{1}{2} v . \left\{ 1 + \frac{(\frac{2}{3} - \alpha)}{2 - \alpha} . \text{tang.}^2 \tfrac{1}{2} v - \frac{(\frac{4}{5} - \alpha) . \alpha}{(2 - \alpha)^2} . \text{tang.}^4 \tfrac{1}{2} v + \&\text{c.} \right\}$$

If the orbit is parabolical, we shall have $\alpha = 0$ [681′, 378*b*], consequently, [682, 690],

[691] $$r = \frac{D}{\cos.^2 \frac{1}{2} v};$$
$$t = \frac{D^{\frac{3}{2}} . \sqrt{2}}{\sqrt{\mu}} . \{ \text{tang.} \tfrac{1}{2} v + \tfrac{1}{3} . \text{tang.}^3 \tfrac{1}{2} v \}.$$

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[*] (477) From [605′] we get $\frac{1}{n} = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}$, and by means of $a = \frac{D}{\alpha}$, [681″], it becomes as in [689]. Substitute this in $t = \frac{u - e . \sin. u}{n}$, [681], and we shall get $t = \frac{D^{\frac{3}{2}}}{\alpha^{\frac{3}{2}} . \sqrt{\mu}} . \left\{ u - e . \sin. u \right\}$. From u [686], and $-e . \sin. u$, [688], put under the following form

$$-e . \sin. u = 2 . \sqrt{\frac{\alpha}{2 - \alpha}} . \text{tang.} \tfrac{1}{2} v . \left\{ \begin{array}{l} -(1-\alpha) + (1-\alpha) . \left(\frac{\alpha}{2-\alpha}\right) . \text{tang.}^2 \tfrac{1}{2} v \\ \qquad - (1-\alpha) . \left(\frac{\alpha}{2-\alpha}\right)^2 . \text{tang.}^4 \tfrac{1}{2} v + \&\text{c.} \end{array} \right\},$$

we shall get

$$u - e . \sin. u = 2 . \sqrt{\frac{\alpha}{2 - \alpha}} . \text{tang.} \tfrac{1}{2} v . \left\{ \begin{array}{l} \alpha + (\frac{2}{3} - \alpha) . \left(\frac{\alpha}{2-\alpha}\right) . \text{tang.}^2 \tfrac{1}{2} v \\ - (\frac{4}{5} - \alpha) . \left(\frac{\alpha}{2-\alpha}\right)^2 . \text{tang.}^4 \tfrac{1}{2} v + \&\text{c.} \end{array} \right\}$$

$$= 2 . \frac{\alpha^{\frac{3}{2}}}{\sqrt{2-\alpha}} . \text{tang.} \tfrac{1}{2} v . \left\{ 1 + \frac{(\frac{2}{3} - \alpha)}{2 - \alpha} . \text{tang.}^2 \tfrac{1}{2} v - \frac{(\frac{4}{5} - \alpha) . \alpha}{(2-\alpha)^2} . \text{tang.}^4 \tfrac{1}{2} v + \&\text{c.} \right\}$$

Substituting this in the preceding expression of t, it will become as in [690].

The time t, the distance D, and the sum of the masses of the sun and comet [691']
μ, are heterogeneous quantities, and to compare them with each other they
ought to be divided by the unity of measure of each species. We shall
therefore suppose that the mean distance of the sun from the earth is *the
unity of distance*, and that D is expressed in parts of that distance. Then if [691'']
we put T for the time of a sidereal revolution of the earth, supposing it to [691''']
commence at the perihelion, we shall have, in the equation $nt = u - e . \sin. u$
[681] $u = 0$ at the commencement of the revolution, and $u = 2\pi$ at the
end; π being the semi-circumference of a circle whose radius is unity. We
shall therefore have $nT = 2\pi$; but we have $n = a^{-\frac{3}{2}} . \sqrt{\mu} = \sqrt{\mu}$ [605']; [691^iv]
because $a = 1$ [691'']; therefore

$$\sqrt{\mu} = \frac{2\pi}{T}. \qquad [692]$$

The value of μ is not exactly the same for the earth as for the comet; since
in the first case, it expresses the sum of the masses of the sun and earth; [692']
whereas in the second case it expresses the sum of the masses of the sun
and comet; but the masses of the earth and comet being much less than that
of the sun, we may neglect them, and suppose that μ is the same for all these
bodies, and that it expresses the mass of the sun. Substituting therefore,
instead of $\sqrt{\mu}$, its value $\frac{2\pi}{T}$ [692], in the preceding expression of t [691]; [692'']
we shall have

$$t = \frac{D^{\frac{3}{2}} . T}{\pi . \sqrt{2}} . \{\text{tang.} \tfrac{1}{2} v + \tfrac{1}{3} . \text{tang.}^3 \tfrac{1}{2} v\}. \qquad [693]$$

This equation now contains only such quantities as are comparable with each
other; and by it t may easily be obtained from v; but to find v when t is
given, it will be necessary to solve an equation of the third degree, which
has but one real root. We may dispense with this solution, by making a [693']
table of the values of v, corresponding to those of t, in a parabola whose
perihelion distance is equal to the mean distance of the earth from the sun,
represented by unity. This table will give the time corresponding to the
anomaly v, in any parabola whose perihelion distance is D, by multiplying by
$D^{\frac{3}{2}}$, the time which corresponds to the same anomaly in the table. We [693'']
shall have the anomaly v, corresponding to the time t, by dividing t by

[693‴] $D^{\frac{3}{2}}$, and seeking in the table the anomaly corresponding to the quotient of this division.*

Suppose now that the anomaly v, corresponding to the time t, is required in a very excentrical ellipsis. If we neglect quantities of the order α^2, and

[693$^{\text{iv}}$] resubstitute $1-e$ for α [681′] the preceding expression [690], of t in v, in the ellipsis, will give†

[694] $$t=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\left\{\begin{array}{l}\text{tang.}\,\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v \\ +(1-e).\text{tang.}\,\tfrac{1}{2}v.\{\tfrac{1}{4}-\tfrac{1}{4}.\text{tang.}^2\tfrac{1}{2}v-\tfrac{1}{5}.\text{tang.}^4\tfrac{1}{2}v\}\end{array}\right\}.$$

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* (478) Let t' be the time corresponding to the anomaly v, in a parabola, whose perihelion distance D is unity, t being the time corresponding to the same angle v and the perihelion distance D. In this case we shall get from [693],

$$t'=\frac{T}{\pi.\sqrt{2}}.\left\{\text{tang.}\,\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v\right\}.$$

[693a] Comparing this with [693] we shall get $t=D^{\frac{3}{2}}.t'$, and $t'=\frac{t}{D^{\frac{3}{2}}}$. Now if v be given,

[693b] we may find t' from the table, and then $t=D^{\frac{3}{2}}.t'$; but if t be given we must find $t'=\frac{t}{D^{\frac{3}{2}}}$, and then find in the table, v from this value of t'.

Delambre computed a table of the values of v corresponding to the argument t, from 0 to 200,000 days, which has been republished in several works on astronomy. Burkhardt has lately made a very useful change in this form of the table, by taking for the argument the logarithm of t. This table was printed by him, in 1814, in an octavo form and is very convenient for use.

† (479) In the expression [690] if we neglect terms of the order α^2 we may put $\frac{1}{2-\alpha}=\frac{1}{2}+\frac{\alpha}{4}$, and $\sqrt{\frac{1}{2-\alpha}}=\frac{1}{\sqrt{2}}.(1+\tfrac{1}{4}\alpha)$, and it becomes

$$t=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.(1+\tfrac{1}{4}\alpha).\{1+(\tfrac{2}{3}-\alpha).(\tfrac{1}{2}+\tfrac{1}{4}\alpha).\text{tang.}^2\tfrac{1}{2}v-\tfrac{1}{5}\alpha.\text{tang.}^4\tfrac{1}{2}v\}.\text{tang.}\,\tfrac{1}{2}v,$$

or by reduction,

$$t=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\{(1+\tfrac{1}{4}\alpha)+(\tfrac{1}{3}-\tfrac{1}{4}\alpha).\text{tang.}^2\tfrac{1}{2}v-\tfrac{1}{5}\alpha.\text{tang.}^4.\tfrac{1}{2}v\}.\text{tang.}\,\tfrac{1}{2}v$$

$$=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\{\text{tang.}\,\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v+\alpha.\text{tang.}\,\tfrac{1}{2}v.(\tfrac{1}{4}-\tfrac{1}{4}.\text{tang.}^2\tfrac{1}{2}v-\tfrac{1}{5}.\text{tang.}^4\tfrac{1}{2}v)\},$$

as in [694].

We must find, by means of the table of the motion of a comet, the anomaly corresponding to the time t, in a parabola in which D is the perihelion distance; let U be this anomaly, and $U+x$ the true anomaly in the ellipsis, [694′] corresponding to the same time, x being a very small angle. If we substitute, in the preceding equation, $U+x$ instead of v, and reduce the second member into a series, arranged according to the powers of x, we shall have, by [694″] neglecting the square of x, and the product of x by $1-e$,*

$$t=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\left\{\begin{array}{l}\{\text{tang.}\,\tfrac{1}{2}U+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U\}+\dfrac{x}{2.\cos.^4\tfrac{1}{2}U}\\[2mm] +\dfrac{1-e}{4}.\text{tang.}\,\tfrac{1}{2}U.\{1-\text{tang.}^2\tfrac{1}{2}U-\tfrac{4}{5}\,\text{tang.}^4\tfrac{1}{2}U\}\end{array}\right\}; \qquad [695]$$

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* (480) Putting $v=U+x$, we obtain from [29] Int.

$$\text{tang.}\,\tfrac{1}{2}v=\text{tang.}\,(\tfrac{1}{2}U+\tfrac{1}{2}x)=\frac{\text{tang.}\,\frac{1}{2}U+\text{tang.}\,\frac{1}{2}x}{1-\text{tang.}\,\frac{1}{2}U.\text{tang.}\,\frac{1}{2}x},$$

and by developing the denominator in a series, neglecting x^2, we get,

$$\text{tang.}\,\tfrac{1}{2}v=\text{tang.}\,\tfrac{1}{2}U+\text{tang.}\,\tfrac{1}{2}x.(1+\text{tang.}^2\tfrac{1}{2}U)=\text{tang.}\,\tfrac{1}{2}U+\frac{\text{tang.}\,\frac{1}{2}x}{\cos.^2\frac{1}{2}U}.$$

The cube of this divided by 3 is

$$\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v=\tfrac{1}{3}.\left\{\text{tang.}\,\tfrac{1}{2}U+\frac{\text{tang.}\,\frac{1}{2}x}{\cos.^2\frac{1}{2}U}\right\}^3=\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U+\text{tang.}^2\tfrac{1}{2}U.\frac{\text{tang.}\,\frac{1}{2}x}{\cos.^2\frac{1}{2}U};$$

hence

$$\text{tang.}\,\tfrac{1}{2}v+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}v=\text{tang.}\,\tfrac{1}{2}U+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U+\frac{\text{tang.}\,\frac{1}{2}x}{\cos.^2\frac{1}{2}U}.\left\{1+\text{tang.}^2\tfrac{1}{2}U\right\}$$

$$=\text{tang.}\,\tfrac{1}{2}U+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U+\frac{\text{tang.}\,\frac{1}{2}x}{\cos.^4\frac{1}{2}U}.$$

And by putting for tang. $\frac{1}{2}x$, the arch itself $\frac{1}{2}x$, it will become

$$\text{tang.}\,\tfrac{1}{2}U+\tfrac{1}{3}.\text{tang.}^3\tfrac{1}{2}U+\frac{x}{2.\cos.^4\frac{1}{2}U}.$$

Substituting this in [694], and in the terms multiplied by α or $1-e$, putting U for v, it will become as in [695]. Making this equal to the value of t, [696] deduced from t [691] by changing v into U, according to the hypothesis [694′], and dividing by $\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}$, we shall get $0=\frac{x}{2.\cos.^4\frac{1}{2}U}+\frac{1-e}{4}.\text{tang.}\,\frac{1}{2}U.\{1-\text{tang.}^2\frac{1}{2}U-\frac{4}{5}.\text{tang.}^4\frac{1}{2}U\}$. Multiplying by $2\cos.^4\frac{1}{2}U$, putting for $\cos.\frac{1}{2}U.\text{tang.}\,\frac{1}{2}U$, its value $\sin.\frac{1}{2}U$, and $\sin.x$ for x, we get

$$\sin.x=\frac{1-e}{2}.\text{tang.}\,\tfrac{1}{2}U.\left\{-\cos.^4\tfrac{1}{2}U+\cos.^2\tfrac{1}{2}U.\sin.^2\tfrac{1}{2}U+\tfrac{4}{5}.\sin.^4\tfrac{1}{2}U\right\},$$

substituting $\sin.^2\frac{1}{2}U=1-\cos.^2\frac{1}{2}U$,

$$\sin.^4\tfrac{1}{2}U=(1-\cos.^2\tfrac{1}{2}U)^2=1-2\cos.^2\tfrac{1}{2}U+\cos.^4\tfrac{1}{2}U,$$

and reducing we obtain [697].

but by hypothesis [694′, 691] we have

[696] $$t=\frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.\{\text{tang.}\,\tfrac{1}{2}\,U+\tfrac{1}{3}.\,\text{tang.}^3\,\tfrac{1}{2}\,U\}\;;$$

we shall therefore have, by substituting for the small arch x its sine,

[697] $$\sin.\,x=\tfrac{1}{10}.(1-e).\,\text{tang.}\,\tfrac{1}{2}\,U.\{4-3.\cos.^2\tfrac{1}{2}\,U-6.\cos.^4\tfrac{1}{2}\,U\}\;;$$

and if we compute a table of the logarithms of the quantity

[698] $$\tfrac{1}{10}.\,\text{tang.}\,\tfrac{1}{2}\,U.\{4-3.\cos.^2\tfrac{1}{2}\,U-6.\cos.^4\tfrac{1}{2}\,U\},$$

it will be only necessary to add this tabular logarithm to the logarithm of $1-e$, to obtain $\log.\sin.x$; in this manner we may find the correction x to [698′] be made in the anomaly U, computed for a parabola, to obtain the corresponding anomaly in a very excentrical ellipsis.

24. It now remains to consider the motion in an hyperbolical orbit. For this purpose we shall observe, that in an hyperbola, the semi-axis a becomes negative, and the excentricity e exceeds unity [378′]. Putting therefore, in [698″] the equations (f) § 20 [606], $a=-a'$, and $u=\frac{u'}{\sqrt{-1}}$, and substituting for sines and cosines their imaginary exponential values, the first of these equations will give*

[699] $$\frac{t.\sqrt{\mu}}{a'^{\frac{3}{2}}}=\frac{e}{2}.\{c^{u'}-c^{-u'}\}-u'.$$

The second will become†

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* (481) Substituting in the first of the equations [606] the value of n [605′], also the exponential value of $\sin.\,u$, [11] Int. it will become

$$\frac{\sqrt{\mu}.t}{a^{\frac{3}{2}}}=u-e.\left\{\frac{c^{u.\sqrt{-1}}-c^{-u.\sqrt{-1}}}{2.\sqrt{-1}}\right\},$$

and if we put $a=-a'$, $u=\frac{u'}{\sqrt{-1}}$, as in [698″], it will change into

$$\frac{\sqrt{\mu}.t}{-a'^{\frac{3}{2}}.\sqrt{-1}}=\frac{u'}{\sqrt{-1}}-e.\left\{\frac{c^{u'}-c^{-u'}}{2\sqrt{-1}}\right\}.$$

Multiplying this by $-\sqrt{-1}$, we shall obtain [699].

† (482) The second equation [606], $r=a.(1-e.\cos.\,u)$, by substituting $-a'$ for a, and $\frac{c^{u.\sqrt{-1}}+c^{-u.\sqrt{-1}}}{2}$, or $\frac{c^{u'}+c^{-u'}}{2}$ for $\cos.\,u$ becomes

$$r=a'.\{\tfrac{1}{2}\,e.(c^{u'}+c^{-u'})-1\},$$

as in [700].

$$r = a' . \{ \tfrac{1}{2} e . (c^{u'} + c^{-u'}) - 1 \}.$$ [700]

Lastly, by taking the sign of the radical of the third equation, so that v and u may increase with t, we shall have*

$$\text{tang.}\ \tfrac{1}{2} v = \sqrt{\frac{e+1}{e-1}} . \left\{ \frac{c^{u'} - 1}{c^{u'} + 1} \right\}.$$ [701]

Suppose in these formulas, $u' = \text{log. tang.}\ (\frac{1}{4} \pi + \frac{1}{2} \varpi)$, π being the semi-circumference of a circle whose radius is unity. The preceding logarithm being hyperbolical, we shall have† [701′]

$$\frac{t . \sqrt{\mu}}{a'^{\frac{3}{2}}} = e . \text{tang.}\ \varpi - \text{log. tang.}\ (\tfrac{1}{4} \pi + \tfrac{1}{2} \varpi)\ ;$$

$$r = a' . \left\{ \frac{e}{\cos. \varpi} - 1 \right\};$$ [702]

$$\text{tang.}\ \tfrac{1}{2} v = \sqrt{\frac{e+1}{e-1}} . \text{tang.}\ \tfrac{1}{2} \varpi.$$

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* (483) From [11, 12] Int. we get

$$\sin.\ \tfrac{1}{2} u = \frac{c^{\frac{1}{2} u . \sqrt{-1}} - c^{-\frac{1}{2} u . \sqrt{-1}}}{2 . \sqrt{-1}} = \frac{c^{\frac{1}{2} u'} - c^{-\frac{1}{2} u'}}{2 . \sqrt{-1}}, \qquad \text{and}$$

$$\cos.\ \tfrac{1}{2} u = \frac{c^{\frac{1}{2} u . \sqrt{-1}} + c^{-\frac{1}{2} u . \sqrt{-1}}}{2} = \frac{c^{\frac{1}{2} u'} + c^{-\frac{1}{2} u'}}{2},$$

hence $\text{tang.}\ \frac{1}{2} u = \frac{\sin. \frac{1}{2} u}{\cos. \frac{1}{2} u} = \frac{c^{\frac{1}{2} u'} - c^{-\frac{1}{2} u'}}{\sqrt{-1} . \{ c^{\frac{1}{2} u'} + c^{-\frac{1}{2} u'} \}}$, or by multiplying the numerator and denominator by $c^{\frac{1}{2} u'}$, $\text{tang.}\ \frac{1}{2} u = \frac{c^{u'} - 1}{\sqrt{-1} . (c^{u'} + 1)}$. Substituting this in the third equation [606], it will become as in [701].

† (484) Putting for brevity $\frac{1}{4} \pi + \frac{1}{2} \varpi = b$, we shall have $u' = \text{log. tang.}\ b$, [701′], [702a] hence $c^{u'} = \text{tang.}\ b = \frac{\sin. b}{\cos. b}$, and $c^{-u'} = \frac{\cos. b}{\sin. b}$, therefore [702b]

$$c^{u'} - c^{-u'} = \frac{\sin. b}{\cos. b} - \frac{\cos. b}{\sin. b} = \frac{\sin.^2 b - \cos.^2 b}{\sin. b . \cos. b};$$

but $\sin.^2 b - \cos.^2 b = - \cos. 2 b$, and $\sin. b . \cos. b = \frac{1}{2} \sin. 2 b$, [31, 32] Int. hence

$$c^{u'} - c^{-u'} = - \frac{\cos. 2 b}{\frac{1}{2} \sin. 2 b} = - 2 . \text{cotang.}\ 2 b = - 2\ \text{cotang.}\ (\tfrac{1}{2} \pi + \varpi) = 2\ \text{tang.}\ \varpi.$$

Substituting this and u' [701′] in [699] we get the first of the equations [702].

[702′] The arch $\frac{t.\sqrt{\mu}}{a'^{\frac{3}{2}}}$, is the mean angular motion of the body m in the time t, supposing it to move in a circular orbit about M, at the distance a'.* This arch may easily be found in parts of the radius; the first of these equations [702] will give, by a few trials, the value of the angle ϖ corresponding to the time t; the other two equations will then give the corresponding values of r and v.

25. Since T [691‴] expresses the time of the sidereal revolution of a planet whose mean distance from the sun is a, the first of the equations (f) [702″] § 20 [606], will give $n\,T = 2\,\pi$ [691$^{\text{iv}}$]; but we have by the same article, [605′], $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}} = n$; we shall therefore find

[703] $$T = \frac{2\,\pi\,.\,a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

If we neglect the masses of the planets, with respect to that of the sun, μ will denote the mass of the sun, and this quantity will be the same for all [703′] the planets; hence for a second planet, in which a' and T' represent the mean distance from the sun, and the time of a sidereal revolution, we shall also have

[704] $$T' = \frac{2\,\pi\,.\,a'^{\frac{3}{2}}}{\sqrt{\mu}};$$

Again the same values of $c^{u'}$, $c^{-u'}$, give

$$c^{u'} + c^{-u'} = \frac{\sin.\,b}{\cos.\,b} + \frac{\cos.\,b}{\sin.\,b} = \frac{\sin.^2 b + \cos.^2 b}{\cos.\,b\,.\sin.\,b} = \frac{1}{\cos.\,b\,.\sin.\,b} = \frac{2}{\sin.\,2\,b} = \frac{2}{\sin.(\frac{1}{2}\,\pi + \varpi)} = \frac{2}{\cos.\,\varpi},$$

which being substituted in [700] gives the second of the equations [702].

Lastly, since tang. $\frac{1}{4}\,\pi = 1$, [701′], and $c^{u'} = \text{tang.}\,b$, [702b], we shall get

$$\frac{c^{u'} - 1}{c^{u'} + 1} = \frac{\text{tang.}\,b - \text{tang.}\,\frac{1}{4}\,\pi}{\text{tang.}\,b\,.\,\text{tang.}\,\frac{1}{4}\,\pi + 1} = \text{tang.}\,(b - \tfrac{1}{4}\,\pi),$$

[30] Int.; and as $b - \frac{1}{4}\,\pi = \frac{1}{2}\,\varpi$, [702$a$] this will become $\frac{c^{u'} - 1}{c^{u'} + 1} = \text{tang.}\,\frac{1}{2}\,\varpi$. Substituting this in [701], we shall obtain the third of the equations [702].

* (485) In a circle, $e = 0$, and [668] gives $v = n\,t$, and by [605′] $\frac{\sqrt{\mu}}{a'^{\frac{3}{2}}} = n$, hence $v = \frac{t.\sqrt{\mu}}{a'^{\frac{3}{2}}}$, as above.

therefore we shall have

$$T^2 : T'^2 :: a^3 : a'^3 \;; \tag{705}$$

that is, the squares of the times of revolution of different planets are as the
cubes of the transverse axes of their orbits; which is one of the laws Kepler's Law.
discovered by Kepler. We see by the preceding analysis, that the law is [705']
not rigorous, and that it exists only in the supposition that the attraction of
the planets upon each other, and upon the sun, is neglected.

If we take the mean motion of the earth for the measure of time, and its
mean distance from the sun for the unity of distance, T in this case will be [705'']
equal to 2π, and we shall have $a = 1$; the preceding expression of T [705''']
[703] will therefore give[*] $\mu = 1$; hence it follows that the mass of the sun
ought to be taken for the unity of mass. We may therefore, in the theory
of the planets and comets, suppose $\mu = 1$, and take for unity of distance, [705^iv]
the mean distance of the earth from the sun; but then the time t will
be measured by the corresponding arch of the mean sidereal motion of the [705^v]
earth.

The equation [703]

$$T = \frac{2\pi . a^{\frac{3}{2}}}{\sqrt{\mu}}, \tag{706}$$

furnishes a very simple method of determining the ratio of the mass of a
planet to that of the sun, in case the planet is accompanied by a satellite.
For by representing the sun's mass by M, if we neglect the mass m of the [706']
planet in comparison with M, we shall have

$$T = \frac{2\pi . a^{\frac{3}{2}}}{\sqrt{M}}. \tag{707}$$

If we then consider the satellite of any planet m'; and put p for the mass of
the satellite, h its mean distance from the centre of m', and T the time of its [707']
sidereal revolution; we shall have

$$\mathrm{T} = \frac{2\pi . h^{\frac{3}{2}}}{\sqrt{m'+p}}\;; \tag{708}$$

therefore

$$\frac{m'+p}{M} = \frac{h^3}{a^3} \cdot \left(\frac{T}{\mathrm{T}}\right)^2. \tag{709}$$

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[*] (486) $T = 2\pi$, and $a = 1$, substituted in the equation [703] evidently gives $\mu = 1$.

This equation gives the ratio of the sum of the masses of the planet *m'* and its satellite, to the mass *M* of the sun; by neglecting therefore the mass of the satellite with respect to that of the planet; or by supposing the ratio of [709'] these masses known; we shall have the ratio of the mass of the planet to that of the sun. We shall give, when treating of the theory of the planets, the values of the masses of those planets about which satellites have been observed.

CHAPTER IV.

DETERMINATION OF THE ELEMENTS OF THE ELLIPTICAL MOTION.

26. After having explained the general theory of the elliptical motion, and the manner of computing it, by converging series, in the two cases of nature, namely, that of orbits nearly circular, and that where they are very excentrical; it now remains to determine the elements of these orbits. If the circumstances of the primitive motions of the heavenly bodies were given, we might from them easily deduce these elements. For, if we [709″] put V for the velocity of m, in its relative motion about M, we shall have [40a]

$$V^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2}; \qquad [710]$$

and the last of the equations (P) § 18 [572], will give

$$V^2 = \mu \,.\left\{ \frac{2}{r} - \frac{1}{a} \right\}. \qquad [711]$$

To eliminate μ from this expression, we shall denote by U the velocity which m would have, if it described about M a circle whose radius is equal to the [711′] unity of distance. In this hypothesis, we shall have $r = a = 1$, consequently* $U^2 = \mu$; therefore

$$V^2 = U^2 .\left\{ \frac{2}{r} - \frac{1}{a} \right\}. \qquad [712]$$

This equation will give the semi-transverse axis a of the orbit, by means of [712′]

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* (487) Substituting $r = a = 1$, and $V = U$, in [711] gives $U^2 = \mu$, hence [711] becomes as in [712].

the primitive velocity of m, and its primitive distance from M.* a is positive in the ellipsis, infinite in the parabola, and negative in the hyperbola; therefore the orbit described by m is an ellipsis if $V < U . \sqrt{\frac{2}{r}}$, a parabola

[712″] if $V = U . \sqrt{\frac{2}{r}}$, and an hyperbola if $V > U . \sqrt{\frac{2}{r}}$. *It is remarkable*

[712‴] *that the direction of the primitive motion has no influence on the species of the conic section described.*

To determine the excentricity of the orbit, we shall observe, that if we put ε for the angle which the direction of the relative motion of m makes

[712^{iv}] with the radius r, we shall have† $\frac{dr^2}{dt^2} = V^2 . \cos.^2 \varepsilon$. Substituting for V^2 its value $\mu . \left(\frac{2}{r} - \frac{1}{a}\right)$ [711], we shall have

[713] $$\frac{dr^2}{dt^2} = \mu . \left(\frac{2}{r} - \frac{1}{a}\right) . \cos. \varepsilon^2 ;$$

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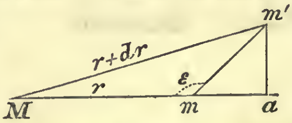
* (488) Putting V for the primitive velocity, and r for the primitive distance, in [712], we shall get $\frac{1}{a} = \frac{2}{r} - \frac{V^2}{U^2}$, from which we may compute the value of $\frac{1}{a}$ or a; and it is evident that $\frac{1}{a}$ is positive if $V < U . \sqrt{\frac{2}{r}}$, $\frac{1}{a} = 0$ if $V = U . \sqrt{\frac{2}{r}}$, and $\frac{1}{a}$ is negative if $V > U . \sqrt{\frac{2}{r}}$, and since a is infinite when $\frac{1}{a} = 0$, it will follow from [378′] that the curve will be an ellipsis if $V < U . \sqrt{\frac{2}{r}}$, a parabola if $V = U . \sqrt{\frac{2}{r}}$, and an hyperbola if $V > U . \sqrt{\frac{2}{r}}$.

† (489) In the adjoined figure let M be the place of the body M, m that of the body m, $m\,m'$ the primitive direction of the body m in its relative motion about M; $m\,m'$ being the space described in that relative orbit in the time dt. Then



$Mm = r$, $Mm' = r + dr$, and taking on Mm, continued, $Ma = Mm'$, $ma = dr$, the angle $Mmm' = \varepsilon$, and $mm' = V dt$. Then in the triangle mam' we have $ma = mm' . \cos. amm' = - mm' . \cos. \varepsilon$, or $dr = - V dt . \cos \varepsilon$, squaring we find $\frac{dr^2}{dt^2} = V^2 . \cos. \varepsilon^2$, hence from V^2, [711], we get [713]. The value of h^2, [599], being but we have by § 19 [598, 599]

$$2\mu r - \frac{\mu r^2}{a} - \frac{r^2\,dr^2}{dt^2} = \mu a\,.\,(1-e^2)\,; \qquad [714]$$

therefore we shall have

$$a\,.\,(1-e^2) = r^2\,.\,\sin.\,\varepsilon^2\,.\left(\frac{2}{r}-\frac{1}{a}\right); \qquad [715]$$

which will give the excentricity of the orbit $a e$. [715′]

The equation of conic sections [378]

$$r = \frac{a\,.\,(1-e^2)}{1+e\,.\cos.\,v}, \qquad [716]$$

gives

$$\cos.\,v = \frac{a\,.\,(1-e^2)-r}{e r}. \qquad [716′]$$

Hence we shall find the angle v, which the radius vector makes with the perihelion, consequently the position of the perihelion will be obtained. The equations (*f*) § 20 [606], will then give the angle u,* and by this means the time of the passage by the perihelion may be found. [716″]

To obtain the position of the orbit with respect to a fixed plane passing through the centre of M, supposed to be at rest; let φ be the inclination of the orbit to this plane, and β the angle which the radius r makes with the line of nodes; also, let z be the given elevation of the body m above the [716‴]

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used in [598] gives [714]. Substituting in [714] this value of $\frac{dr^2}{dt^2}$, we find

$$2\mu r - \frac{\mu r^2}{a} - r^2\mu\,.\left(\frac{2}{r}-\frac{1}{a}\right).\cos.\,\varepsilon^2 = \mu a\,.\,(1-e^2).$$

Dividing by μ and reducing the first member, it becomes

$$\left(\frac{2}{r}-\frac{1}{a}\right).\,r^2\,.\,(1-\cos.\,\varepsilon^2) = a\,.\,(1-e^2),$$

and as $1-\cos.\,\varepsilon^2 = \sin.\,\varepsilon^2$, it changes into [715]. Now as the primitive value of r is given, and a is known by [712], we shall easily obtain e from [715].

* (490) The last of the equations [606] gives u by means of e, v, which had been previously computed [715′, 716′]. Having u, we may obtain t by means of the first equation [606].

fixed plane, at the commencement of the motion ; we shall have*

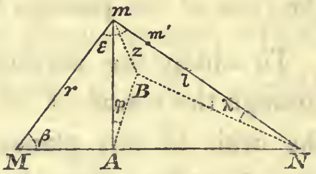
[717] $$r\,.\,\sin.\,\beta\,.\,\sin.\,\varphi = z\;;$$

hence the inclination of the orbit φ will be known when β shall be determined. For this purpose let λ be the angle which the primitive direction of the
[717′] relative motion of m makes with the fixed plane, this angle being supposed to be given ; if we refer to the triangle formed by the line of nodes, the radius r, and the line of the primitive direction of the motion continued till
[717″] it meets the line of nodes, and put l for the side of the triangle opposite to the angle β, we shall have†

[718] $$l = \frac{r\,.\,\sin.\,\beta}{\sin.\,(\beta+\varepsilon)}\;;$$

[718′] also $\frac{z}{l} = \sin.\,\lambda$; hence we shall get

[719] $$\text{tang.}\,\beta = \frac{z\,.\,\sin.\,\varepsilon}{r\,.\,\sin.\,\lambda - z\,.\,\cos.\,\varepsilon}\,.$$



* (491) Let M, m, be the places of the bodies M, m, MN the line of nodes; mm', the line of the primitive direction of the relative motion of the body m about M, which line being continued meets MN in N. Draw mA perpendicular to MN, and mB perpendicular to the fixed plane, to meet it in B. Then $Mm = r$, $mB = z$, $Nm = l$, angle $NMm = \beta$, angle $MmN = \varepsilon$, consequently the angle $MNm = \pi - \beta - \varepsilon$, π being equal to two right angles, angle $BAm = \varphi$, angle $BNm = \lambda$. Then in the right angled plane triangles MAm, ABm, we have $Am = Mm\,.\,\sin.\,NMm = r\,.\,\sin.\,\beta$, and mB or $z = Am\,.\,\sin.\,BAm = Am\,.\,\sin.\,\varphi = r\,.\,\sin.\,\beta\,.\,\sin.\,\varphi$, [717].

† (492) In the plane triangle MNm, we have $\sin.\,MNm : Mm :: \sin.\,NMm : Nm$, or in symbols $\sin.\,(\pi-\beta-\varepsilon) : r :: \sin.\,\beta : l$, hence $l = \frac{r\,.\,\sin.\,\beta}{\sin.\,(\beta+\varepsilon)}$, as in [718]. Again, in the right angled plane triangle NBm, we have $mB = Nm\,.\,\sin.\,BNm$, which in symbols is $z = l\,.\,\sin.\,\lambda$, or $l = \frac{z}{\sin.\,\lambda}$. Substituting this in [718] we get $\sin.\,\lambda = \frac{z\,.\,\sin.\,(\beta+\varepsilon)}{r\,.\,\sin.\,\beta} = \frac{z\,.\,(\sin.\,\beta\,.\,\cos.\,\varepsilon + \cos.\,\beta\,.\,\sin.\,\varepsilon)}{r\,.\,\sin.\,\beta}$, [21] Int., and if we divide the numerator and denominator by $\cos.\,\beta$ it becomes $\sin.\,\lambda = \frac{z\,.\,(\text{tang.}\,\beta\,.\,\cos.\,\varepsilon + \sin.\,\varepsilon)}{r\,.\,\text{tang.}\,\beta}$, hence we easily deduce $\text{tang.}\,\beta$, [719].

The elements of the orbit of the planet being determined, by these formulas, in functions of the radius r, the elevation z, the velocity of the planet, and the direction of its motion; we can find the variations of these elements, corresponding to any supposed variation in the velocity or in the direction of the motion; and it will be easy, by the method we shall hereafter give, to deduce therefrom the differential variations of these elements, arising from the actions of the disturbing forces.

We shall now resume the equation [712]

$$V^2 = U^2 \cdot \left\{ \frac{2}{r} - \frac{1}{a} \right\}. \qquad [720]$$

In the circle $a = r$, consequently $V = U \cdot \sqrt{\frac{1}{r}}$; hence *the velocities of planets in different circles are* inversely proportional to the square roots of their radii.* [720′]

In the parabola $a = \infty$, hence $V = U \cdot \sqrt{\frac{2}{r}}$ [720]; therefore the velocity in any point whatever of the orbit is inversely proportional to the square root of the corresponding radius vector r; and the velocity of the comet will be to that of a planet, which should revolve about the sun in a circular orbit at the same distance r, as $\sqrt{2}$ to 1.† [720″] [720‴]

An ellipsis infinitely flattened becomes a right line; and in this case V would express the velocity of m, if it should fall in a right line directly towards M. Suppose that m should fall from a state of rest, and that its distance from M at the commencement of motion was r, and when it has arrived at the distance r', it should have acquired the velocity V'; the preceding expression of the velocity will give the two following equations:‡ [720^{iv}] [720^v]

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* (493) In the original the word inversely was accidentally omitted.

† (494) The velocity in the parabola having been found to be $U \cdot \sqrt{\frac{2}{r}}$, [720″], and in the circle $U \cdot \sqrt{\frac{1}{r}}$, [720′], these are evidently to each other as $\sqrt{2} : 1$.

‡ (495) The first of these equations is found by putting $V = 0$, [720], at the commencement of the motion, this gives $\frac{2}{r} - \frac{1}{a} = 0$, hence $\frac{1}{a} = \frac{2}{r}$, which being

[721] $$0=\frac{2}{r}-\frac{1}{a}; \qquad V'^2=U^2.\left\{\frac{2}{r'}-\frac{1}{a}\right\};$$

hence we deduce

[722] $$V'=U.\sqrt{\frac{2.(r-r')}{rr'}};$$

which is the expression of the relative velocity acquired by m, in falling from
[722'] the height r, towards M, through the space $r-r'$. We can determine easily, by means of this formula, from what height the body m, moving in a conic section, ought to fall towards M, to acquire in falling from the extremity of the radius vector r, a relative velocity equal to that which it has at that extremity; for V being this last velocity, we shall have

[723] $$V^2=U^2.\left\{\frac{2}{r}-\frac{1}{a}\right\};$$

but the square of the velocity acquired by falling from the height $r-r'$, is
[723'] $2\,U^2.\frac{(r-r')}{rr'}$ [722]; putting these two expressions equal to each other, we shall find*

[724] $$r-r'=\frac{r.(2a-r)}{4a-r}.$$

[724'] In the circle, $a=r$, and then $r-r'=\frac{1}{3}r$ [724]; in the ellipsis, we have† $r-r'<\frac{1}{2}r$; in the parabola a is infinite, and we have $r-r'=\frac{1}{2}r$; and in the hyperbola, where a is negative, we have $r-r'>\frac{1}{2}r$.

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substituted in the second equation [721] $V'^2=U^2.\left\{\frac{2}{r}-\frac{1}{a}\right\}$, it becomes

$$V'^2=U^2.\left(\frac{2}{r'}-\frac{2}{r}\right)=U^2.\frac{2.(r-r')}{rr'}, \qquad \text{hence} \qquad V'=U.\sqrt{\frac{2.(r-r')}{rr'}},$$

as in [722].

* (496) This equation, by rejecting the common factor U^2, becomes $\frac{2.(r-r')}{rr'}=\frac{2}{r}-\frac{1}{a}$, hence $r'=\frac{2ar}{4a-r}$, and $r-r'=\frac{r.(2a-r)}{4a-r}$, as in [724].

† (497) The expression [724] may be put under the form $r-r'=\frac{1}{2}r-\frac{r^2}{2.(4a-r)}$, and in the ellipsis where $4a-r$ is always positive, the last term must be negative, consequently $r-r'<\frac{1}{2}r$. In the parabola, where $a=\infty$, that term vanishes, and we get $r-r'=\frac{1}{2}r$. In the hyperbola a becomes negative, and then by putting $a=-a'$, it becomes $r-r'=\frac{1}{2}r+\frac{r^2}{2.(4a'+r)}$, which evidently exceeds $\frac{1}{2}r$.

27. The equation [572]

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - \mu . \left(\frac{2}{r} - \frac{1}{a}\right), \qquad [725]$$

is remarkable, because it gives the velocity independently of the excentricity of the orbit. It is comprised in a more general equation, between the transverse axis of the orbit, the chord of an elliptical arch, the sum of the extreme radii vectores of the arch, and the time of describing the same arch. To obtain this last equation, we shall resume the equations of the elliptical motion, given in § 20 [603—606], supposing for greater simplicity $\mu = 1$. These equations thus become* [725'] [725'']

$$r = \frac{a . (1 - e^2)}{1 + e . \cos . v};$$

$$r = a . (1 - e . \cos . u); \qquad [726]$$

$$t = a^{\frac{3}{2}} . (u - e . \sin . u).$$

Suppose that $r, v, u, t,$ correspond to the first extremity of the elliptical arch, and $r', v', u', t',$ to the other extremity; we shall have [726']

$$r' = \frac{a . (1 - e^2)}{1 + e . \cos . v'};$$

$$r' = a . (1 - e . \cos . u'); \qquad [727]$$

$$t' = a^{\frac{3}{2}} . (u' - e . \sin . u').$$

Put

$$t' - t = \text{T}; \qquad \frac{u' - u}{2} = \beta; \qquad \frac{u' + u}{2} = \beta'; \qquad r' + r = R. \qquad [728]$$

If we subtract the expression of t [726], from that of t' [727], observing that†

$$\sin . u' - \sin . u = 2 . \sin . \beta . \cos . \beta'; \qquad [729]$$

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* (499) The first of these equations is as in [603]. The second is like the second of [606]. The last is the same as the first of [606], substituting $n = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$, [605'], putting $\mu = 1$, [725''], and multiplying by $a^{\frac{3}{2}}$. Accenting the letters $r, v, u, t,$ [726] we obtain [727].

† (500) By [26, 27] Int. we have $\sin . u' - \sin . u = 2 . \sin . \left(\frac{u' - u}{2}\right) . \cos . \left(\frac{u' + u}{2}\right)$, and $\cos . u' + \cos . u = 2 . \cos . \left(\frac{u' + u}{2}\right) . \cos . \left(\frac{u' - u}{2}\right)$, which by using the values of

we shall have

[730] $$\mathrm{T} = 2\,a^{\frac{3}{2}} \,.\, \{\beta - e \,.\, \sin.\,\beta \,.\, \cos.\,\beta'\}.$$

If we add together the two expressions of r and r' [726, 727], in u and u', observing that

[731] $$\cos.\,u' + \cos.\,u = 2 \,.\, \cos.\,\beta \,.\, \cos\,\beta',$$

we shall have

[732] $$R = 2\,a \,.\, (1 - e \,.\, \cos.\,\beta \,.\, \cos.\,\beta').$$

Now c being the chord of the elliptical arch, we shall have*

[733] $$c^2 = r^2 + r'^2 - 2\,r\,r' \,.\, \cos.\,(v - v')\;;$$

but the two equations [726]

[734] $$r = \frac{a \,.\, (1 - e^2)}{1 + e \,.\, \cos.\,v}\,; \qquad r = a \,.\, (1 - e \,.\, \cos.\,u)\;;$$

give†

[735] $$\cos.\,v = a \,.\, \frac{\{\cos.\,u - e\}}{r}\;; \qquad \sin.\,v = \frac{a \,.\, \sqrt{1 - e^2} \,.\, \sin.\,u}{r}\,.$$

And in like manner

[736] $$\cos.\,v' = a \,.\, \frac{\{\cos.\,u' - e\}}{r'}\;; \qquad \sin.\,v' = \frac{a \,.\, \sqrt{1 - e^2} \,.\, \sin.\,u'}{r'}\;;$$

β, β', [728] become as in [729, 731]. These being substituted in $t' - t$, and $r' + r$, deduced from [726, 727] give [730, 732].

* (501) As in [471] or [63] Int., putting, in the second figure, page 292, $\mathcal{A}\,C = r$, $C\,\mathcal{M} = r'$, angle $\mathcal{A}\,C\,\mathcal{M} = v - v'$.

† (502) The first value of r, [734], gives $\cos.\,v = \frac{a \,.\, (1 - e^2) - r}{e\,r}$, and by substituting in the numerator the second value of r, [734] it becomes $\cos.\,v = a \,.\, \frac{\{\cos.\,u - e\}}{r}$; hence

$$\sin.\,v = \sqrt{1 - \cos.^2 v} = \sqrt{1 - \frac{a^2}{r^2} \,.\, (\cos.\,u - e)^2} = \frac{1}{r} \,.\, \sqrt{r^2 - a^2 \,.\, (\cos.\,u - e)^2}$$

Substituting $r = a \,.\, (1 - e \,.\, \cos.\,u)$, [726], in the radical, it becomes

$$\sin.\,v = \frac{a}{r} \,.\, \sqrt{(1 - \cos.^2 u) \,.\, (1 - e\,e)} = \frac{a \,.\, \sqrt{1 - e^2} \,.\, \sin.\,u}{r}\,,$$

as in [735]; and by accenting the letters r, u, v, [735], we get [736].

therefore we shall have*

$$r\,r'\,.\,\cos.\,(v-v')=a^2.(e-\cos.\,u).(e-\cos.\,u')+a^2.(1-e^2).\sin.u.\sin.u'\,; \qquad [737]$$

consequently

$$c^2=2a^2.(1-e^2).\{1-\sin.u.\sin.u'-\cos.u.\cos.u'\}+a^2e^2.(\cos.u-\cos.u')^2\,; \qquad [738]$$

now we have

$$\begin{aligned}\sin.\,u\,.\,\sin.\,u'+\cos.\,u\,.\,\cos.\,u'&=2\,.\,\cos.\,\beta^2-1\,;\\ \cos.\,u-\cos.\,u'&=2\,.\,\sin.\,\beta\,.\,\sin.\,\beta'\,;\end{aligned} \qquad [739]$$

therefore

$$c^2=4\,a^2.\sin.^2\beta\,.\,(1-e^2.\cos.^2\beta')\,; \qquad [740]$$

hence we have the three following equations [732, 730, 740]:

$$\begin{aligned}R&=2\,a\,.\{1-e\,.\,\cos.\,\beta\,.\,\cos.\,\beta'\}\,;\\ \mathrm{T}&=2\,a^{\frac{3}{2}}.\{\beta-e\,.\,\sin.\,\beta\,.\,\cos.\,\beta'\}\,;\\ c^2&=4\,a^2.\sin.^2\beta\,.\{1-e^2.\cos.^2\beta'\}.\end{aligned} \qquad [741]$$

The first of these equations gives

$$e\,.\,\cos.\,\beta'=\frac{2\,a-R}{2\,a\,.\,\cos.\,\beta}\,; \qquad [742]$$

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* (503) Since $\cos.\,(v-v')=\cos.\,v\,.\,\cos.\,v'+\sin.\,v\,.\,\sin.\,v'$, [24] Int., by using the values [735, 736], we shall get,

$$r\,r'\,.\,\cos.\,(v-v')=a^2\,.\,(e-\cos.\,u)\,.\,(e-\cos.\,u')+a^2\,.\,(1-e^2)\,.\,\sin.\,u\,.\,\sin.\,u',$$

[737]; substituting this, and $r=a\,.\,(1-e\,.\,\cos.\,u)$, $r'=a\,.\,(1-e\,.\,\cos.\,u')$, [734], in [733], it becomes

$$\begin{aligned}c^2&=a^2.(1-e.\cos.u)^2+a^2.(1-e.\cos.u')^2-2a^2.(e-\cos.u).(e-\cos.u')-2a^2.(1-e^2).\sin.u.\sin.u'\\ &=2a^2-2a^2e^2-2a^2.(1-e^2).\sin.u.\sin.u'+a^2e^2.\cos.^2u-2a^2.\cos.u.\cos.u'+a^2e^2.\cos.^2u',\end{aligned}$$

in which the three last terms

$$a^2e^2.\cos.^2u-2a^2.\cos.u.\cos.u'+a^2e^2.\cos.^2u'=a^2e^2.(\cos.u-\cos.u')^2-2\,a^2.(1-e\,e)\,.\,\cos.u.\cos.u',$$

being substituted we get

$$c^2=2\,a^2\,.\,(1-e^2)\,.\,\{1-\sin.\,u\,.\,\sin.\,u'-\cos.\,u\,.\,\cos.\,u'\}+a^2\,e^2\,.\,(\cos.\,u-\cos.\,u')^2,$$

as in [738], but $\sin.\,u\,.\,\sin.\,u'+\cos.\,u\,.\,\cos.\,u'=\cos.\,(u'-u)$, [24] Int., and this by using [728], is $=\cos.\,2\,\beta=2\,.\,\cos.^2\beta-1$, [6] Int. Also by [17] Int.

$$2\,.\,\sin.\,\beta\,.\,\sin.\,\beta'=\cos.\,(\beta'-\beta)-\cos.\,(\beta'+\beta)=\cos.\,u-\cos.\,u', \quad [739].$$

These being substituted in [738], we get

$$c^2=2\,a^2\,.\,(1-e^2)\,.\,\{2-2\,.\,\cos.^2\beta\}+a^2\,e^2\,.\,(2\,.\,\sin.\,\beta\,.\,\sin.\,\beta')^2,$$

and by putting $2-2\,.\,\cos.^2\beta=2\,.\,\sin.^2\beta$, it becomes

$$c^2=4\,a^2\,.\,\sin.^2\beta\,.\,\{1-e^2+e^2\,.\,\sin.^2\beta'\}=4\,a^2\,.\,\sin.^2\beta\,.\,\{1-e^2\,.\,\cos.^2\beta'\}, \quad [740].$$

substituting this value of $e . \cos. \beta'$ in the other two equations, we shall have*

[743]
$$\mathrm{T} = 2\,a^{\frac{3}{2}} . \left\{ \beta + \left(\frac{R-2\,a}{2\,a}\right) . \mathrm{tang.}\,\beta \right\};$$
$$c^2 = 4\,a^2 . \mathrm{tang.}^2\beta . \left\{ \cos.^2\beta - \left(\frac{2\,a-R}{2\,a}\right)^2 \right\}.$$

These two equations do not contain the excentricity e; and if in the first, we substitute for β its value given by the second, we shall have T in a function of c, R, a. [743′] Hence we see that the time T depends only on the semi-transverse axis, the chord c, and the sum R of the extreme radii vectores.

If we put

[744]
$$z = \frac{2\,a-R+c}{2\,a}; \qquad z' = \frac{2\,a-R-c}{2\,a};$$

the last of the preceding equations will give†

[745]
$$\cos.\,2\,\beta = z\,z' + \sqrt{(1-z^2).(1-z'^2)};$$

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* (504) Substituting the value of $e . \cos. \beta'$, [742], in c^2, [741], we shall find

$$c^2 = 4\,a^2 . \left\{ \sin.^2\beta - \left(\frac{2\,a-R}{2\,a}\right)^2 . \frac{\sin.^2\beta}{\cos.^2\beta} \right\},$$

and by putting in the first term $\sin.\,\beta = \cos.\,\beta . \mathrm{tang.}\,\beta$, and in the last $\frac{\sin.\,\beta}{\cos.\,\beta} = \mathrm{tang.}\,\beta$, it becomes as in [743].

[745a] † (505) From [744] we get $z - z' = \frac{c}{a}$, $\frac{1}{2}(z+z') = \frac{2\,a-R}{2\,a}$, which, being substituted in $\frac{c^2}{a^2}$, [743], give

$$(z-z')^2 = 4 . \mathrm{tang.}^2\beta . \{\cos.^2\beta - \tfrac{1}{4}(z+z')^2\} = 4 . \sin.^2\beta - (z+z')^2 . \mathrm{tang.}^2\beta.$$

Now by putting $\cos.\,2\,\beta = v$, we shall have $\sin.^2\beta = \frac{1}{2} - \frac{1}{2} . \cos.\,2\,\beta = \frac{1}{2} - \frac{1}{2}\,v$, [1] Int. $\mathrm{tang.}^2\beta = \frac{1-\cos.\,2\,\beta}{1+\cos.\,2\,\beta} = \frac{1-v}{1+v}$, [40] Int.; hence $(z-z')^2 = 4 . (\frac{1}{2} - \frac{1}{2}\,v) - (z+z')^2 . \frac{1-v}{1+v}$. Multiplying by $\frac{1}{2}(1+v)$ we shall get

$$1 - v^2 = \tfrac{1}{2}(1+v) . (z-z')^2 + \tfrac{1}{2}(1-v) . (z+z')^2 = z^2 + z'^2 - 2\,v\,z\,z', \qquad \text{or}$$
$$v^2 - 2\,v\,z\,z' + z^2\,z'^2 = 1 - z^2 - z'^2 + z^2\,z'^2 = (1-z^2) . (1-z'^2),$$

extracting the square root, we shall find $v - z\,z' = \sqrt{(1-z^2).(1-z'^2)}$, and by resubstituting

hence we deduce

$$2\beta = \text{arc. cos. } z' - \text{arc. cos. } z\,; \qquad [746]$$

arc. cos. z denoting the arch which has z for its cosine; hence we shall have*

$$\text{tang. } \beta = \frac{\sin.\,(\text{arc. cos. } z') - \sin.\,(\text{arc. cos. } z)}{z+z'}\,; \qquad [747]$$

we shall have also $z+z'=\frac{2a-R}{a}$ [744]; the expression of T will
therefore become, by observing that if T be the time of a sidereal revolution
of the earth, and the mean distance of the earth from the sun be taken for [747′]
unity, we shall have by § 16, $T=2\pi$ [705″],†

$$\mathrm{T}=\frac{a^{\frac{3}{2}}.T}{2\pi}.\{\text{arc.cos.}z'-\text{arc.cos.}z-\sin.(\text{arc.cos.}z')+\sin.(\text{arc.cos.}z)\}. \qquad (a) \qquad [748]$$

As the same cosine may appertain to several arcs, this expression of T is ambiguous, and we must carefully distinguish the arcs to which the cosines z and z' correspond.

$v=\cos.\,2\beta$, we shall obtain $\cos.\,2\beta = zz' + \sqrt{(1-z^2).(1-z'^2)}$, as in [745]. Now
if we put $z=\cos.\,A$, $z'=\cos.\,B$, this will become [745*b*]

$$\cos.\,2\beta = \cos.\,A\,.\,\cos.\,B + \sin.\,A\,.\,\sin.\,B = \cos.\,(B-A), \qquad [24]\ \text{Int.}$$

Hence $\cos.\,2\beta=\cos.\,(B-A)$, therefore we may put

$$2\beta = B - A = \text{arc. cos. } z' - \text{arc. cos. } z, \qquad \text{as in [746].} \qquad [745c]$$

* (506) We have $\text{tang.}\tfrac{1}{2}.(B-A)=\frac{\sin.B-\sin.A}{\cos.B+\cos.A}$, [36] Int., also, $\tfrac{1}{2}.(B-A)=\beta$, $\cos.\,B=z'$, $\cos.\,A=z$, [745*c*, *b*], hence we find

$$\text{tang. } \beta = \frac{\sin.\,(\text{arc. cos. } z')-\sin.\,(\text{arc. cos. } z)}{z+z'},$$

as in [747].

† (507) Having by [705″], $T=2\pi$, the expression of T, [743], becomes by multiplying by $\frac{T}{2\pi}$, which is equal to unity, $\mathrm{T}=\frac{a^{\frac{3}{2}}T}{2\pi}.\left\{2\beta+\left(\frac{R-2a}{a}\right).\text{tang. }\beta\right\}$; and by [745*c*, *a*], $2\beta=\text{arc. cos. } z'-\text{arc. cos. } z$, $\left(\frac{R-2a}{a}\right)=-(z+z')$; using these and tang. β, [747] we obtain [748].

In the parabola the semi-transverse axis a is infinite, and we shall have*

[749] $$\text{arc.}\cos.z'-\sin.(\text{arc.}\cos.z')=\tfrac{1}{6}.\left(\frac{R+c}{a}\right)^{\frac{3}{2}}.$$

By making c negative, we shall have the value of $\text{arc.}\cos.z-\sin.(\text{arc.}\cos.z)$; the formula ($a$) [748] will therefore give, for the time T employed in describing the arch subtended by the chord c,

[750] $$\mathrm{T}=\frac{T}{12\pi}.\{(r+r'+c)^{\frac{3}{2}}\mp(r+r'-c)^{\frac{3}{2}}\};$$

the sign — taking place when the two extremities of the parabolic arch are situated on the same side of the axis of the parabola, or when one of them being below,† the angle formed by the two radii vectores is turned towards

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* (508) The transverse axis $2a$ being very great in comparison with R, c, the values of z, z', [744], must be very nearly equal to unity; and A, B, [745b] may be considered as very small, therefore if we neglect B^4 and its higher powers, we shall have

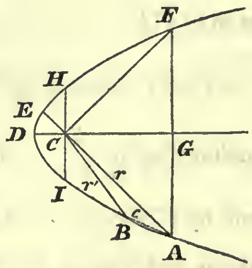
$$\sin.B=B-\tfrac{1}{6}B^3. \qquad \cos.B=1-\tfrac{1}{2}B^2, \quad [43, 44] \text{ Int.}$$

but by [744, 745b], $\cos.B=z'=\frac{2a-R-c}{2a}$, hence

$$1-\tfrac{1}{2}B^2=\frac{2a-R-c}{2a}=1-\frac{(R+c)}{2a},$$

therefore $B^2=\frac{R+c}{a}$ nearly. This gives $\frac{1}{6}B^3=\frac{1}{6}.\left(\frac{R+c}{a}\right)^{\frac{3}{2}}$, which being substituted in the preceding value of $\sin.B$, gives $B-\sin.B=\frac{1}{6}.\left(\frac{R+c}{a}\right)^{\frac{3}{2}}$, as in [749]. Again, since z may be derived from z', [744], and therefore A from B, [745b], by changing the sign of c, we may, from the preceding expression of $B-\sin.B$, obtain the value of $A-\sin.A=\frac{1}{6}.\left(\frac{R-c}{a}\right)^{\frac{3}{2}}$, corresponding to z. These being substituted in [748], putting also $R=r+r'$, [728], it will become as in [750].

† (509) Let $ABDEF$ be a parabola, whose axis is DCG, vertex D, focus C, AB the proposed parabolic arch, whose chord $AB=c$, $CA=r$, $CB=r'$, and suppose AC to be continued to E; A being always taken for the point most distant from the vertex. Now to ascertain the sign of the terms $\mp(r+r'-c)^{\frac{3}{2}}$ in T, we shall observe that when the time is very small, it must be nearly proportional to the chord AB, which will be small in comparison with r, r', and if we develop T, [750],



the perihelion; in other cases we must use the sign $+$. T being equal to $365^d,25638$, we have $\frac{T}{12\pi}=9^d,688724$. [750′]

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according to the powers of c, neglecting c^2, c^3, &c. we shall have, by using $R=r+r'$, for brevity, $\mathrm{T}=\frac{T}{12\pi}\cdot\left\{R^{\frac{3}{2}}+\tfrac{3}{2}R^{\frac{1}{2}}c\mp R^{\frac{3}{2}}\pm\tfrac{3}{2}R^{\frac{1}{2}}c\right\}$, which cannot be proportional to c, unless the two terms $R^{\frac{3}{2}}\mp R^{\frac{3}{2}}$ destroy each other, which is the case when the upper sign takes place; therefore when c is small we shall have [750a]

$$\mathrm{T}=\frac{T}{12\pi}\cdot\left\{(r+r'+c)^{\frac{3}{2}}-(r+r'-c)^{\frac{3}{2}}\right\}. \qquad [750b]$$

The most distant point A from the vertex being fixed, suppose the other point B, to move from A towards E, the terms $r+r'+c$, and $r+r'-c$, will always be positive, because in the triangle BCA we have $AC+BC>AB$, or $AC+BC-AB>0$, that is $r+r'-c>0$. At the point E we shall have $r+r'-c=0$. In proceeding from E towards F, and beyond F, $r+r'-c$ always exceeds 0, so that there can be no change of sign except when the point B passes through E; and if a change of sign then take place, we shall have in the branch EF, $\mathrm{T}=\frac{T}{12\pi}\cdot\left\{(r+r'+c)^{\frac{3}{2}}+(r+r'-c)^{\frac{3}{2}}\right\}$; [750c] and the rules for applying the signs will agree with the above. It only remains therefore to examine whether this formula is exact for any one point of the branch EF. Now putting $CD=e$, we shall take the points A, F, so that the absciss $DG=4e$, and $CA=CF=r=r'$. Then by the nature of the parabola the ordinate $CH=CI=2e$, the ordinate $FG=AG=4e$; and $CF=5e$. Hence $r=r'=5e$, $c=2.FG=8e$; $r+r'+c=18e$, $r+r'-c=2e$, and [750] becomes

$$\mathrm{T}=\frac{T}{12\pi}\cdot\left\{(18e)^{\frac{3}{2}}\mp(2e)^{\frac{3}{2}}\right\}=\frac{T}{12\pi}\cdot(2e)^{\frac{3}{2}}\cdot\left\{9^{\frac{3}{2}}\mp1\right\}=\frac{T}{12\pi}\cdot(2e)^{\frac{3}{2}}\cdot\left\{27\mp1\right\}. \qquad [750d]$$

In a similar manner we may find the time of describing IDH, by putting $CI=r=2e$, $CH=r'=2e$, $HI=c=4e$, hence $r+r'+c=8e$, $r+r'-c=0$, and from [750] $\mathrm{T}=\frac{T}{12\pi}\cdot\left\{(8e)^{\frac{3}{2}}\right\}=\frac{T}{12\pi}\cdot(2e)^{\frac{3}{2}}.8$. This value is to the former [750d], [750e] as $8:27\mp1$, which ought to represent the ratio of the areas IDH, $ADFCA$, [365′]; the former of which is $=\frac{2}{3}.HI.DC=\frac{2}{3}.4e.e=\frac{8}{3}e^2$. The latter is equal to the parabolic space $ADFG$ less the triangle ACF. This parabolic space is $=\frac{2}{3}.DG.AF=\frac{2}{3}.4e.8e=\frac{64}{3}e^2$, the triangle $ACF=\frac{1}{2}.CG.AF=\frac{1}{2}.3e.8e=12e^2$, and $\frac{64}{3}e^2-12e^2=\frac{28e^2}{3}$. Hence the space IDH : space $ADFCA::\frac{8}{3}e^2:\frac{28e^2}{3}::8:28$, and as this ratio ought to be the same as $8:27\mp1$, the lower sign must take place at the point F, which was to be proved.

In the hyperbola, a is negative, [378′]; z and z' become greater than unity;* the arches arc. cos. z, and arc. cos. z', are imaginary; and we shall have, by using hyperbolic logarithms,†

[751]
$$\text{arc. cos. } z = \frac{1}{\sqrt{-1}} . \log. (z + \sqrt{z^2 - 1});$$
$$\text{arc. cos. } z' = \frac{1}{\sqrt{-1}} . \log. (z' + \sqrt{z'^2 - 1});$$

the formula (a) [748] thus becomes, by changing a into $-a$,

[752]
$$\text{T} = \frac{a^{\frac{3}{2}} . T}{2\pi} . \{\sqrt{z'^2 - 1} \mp \sqrt{z^2 - 1} - \log. (z' + \sqrt{z'^2 - 1}) \pm \log. (z + \sqrt{z^2 - 1})\}.$$

The formula (a) [748] gives the time employed by a body in descending in a right line towards the focus, setting out from a given point with a given velocity; to obtain this, we must suppose the ellipsis to be infinitely flattened. If we suppose, for example, that the body sets out from a point at rest at the distance $2a$ from the focus, and the time T be required, in which it would fall through the distance c; we shall have, in this case, $R = 2a + r$; $r = 2a - c$; hence‡ $z' = -1$; $z = \frac{c - a}{a}$; the formula (a) [748] will therefore give

[753]
$$\text{T} = \frac{a^{\frac{3}{2}} . T}{2\pi} . \left\{ \pi - \text{arc. cos. } \left(\frac{c - a}{a}\right) + \sqrt{\frac{2ac - c^2}{a^2}} \right\}.$$

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* (510) Putting a negative in the values of z, z', [744], they become

$$z = \frac{-2a - R + c}{-2a} = 1 + \frac{R - c}{2a}; \qquad z' = \frac{-2a - R - c}{-2a} = 1 + \frac{R + c}{2a};$$

and as $R = CA + CB$, [728], always excels AB or c, the term $R - c$ must be positive, consequently z, z', exceed unity.

† (511) We have $c^{A.\sqrt{-1}} = \cos. A + \sqrt{-1} . \sin. A$, [13] Int. whose logarithm divided by $\sqrt{-1}$ gives $A = \frac{1}{\sqrt{-1}} . \log. \left\{\cos. A + \sqrt{-1} . \sin. A\right\}$; and as $z = \cos. A$, [745b], this becomes $\text{arc. cos. } z = \frac{1}{\sqrt{-1}} . \log. \left\{z + \sqrt{z^2 - 1}\right\}$. The expression of arc. cos. z', is found in a similar manner; these agree with [751].

‡ (512) These values of R, r give $R = 4a - c$, which being substituted in z, z', [744], they become $z' = -1$, $z = \frac{c - a}{a}$, and as $\cos. A = z$, $\cos. B = z'$, [745b],

There is however an essential difference between the direct motion towards the focus, and the motion in an infinitely flattened ellipsis. In the first case the body having arrived at the focus, passes through it, and ascends to the same distance on the opposite side; in the second case, the body [753'] having arrived at the focus, returns back to the point from which it set out. A tangential velocity at the aphelion, however small it might be, would be sufficient to produce this difference; and such a change in the velocity would have no effect in altering the time of descent to the focus.

28. As the circumstances of the original motions of the heavenly bodies are not known from observations, we cannot determine, by the formulas of § 26, the elements of their orbits. It is necessary for this purpose to compare their respective positions, found by observations, at different epochs; this is rendered more difficult by the observations not being made from the centre of their motions. With respect to the planets, we may, by means of their oppositions or conjunctions, obtain their longitudes, as if they were observed from the centre of the sun. This circumstance, taken in connexion [753''] with the smallness of the excentricities, and the inclinations of the orbits to the ecliptic, furnishes a very simple method of obtaining their elements. In the present state of astronomy, the elements of these orbits require but very small corrections; and as the variations of the distances of the planets [753'''] from the earth, are never so great as to render them invisible, we may observe them at all times, and by comparing a great number of observations, we may rectify the elements of their orbits, and correct the effect of small errors to which the observations are liable. This is not the case with comets; we see them only when near the perihelion, and if the observations made at the time of their appearance are not sufficient to determine their elements, we shall have no method of tracing in our minds the paths of these bodies in the immensity of space; and when in the course of ages they shall approach again towards [753iv] the sun, it will be impossible to recognise them. It is therefore important

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we shall have $B=\pi$, whose cosine $=-1$, and sine $=0$, also $\cos. A=z=\frac{c-a}{a}$, hence $\sin. A=\sqrt{1-\left(\frac{c-a}{a}\right)^2}=\sqrt{\frac{2ac-c^2}{a^2}}$. These values being substituted in [748] it becomes as in [753].

to determine the elements of the orbit, by the observations made during the appearance of the comet; but this problem, taken rigorously, exceeds the power of analysis, and we are obliged to have recourse to methods of approximation, to obtain the first values of the elements, which may afterwards be corrected with all the precision that the observations may require.

If we use observations taken at distant intervals, the elimination of the unknown quantities leads to impracticable calculations; we must therefore confine ourselves to observations made near to each other, and even with this
[753^{v}] restriction, the problem is extremely difficult. After having reflected on the subject, it has appeared to me that instead of using directly the observations, it would be better to deduce from them certain quantities which would furnish an exact and simple result; and I am convinced that the quantities which best fulfil this condition, are the geocentric longitude and latitude of the comet, at a given time, and their first and second differentials divided by the corresponding powers of the element of the time; for by means of these given quantities, we may determine the elements, rigorously and with
[753^{vi}] facility, without any integration, using merely the differential equations of the orbit. This manner of considering the problem allows us to use a great number of observations taken near to each other, but comprising a considerable interval between the extreme observations, which is very useful
[753^{vii}] in diminishing the influence of the errors to which these observations are always liable, on account of the nebulous appearance surrounding comets. I shall, in the first place, give the necessary formulas to determine the first differentials of the longitude and latitude, from any number of observations taken at short intervals; I shall then determine the elements of the orbit of
[753^{viii}] a comet, by means of these first differentials; lastly, I shall explain the method which appears to me the most simple, to correct these elements, by three observations taken at distant intervals.

[753^{ix}] 29. At a given epoch, let α be the geocentric longitude of a comet, θ its northern geocentric latitude, the southern latitudes being supposed negative. If we denote by s, the number of days elapsed since the epoch, the
[753^{x}] geocentric longitude and latitude of the comet, after that interval, will be expressed by means of the formula (*i*) of § 21 [617], by the two series

$$\alpha+s.\left(\frac{d\alpha}{ds}\right)+\frac{s^2}{1.2}.\left(\frac{d^2\alpha}{ds^2}\right)+\frac{s^3}{1.2.3}.\left(\frac{d^3\alpha}{ds^3}\right)+\&c.;$$ [754]

$$\theta+s.\left(\frac{d\theta}{ds}\right)+\frac{s^2}{1.2}.\left(\frac{d^2\theta}{ds^2}\right)+\frac{s^3}{1.2.3}.\left(\frac{d^3\theta}{ds^3}\right)+\&c.$$

We shall determine the values of α, $\left(\frac{d\alpha}{ds}\right)$, $\left(\frac{d^2\alpha}{ds^2}\right)$, &c.; θ, $\left(\frac{d\theta}{ds}\right)$, &c.; by means of several observed geocentric longitudes and latitudes. To obtain these quantities in the most simple manner, we shall consider the infinite series expressing the geocentric longitude. The coefficients of the powers of s in this series, ought to be determined by the condition that it will [754'] represent each observed longitude, by substituting for s, the number of days which corresponds to it; we shall thus have as many equations as observations, and if the number of observations be n, we can determine, by means of them, only n quantities of the infinite series α, $\left(\frac{d\alpha}{ds}\right)$, &c. But [754''] we ought to observe, that s being supposed very small, we may neglect the terms multiplied by s^n, s^{n+1}, &c.; this will reduce the infinite series to its n first terms, which may be determined by the n observations. These values will be merely approximations towards the truth, and the degree of correctness will depend on the smallness of the neglected terms. They will become more correct by decreasing s, and by increasing the number of [754'''] observations. The theory of interpolations is reduced by this means to the finding of a rational and integral* function of s, of such form, that by substituting for s the number of days corresponding to each observation, it [754^{iv}] will become equal to the observed longitude.

We shall represent by β, β', β'', &c., the observed longitudes of the comet, and by i, i', i'', &c., the number of days they fall after the given epoch; [754^{v}] these numbers being supposed negative for observations made before the epoch. If we put

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* (513) A rational and integral function of s, is of the form $A+Bs+Cs^2+Ds^3+\&c.$ depending only on integral positive powers of s, without surds, and without fractions containing s in the denominators.

$$\frac{\beta'-\beta}{i'-i}=\delta\beta\,; \qquad \frac{\beta''-\beta'}{i''-i'}=\delta\beta'\,; \qquad \frac{\beta'''-\beta''}{i'''-i''}=\delta\beta''\,; \qquad \&c.\,;$$

[755]

$$\frac{\delta\beta'-\delta\beta}{i''-i}=\delta^2\beta\,; \qquad \frac{\delta\beta''-\delta\beta'}{i'''-i'}=\delta^2\beta'\,; \qquad \&c.\,;$$

$$\frac{\delta^2\beta'-\delta^2\beta}{i'''-i}=\delta^3\beta\,; \qquad \&c.\,;$$

$$\&c.\,;$$

the required function will be

[756]

$$\beta+(s-i).\delta\beta+(s-i).(s-i').\delta^2\beta+(s-i).(s-i').(s-i'').\delta^3\beta+\&c.\,;$$

for it is easy to prove that if we put in succession $s=i$, $s=i'$, $s=i''$, &c.; it will become β, β', β'', &c.*

If we now compare the preceding function with the following, [754],

[757]

$$\alpha+s.\left(\frac{d\alpha}{ds}\right)+\frac{s^2}{1.2}.\left(\frac{d^2\alpha}{ds^2}\right)+\&c.,$$

we shall have, by putting the coefficients of like powers of s equal to each other,

[758]

$$\alpha=\beta-i.\delta\beta+i.i'.\delta^2\beta-i.i'.i''.\delta^3\beta+\&c.\,;$$

$$\left(\frac{d\alpha}{ds}\right)=\delta\beta-(i+i').\delta^2\beta+(ii'+ii''+i'i'').\delta^3\beta-\&c.\,;$$

$$\tfrac{1}{2}.\left(\frac{d^2\alpha}{ds^2}\right)=\delta^2\beta-(i+i'+i'').\delta^3\beta+\&c.\,;$$

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* (514) Thus if $s=i$, this becomes β. If $s=i'$, it becomes $\beta+(i'-i).\delta\beta$, all the other terms vanishing; and by substituting $\delta\beta=\frac{\beta'-\beta}{i'-i}$, it changes into $\beta+(\beta'-\beta)$, or simply β'. If $s=i''$, it becomes $\beta+(i''-i).\delta\beta+(i''-i).(i''-i').\delta^2\beta$, and by substituting for $(i''-i).\delta^2\beta$ its value [755], $\delta\beta'-\delta\beta$, it becomes

$$\beta+(i''-i).\delta\beta+(i''-i').(\delta\beta'-\delta\beta) \qquad \text{or} \qquad \beta+(i'-i).\delta\beta+(i''-i').\delta\beta',$$

which by using the values of $\delta\beta$, $\delta\beta'$, [755], changes into $\beta+(\beta'-\beta)+(\beta''-\beta')$, or simply β''; and in the same manner the others may be proved. It is to be observed that this is the usual rule of interpolation, as given by Newton, in page 129, Vol. III, of Horsley's edition of his works. This appears by changing the symbols of Newton into those of the present section in the following manner,

[756a]
for a; b, $2b$ &c.; c, $2c$ &c.; d, $2d$ &c.; e, $2e$ &c.;
write β; $-\delta\beta$, $-\delta\beta'$ &c.; $\delta^2\beta$, $\delta^2\beta'$ &c.; $-\delta^3\beta$, $-\delta^3\beta'$ &c.; $\delta^4\beta$, $\delta^4\beta'$ &c.;

the differential coefficients of higher orders will not be of any use.* The
coefficients of these expressions are alternately positive and negative; the [758′]
coefficient of $d^r\beta$, neglecting its sign, is the product of r quantities i, i', [758″]
$i'' \ldots i^{(r-1)}$, taken r by r, in the value of α; it is the sum of the products of
the same quantities, taken $r-1$ by $r-1$, in the value of $\left(\frac{d\alpha}{ds}\right)$; lastly,
it is the sum of the products of these quantities, taken $r-2$ by $r-2$, in [758‴]
the value of $\frac{1}{2}\cdot\left(\frac{d^2\alpha}{ds^2}\right)$.

If we put $\gamma, \gamma', \gamma''$, &c., for the observed geocentric latitudes of the comet; [758^{iv}]
we shall have the values of θ, $\left(\frac{d\theta}{ds}\right)$, $\left(\frac{d^2\theta}{ds^2}\right)$, &c., by changing in the
preceding expressions of α, $\left(\frac{d\alpha}{ds}\right)$, $\left(\frac{d^2\alpha}{ds^2}\right)$, &c., the quantities β, β', β'', [758^v]
&c., into $\gamma, \gamma', \gamma''$, &c.

These expressions are rendered more accurate by increasing the number
of observations, and decreasing the intervals between them; we might
therefore use all the observations near the epoch, if they were accurate; but [758^{vi}]
the errors to which they are liable would lead to an inaccurate result; to
diminish the influence of these errors, we must therefore increase the interval
of the extreme observations, when we augment their number. We may in
this way, with five observations, embrace an interval of thirty-five or forty [758^{vii}]
degrees,† which ought to give with considerable exactness the geocentric
longitudes and latitudes, and their first and second differential coefficients.

If the epoch made choice of, is such that there is an equal number of [758^{viii}]
observations before and after it, so that each longitude after the epoch, has a

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for	p;	q;	r,	&c.;
write	$-(s-i)$;	$(s-i)\cdot(s-i')$;	$-(s-i)\cdot(s-i')\cdot(s-i'')$,	&c.;

[756b]

and then Newton's value of $RS = a + bp + cq + dr + \&c.$ will become the same as [756].

* (515) It will be seen in the final equations [806], that no differentials of α, θ, higher than the second order occur, it will therefore be of no use to compute them.

† (516) The degrees here mentioned are of the centesimal division, they correspond to $31^d\frac{1}{2}$ and 36^d, in sexagesimals. To form a rough estimate of the degree of accuracy of this [757a]

corresponding one at an equal interval before the epoch; this condition will render the values of α, $\left(\frac{d\,\alpha}{d\,s}\right)$, $\tfrac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,s^2}\right)$, more correct; and it is easy to prove that additional observations, taken at equal distances on each side of

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method, the formula [756] was applied to the values of β, β', β'', β''', β^{iv}, β^{v}, given in the following table, in sexagesimals,

	days	β	1st diff.	2d diff.	3d diff.	4th dif	5th dif.
		d m s	d m s	m s	m s	m s	s
i	0	$\beta\;=\;$ 0 00 00,0	8 20 01,3				
i'	6	$\beta'=\;$ 8 20 01,3	8 09 41,2	— 10 20,1	— 9 00,5		
i''	12	$\beta''=$ 16 29 42,5	7 50 20,6	— 19 20,6	— 6 46,4	2 14,1	20,5
i'''	18	$\beta'''=$ 24 20 03,1	7 24 13,6	— 26 07,0	— 4 11,8	2 34,6	
i^{iv}	24	$\beta^{\text{iv}}=$ 31 44 16,7	6 53 54,8	— 30 18,8			
i^{v}	30	$\beta^{\text{v}}=$ 38 38 11,5					

which numbers were taken for intervals of six days, from Delambre's table of the heliocentric motion of a comet in a parabola, whose perihelion distance is equal to the mean distance of the earth from the sun; it being supposed that if these heliocentric values were assumed for the geocentric longitudes, in the formula [756], the errors from the neglect of any of the terms of that formula, would generally be of the same order in these heliocentric longitudes as would occur in the corresponding geocentric places of the comet; we shall therefore, in the rest of this note, suppose β, β', &c. to represent the observed geocentric longitudes of the comet at intervals of six days. In this case the first observation is $\beta=0$, the fifth $\beta^{\text{iv}}=31^d\,44^m\,16^s{,}7$, the interval being greater than $31^d\tfrac{1}{2}$, one of the limits mentioned in [757a] for five observations. The sixth term, or $\beta^{\text{v}}=38^d\,38^m\,11^s{,}5$, was added so as to include the differences of the fifth order. Then the intervals

$$i'-i=i''-i'=i'''-i''=i''''-i'''=6;\qquad i''-i=i'''-i'=\&\text{c.}=12;$$
$$i'''-i=i''''-i'=\&\text{c.}=18.$$

Therefore, by dividing the numbers in the column of first differences by 6, we shall obtain $\delta\,\beta$, $\delta\,\beta'$, $\delta\,\beta''$, &c. [755]. Those of the column of second differences divided by $6\,.\,12$ or 72 give $\delta^2\beta$, $\delta^2\beta'$, &c. Those of the column of third differences divided by $6\,.\,12\,.\,18$, or 1296 give $\delta^3\beta$, $\delta^3\beta'$, &c.; and in the same manner $\delta^5\beta=\frac{20^s{,}5}{6.12.18.24.30}$; so that by taking only the five first observations, and rejecting the sixth, we should neglect this value of $\delta^5\beta$. We shall now compute the effect of this neglected term, in the values of α, $\left(\frac{d\,\alpha}{d\,s}\right)$, and $\tfrac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,s^2}\right)$. The term produced in α, [758], is, $-i.i'.i''.i'''.i''''.\delta^5\beta$, which becomes 0, if we take the epoch at the time of one of the observations; for then one of the quantities i, i', i'', &c. will be nothing, and the expression will become 0. If we suppose the epoch to be taken at the middle time between the two first observations we shall have $i=-3$, $i'=3$, $i''=9$, $i'''=15$, $i''''=21$, and the preceding expression will

[758a]

[758b]

the epoch, will only add to these values, quantities which will be to their last terms, of the same order as the ratio of $s^2 . \left(\frac{d^2 \alpha}{d s^2}\right)$ to α.* This [758^{ix}]

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become $\frac{3.3.9.15.21}{6.12.18.24.30} . 20^s,5 = 0'',5$. If we take the epoch at the middle time between the extreme observations it will be less, and in no case will this term amount to a second, when the epoch is taken between the extreme observations. The coefficient of $\delta^5 \beta$ in the value of $\left(\frac{d \alpha}{d s}\right)$ is $i . (i'.i''.i''' + i'.i''.i'''' + i'.i'''.i'''' + i''.i'''.i'''') + i'.i''.i'''.i''''$. This, in the case mentioned in [758*b*], where $i = -3$, $i' = 3$, $i'' = 9$, $i''' = 15$, $i'''' = 21$, becomes -5751, and the corresponding term of $\left(\frac{d \alpha}{d s}\right)$ is $\frac{-5751}{6.12.18.24.30} . 20^s,5 = -0^s,1$. Taking the epoch any where between the first and fifth observations, it is evident that no one of the terms of this coefficient of $\left(\frac{d \alpha}{d s}\right)$, as i, i', i'', i''', can exceed $6 . 12 . 18 . 24$, so that the five terms, which compose this coefficient, cannot be so great, and in general must be much less than $5 . (6 . 12 . 18 . 24)$; therefore the term of $\left(\frac{d \alpha}{d s}\right)$ produced by $\delta^5 \beta$ must be much less than this quantity multiplied by $\delta^5 \beta$, that is, it must be much less than $\frac{5.6.12.18.24}{6.12.18.24.30} . 20^s,5 = \frac{20^s,5}{6} = 3^s,4$. Hence it is evident that it must be very small. In like manner the coefficient of $\delta^5 \beta$ in the expression of $\frac{1}{2} . \left(\frac{d^2 \alpha}{d s^2}\right)$ must, independent of its sign, consist of the sum of the products of the five quantities i, i', i'', i''', i'''', taken three and three, thus $i . i' . i'' + i . i' . i''' +$ &c. The number of terms of this series, by the doctrine of combinations, is $\frac{1.2.3.4.5}{1.2.3} = 20$, and as the greatest term, when the epoch is between the first and fifth observations, cannot exceed $12 . 18 . 24$, the whole sum must be much less than $20 . (12 . 18 . 24)$, and the corresponding term of $\frac{1}{2} . \left(\frac{d^2 \alpha}{d s^2}\right)$, independent of its sign, must be much less than this quantity multiplied by $\delta^5 \beta$, or $\frac{20.12.18.24}{6.12.18.24.30} . 20^s,5 = 2^s,2$; consequently this term must be very small. From this rough essay we perceive that, with the limits assigned by the author, in the length of the described arch, and in the number of observations, the errors of the formula must be very small.

* (517) This ratio is not generally correct for any one of the quantities α, $\left(\frac{d \alpha}{d s}\right)$, $\frac{1}{2} . \left(\frac{d^2 \alpha}{d s^2}\right)$. It is however correct for the last of them, when the epoch is taken at the middle

symmetrical form takes place when all the observations are equidistant, and the epoch is placed at the middle of the interval comprised by the [758x] observations; it is therefore advantageous to use such observations. In

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observation, and an equal number of equidistant observations are taken on each side of the epoch. In general the neglected terms are rather greater than is stated by the author. This may be proved in the following manner. The general expression of the longitude, which we shall call l, is

[758c] $$l=\beta+(s-i)\,.\,\delta\beta+(s-i)\,.\,(s-i')\,.\,\delta^2\beta+(s-i)\,.\,(s-i')\,.\,(s-i'')\,.\,\delta^3\beta+\&c.$$

[756], which may also be put under this form

[758d] $$\begin{aligned}l=\beta'&+(s-i')\,.\,\delta\beta'+(s-i')\,.\,(s-i'')\,.\,\delta^2\beta'+(s-i')\,.\,(s-i'')\,.\,(s-i''')\,.\,\delta^3\beta'\\&+(s-i')\,.\,(s-i'')\,.\,(s-i''')\,.\,(s-i'''')\,.\,\delta^4\beta'+\&c.\end{aligned}$$

The only difference in these two expressions consists in commencing the series of longitudes β, β', &c. and times i, i', &c. at β', i', instead of β, i. Now supposing in the first place, that there were five observations, β, β', β'', β''', β'''', and that if the series were extended on either side, their fourth differences would be constant, the expression of l, [758d] would give the true longitude, neglecting $\delta^5\beta'$, &c. which would vanish, because the fourth differences [758e] are constant. If we suppose only the three middle observations, β', β'', β''', to have been made, the formula [758c] would, according to this method of calculation, give the value of l, by putting another accent on β and i, because the first terms of β', β'', β''', would commence with β' and i', instead of β and i; this value of l would therefore be

[758f] $$\begin{aligned}l&=\beta'+(s-i')\,.\,\delta\beta'+(s-i')\,.\,(s-i'')\,.\,\delta^2\beta'\\&=\{\beta'-i'\,.\,\delta\beta'+i'\,.\,i''\,.\,\delta^2\beta'\}+s\,.\,\{\delta\beta'-(i'+i'')\,.\,\delta^2\beta'\}+s^2\,.\,\delta^2\beta'.\end{aligned}$$

Hence we see that by taking three observations β', β'', β''', and afterwards adding another observation at each extreme, as β, β'''', the value of l will be increased by the terms depending on $\delta^3\beta'$, $\delta^4\beta'$ [758d], which we shall denote by L, and we shall have

[758g] $$\begin{aligned}L&=(s-i')\,.\,(s-i'')\,.\,(s-i''')\,.\,\delta^3\beta'+(s-i')\,.\,(s-i'')\,.\,(s-i''')\,.\,(s-i'''')\,.\,\delta^4\beta'\\&=\{-i'\,.\,i''\,.\,i'''\,.\,\delta^3\beta'+i'\,.\,i''\,.\,i'''\,.\,i''''\,.\,\delta^4\beta'\}\\&+s\,.\,\{(i'\,.\,i''+i'\,.\,i'''+i''\,.\,i''')\,.\,\delta^3\beta'-(i'\,.\,i''\,.\,i'''+i'\,.\,i''\,.\,i''''+i'\,.\,i'''\,.\,i''''+i''\,.\,i'''\,.\,i'''')\,.\,\delta^4\beta'\}\\&+s^2\,.\,\{-(i'+i''+i''')\,.\,\delta^3\beta'+(i'.i''+i'.i'''+i'.i''''+i''.i'''+i''.i''''+i'''.i'''')\,.\,\delta^4\beta'\}+\&c.\end{aligned}$$

If we compare the value of l, [758f], with the general formula [757],

$$l=\alpha+s\,.\,\left(\frac{d\alpha}{ds}\right)+\tfrac{1}{2}\,.\,s^2\,.\,\left(\frac{d^2\alpha}{ds^2}\right)+\&c.$$

[758h] we shall get $\alpha=\beta'-i'\,.\,\delta\beta'+i'\,.\,i''\,.\,\delta^2\beta'$, $\left(\frac{d\alpha}{ds}\right)=\delta\beta'-(i'+i'')\,.\,\delta^2\beta'$, $\left(\frac{d^2\alpha}{ds^2}\right)=2\,\delta^2\beta'$. And if we denote by α', $\left(\frac{d\alpha'}{ds}\right)$, and $\left(\frac{d^2\alpha'}{ds^2}\right)$, the increments of the

general, it will be useful to fix the epoch nearly in the middle of this interval; because the number of days from that time to the extreme observations will be less, which will render the series more converging. The calculation may [758xi]

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preceding terms respectively, arising from the introduction of the terms of L, [758g], we shall have $\alpha' = -i' . i'' . i''' . \delta^3 \beta' + i' . i'' . i''' . i'''' . \delta^4 \beta'$;

$$\left(\frac{d\alpha'}{ds}\right) = (i'.i''+i'.i'''+i''.i''').\delta^3\beta' - (i'.i''.i'''+i'.i''.i''''+i'.i'''.i''''+i''.i'''.i'''').\delta^4\beta';$$

$$\left(\frac{d^2\alpha'}{ds^2}\right) = -(i'+i''+i''').\delta^3\beta' + (i'.i''+i'.i'''+i'.i''''+i''.i'''+i''.i''''+i'''.i'''').\delta^4\beta'.$$

Putting also α'', $\left(\frac{d\alpha''}{ds}\right)$, $\left(\frac{d^2\alpha''}{ds^2}\right)$, for the last terms of α, $\left(\frac{d\alpha}{ds}\right)$, $\left(\frac{d^2\alpha}{ds^2}\right)$, [758h], we shall have $\alpha'' = i' . i'' . \delta^2 \beta'$; $\left(\frac{d\alpha''}{ds}\right) = -(i'+i'') . \delta^2 \beta'$, $\left(\frac{d^2\alpha''}{ds^2}\right) = 2\,\delta^2 \beta'$. Now if the epoch be taken at the time of the second, third, or fourth observation, we shall have one of the quantities i', i'', i''', equal to nothing, consequently $\alpha' = 0$. In general we shall have the ratio of α' to α'' expressed by $(-i''' . \delta^3 \beta' + i''' . i'''' . \delta^4 \beta') : \delta^2 \beta$, and as $\delta^4 \beta'$ is of a less order than $\delta^3 \beta'$ it may be neglected, this ratio will become $-i''' . \delta^3 \beta' : \delta^2 \beta'$, and since i''' is of the same order as s, it will be of the same order as $s\,\delta^3 \beta' : \delta^2 \beta'$. This may be expressed in a different manner, by observing that from [758h], α is of the finite order β', $\left(\frac{d\alpha}{ds}\right)$ is of the first order of differences $\delta \beta'$, and $\tfrac{1}{2} . \left(\frac{d^2\alpha}{ds^2}\right)$ is of the second order $\delta^2 \beta'$, and as $\delta^3 \beta'$ is of the next higher order to $\delta^2 \beta'$, we may, in counting the order of the terms, consider the ratio of $\delta^3 \beta'$ to $\delta^2 \beta'$, to be of the same order as $\left(\frac{d\alpha}{ds}\right)$ to α, consequently $\alpha' : \alpha''$ is of the same order as $s . \left(\frac{d\alpha}{ds}\right) : \alpha$, instead of $s^2 . \left(\frac{d^2\alpha}{ds^2}\right) : \alpha$, as was stated by the author in [758ix]. In like manner

$$\left(\frac{d\alpha'}{ds}\right) : \left(\frac{d\alpha''}{ds}\right) :: (i' . i'' + i' . i''' + i'' . i''') . \delta^3 \beta'$$
$$- (i' . i'' . i''' + i' . i'' . i'''' + i' . i''' . i'''' + i'' . i''' . i'''') . \delta^4 \beta' : -(i' + i'') . \delta^2 \beta',$$

and by neglecting the term $\delta^4 \beta'$, as of a less order than $\delta^3 \beta'$, and observing that i', i'', i''', are each of the order s, and the ratio of $\delta^3 \beta'$ to $\delta^2 \beta'$ of the same order as $\left(\frac{d\alpha}{ds}\right)$ to α, this ratio of $\left(\frac{d\alpha'}{ds}\right) : \left(\frac{d\alpha''}{ds}\right)$ will become of the same order as $s . \left(\frac{d\alpha}{ds}\right) : \alpha$, which also differs from [758ix]. Lastly,

$$\left(\frac{d^2\alpha'}{ds^2}\right) : \left(\frac{d^2\alpha''}{ds^2}\right) :: -(i'+i''+i''').\delta^3\beta' + (i'.i''+i'.i'''+i'.i''''+i''.i'''+i''.i''''+i'''.i'''').\delta^4\beta' : 2\delta^2\beta'.$$

If in this we take the epoch at the middle observation, making $i'' = 0$, and take $i' = -i'''$, also be simplified by fixing the epoch at the instant of one of the observations; which will give directly the values of α and θ.

When we shall have found, in the preceding manner, $\left(\frac{d\alpha}{ds}\right)$, $\left(\frac{d^2\alpha}{ds^2}\right)$,
[758xii] $\left(\frac{d\theta}{ds}\right)$, and $\left(\frac{d^2\theta}{ds^2}\right)$, we may deduce from them the first and second
differentials of α and θ, divided by the corresponding powers of the element
of the time, in the following manner. If we neglect the masses of the
[758xiii] planets and comets, in comparison with that of the sun, taken as the *unity*
of mass; and take also for the *unity* of distance, the mean distance of the
earth from the sun; the mean motion of the earth about the sun will be by
[758xiv] § 25 [705''], the measure of the time t. Therefore let λ be the number of
seconds which the earth describes in a day, by means of its mean sidereal
motion; the time t, corresponding to the number of days s, will be λs; we
shall therefore have*

[759] $$\left(\frac{d\alpha}{dt}\right) = \frac{1}{\lambda}\cdot\left(\frac{d\alpha}{ds}\right); \qquad \left(\frac{d^2\alpha}{dt^2}\right) = \frac{1}{\lambda^2}\cdot\left(\frac{d^2\alpha}{ds^2}\right).$$

Using common logarithms, we have by observation,† $\log.\lambda = 4{,}0394622$,
[759'] [or 3,5500072 sex.]; also $\log.\lambda^2 = \log.\lambda + \log.\frac{\lambda}{R}$, R being the radius

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it will become simply $i'.i'''.\delta^4\beta' : 2\,\delta^2\beta'$, and by putting $i'.i'''$, of the order s^2, and the ratio of $\delta^4\beta'$ to $\delta^2\beta'$, of the same order as that of $\left(\frac{d^2\alpha}{ds^2}\right)$ to α, it will become of the same order as $s^2.\left(\frac{d^2\alpha}{ds^2}\right):\alpha$, as is stated in [758ix]; but this takes place only when $i'+i''+i'''=0$, for if this quantity is finite, and of the order s, the term $\delta^3\beta'$ will not vanish, and we shall have $\left(\frac{d^2\alpha'}{ds^2}\right) : \left(\frac{d^2\alpha''}{ds^2}\right)$ of the same order as $s.\left(\frac{d\alpha}{ds}\right):\alpha$. Thus the ratio of $s^2.\left(\frac{d^2\alpha}{ds^2}\right)$ to α, given in [758ix], can hardly be said to be correct, in any point of view, in the example we have now computed, for three and five observations; and it is evident that the same reasoning will apply with scarcely any alteration to a greater number of observations.

* (517a) The second members of the equations [759] are deduced from the first, by changing dt into $\lambda\,ds$, as in [758iv].

† (518) Using the centesimal division of the circle and day, the number of seconds in the whole circumference is 4000000'', the number of days in a sidereal year, 365,25638,

of the circle reduced to seconds; hence we have $\log. \lambda^2 = 2{,}2750444$, [or [759'']
1,7855894 sex.]; therefore if we reduce into seconds the values of $\left(\frac{d\alpha}{ds}\right)$
and $\left(\frac{d^2\alpha}{ds^2}\right)$, we shall have the logarithms of $\left(\frac{d\alpha}{dt}\right)$ and $\left(\frac{d^2\alpha}{dt^2}\right)$, by
subtracting from the logarithms of these values, the logarithms 4,0394622, [759''']
and 2,2750444 respectively, [3,5500072, and 1,7855894 sex.] In like
manner we shall have the logarithms of $\left(\frac{d\theta}{dt}\right)$ and $\left(\frac{d^2\theta}{dt^2}\right)$, by subtracting [759^iv]
the same logarithms respectively from the logarithms of their values reduced
to seconds.

It is on the precision of the values of α, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, θ, $\left(\frac{d\theta}{dt}\right)$, and [759^v]
$\left(\frac{d^2\theta}{dt^2}\right)$, that the exactness of the following rules depends, and as their
computation is very simple, we must select and augment the number of
observations, so as to obtain these quantities with the greatest correctness.
We shall now determine, by means of these values, the elements of the orbit
of the comet; and to generalize the results, we shall consider the motion
of a system of bodies acted upon by any forces whatever.

30. Put x, y, z, for the rectangular co-ordinates of the first body; x', y', z', [759^vi]
for those of the second body, and so on for the rest. Suppose that the first
body is urged in directions parallel to the axes x, y, z, by the forces X, Y, Z, [759^vii]
respectively, *tending to decrease these co-ordinates*; that the second body is [759^viii]
urged in directions parallel to the same axes, by the forces X', Y', Z'; and

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hence $\log. \lambda = \log. 4000000 - \log. 365{,}25638 = 6{,}6020600 - 2{,}5625978 = 4{,}0394622$.
The radius being 1, the semi-circumference is $\pi = 3{,}1459$, &c. whose log. is 0,4971499,
and as the number of seconds in the semi-circumference is 2000000 whose log. is 6,3010300,
we have log. radius in seconds $= 6{,}3010300 - 0{,}4971499 = 5{,}8038801 = \log. R$. Hence [759a]
$\log. \lambda^2 = 2 \log. \lambda - \log. R = 2 \times 4{,}0394622 - 5{,}8038801 = 2{,}2750443$, nearly. If we wish to
use the common division of the circle into 360^d, or 1296000^s instead of $4000000''$, we must add
to the preceding logarithms the logarithm of $\frac{1296000}{4000000} = \log.$ of $0{,}324 = 9{,}5105450$; adding [759b]
this to 4,0394622, 2,2750444, and neglecting 10 in the index, they become respectively [759c]
3,5500072 and 1,7855894.

so on for the others. The motions of all these bodies will be given by the differential equations of the second order,*

[760]
$$0=\frac{ddx}{dt^2}+X; \qquad 0=\frac{ddy}{dt^2}+Y; \qquad 0=\frac{ddz}{dt^2}+Z;$$
$$0=\frac{ddx'}{dt^2}+X'; \qquad 0=\frac{ddy'}{dt^2}+Y'; \qquad 0=\frac{ddz'}{dt^2}+Z';$$

&c.

If the number of these bodies is n, the number of equations will be $3n$, and
[760′] their finite integrals will contain $6n$ arbitrary terms, which will be the elements of the orbits of the different bodies.

To determine these elements by observation, we shall transform the
[760″] co-ordinates of each body into others whose origin is at the place of the observer. Suppose therefore a plane to pass through the eye of the observer,
[760‴] and to maintain a situation parallel to itself, while the observer moves on a given curve; we shall call ρ, ρ', ρ'', &c., the distances from the observer to the different bodies, projected on this plane; α, α', α'', &c., the apparent
[760$^{\text{iv}}$] longitudes of these bodies, referred to the same plane, and θ, θ', θ'', &c., their apparent latitudes. The variable quantities x, y, z, will be given in functions of ρ, α, θ, and of the co-ordinates of the observer. In like manner x', y', z',
[760$^{\text{v}}$] will be given in functions of ρ', α', θ', and of the co-ordinates of the observer, and so on for the rest. Also if we suppose the forces X, Y, Z, X', Y', Z', &c., to depend on the reciprocal action of the bodies of the system, and upon external attractions, they will be given in functions of ρ, ρ', ρ'', &c., α, α', α'',
[760$^{\text{vi}}$] &c., θ, θ', θ'', &c., and of known quantities; the preceding differential equations will thus correspond to these new variable quantities, and their first and second differentials; now by observations we can find, for any given
[760$^{\text{vii}}$] instant, the values of α, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$ [759]; θ, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$; α', $\left(\frac{d\alpha'}{dt}\right)$, &c.; there will therefore remain unknown only the quantities

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* (519) These are similar to the equations [38] or to those deduced from [142], by changing $-P$, $-Q$, $-R$, &c. into X, Y, Z, &c. because the forces P, Q, R, in note 66, page 96, are supposed to *increase* the co-ordinates, but the forces X, Y, &c. [759$^{\text{viii}}$], are supposed to *decrease* them, consequently P, Q, &c. ought to have different signs from X, Y, &c.

ρ, ρ', ρ'', &c., and their first and second differentials. The number of these unknown quantities will be $3n$, and as we have $3n$ differential equations, [760viii] we shall be able to determine them. We shall have even this advantage, that the first and second differentials of ρ, ρ', ρ'', &c., will appear only under a linear form* in these equations.

The quantities α, θ, ρ, α', θ', ρ', &c., and their first differentials divided by dt, being known; we shall have for a given instant, the values of x, y, z, x', y', z', [760ix] &c., and their first differentials divided by dt. If we substitute these values in the $3n$ finite integrals of the preceding equations, and in the first differentials of these integrals, we shall have $6n$ equations, by means of [760x] which we can determine the $6n$ arbitrary constant quantities of these integrals, or the elements of the orbits of the different bodies.

31. We shall now apply this method to the motion of comets. For this purpose we shall observe that the principal force which acts on them is the sun's attraction; we may therefore neglect the other forces. However, if a comet should pass so near to a great planet as to be sensibly disturbed by it, [760xi] the preceding method would give the velocity of the comet, and its distance from the earth; but as this case very rarely occurs, we shall in what follows, only take notice of the action of the sun.

If we take the mass of the sun for the unity of mass; the mean distance [760xii] of the earth from the sun for the unity of distance; and fix at the centre of the sun, the origin of the co-ordinates x, y, z, of a comet, whose radius vector is r; the differential equations (O) § 17 [545] will become, by [760xiii] neglecting the mass of the comet in comparison with that of the sun,†

$$\left.\begin{aligned} 0 &= \frac{ddx}{dt^2} + \frac{x}{r^3} \\ 0 &= \frac{ddy}{dt^2} + \frac{y}{r^3} \\ 0 &= \frac{ddz}{dt^2} + \frac{z}{r^3} \end{aligned}\right\}. \qquad (k) \qquad [761]$$

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* (520) This is evident from the form of x, y, z, &c. [762], which give dx, ddx, dy, ddy, dz, ddz, &c. in terms of ρ, $d\rho$, $dd\rho$, in a linear form [125a], which being substituted in [760] produce the linear equations mentioned above.

† (520a) In this case $M+m=\mu$, [544v], and by putting $M=1$, [760xii], and neglecting the mass m of the comet, we get $\mu=1$.

[761'] Suppose the plane of x and y to be the plane of the ecliptic; the axis of x to
be the line drawn from the centre of the sun to the first point of aries, at a
[761''] given epoch; the axis of y to be the line drawn from the centre of the sun to
the first point of cancer, at the same epoch, and the axis of z to be directed
[761'''] towards the north pole of the ecliptic. Then put x' and y' for the co-ordinates
of the earth, and R for its radius vector; this being premised,

We shall transform the co-ordinates x, y, z, into others referred to the place of the observer. For this purpose put

$\alpha =$ the geocentric longitude of the comet;

[761iv] $\theta =$ the geocentric latitude of the comet;

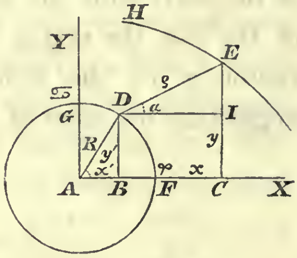
$\rho =$ the distance of the comet from the earth, projected upon the plane of the ecliptic;

we shall have*

[762] $$x = x' + \rho \,.\, \cos.\, \alpha\,; \qquad y = y' + \rho \,.\, \sin.\, \alpha\,; \qquad z = \rho \,.\, \mathrm{tang.}\, \theta.$$

If we multiply the first of the equations (k) [761] by sin. α, and subtract from it the second multiplied by cos. α, we shall have

[763] $$0 = \sin.\, \alpha \,.\, \frac{ddx}{dt^2} - \cos.\, \alpha \,.\, \frac{ddy}{dt^2} + \frac{x \,.\, \sin.\, \alpha - y \,.\, \cos.\, \alpha}{r^3}\,;$$



* (521) Let A be the sun supposed at rest, FDG the orbit of the earth, AFX the axis of x, drawn through the first point of aries; AGY the axis of y, drawn through the first point of cancer; EH the orbit of the comet projected upon the plane of the ecliptic, the earth being at D when the projected place of the comet is at E; we shall have [761'—761iv], the co-ordinates $AB = x'$, $BD = y'$, $AC = x$, $CE = y$, $AD = R$, $DE = \rho$, the angle $XAD = A$; then if we draw DI parallel to AX to meet CE in I; and put the angle $EDI = \alpha$, we shall have in the rectangular triangle DIE,

$$DI = DE \,.\, \cos.\, EDI = \rho \,.\, \cos.\, \alpha, \qquad EI = DE \,.\, \sin.\, EDI = \rho \,.\, \sin.\, \alpha.$$

Now $AC = AB + BC = AB + DI$, hence $x = x' + \rho \,.\, \cos.\, \alpha$, [762], also $CE = CI + EI = BD + EI$ becomes $y = y' + \rho \,.\, \sin.\, \alpha$, [762]; and as the comet is elevated above the plane by an angle which, viewed from D, is equal to θ, the distance DE being equal to ρ, the actual elevation will be equal to $\rho \,.\, \mathrm{tang}\, \theta$, hence $z = \rho \,.\, \mathrm{tang.}\, \theta$, as in [762].

by substituting in it the values of x, y, given by the preceding equations, we obtain*

$$0=\sin.\alpha.\frac{ddx'}{dt^2}-\cos.\alpha.\frac{ddy'}{dt^2}+\frac{x'.\sin.\alpha-y'.\cos.\alpha}{r^3}-2.\left(\frac{d\rho}{dt}\right).\left(\frac{d\alpha}{dt}\right)-\rho.\left(\frac{dd\alpha}{dt^2}\right). \qquad [764]$$

The earth being retained in its orbit, like the comet, by the attraction of the sun, we shall have†

$$0=\frac{ddx'}{dt^2}+\frac{x'}{R^3}; \qquad 0=\frac{ddy'}{dt^2}+\frac{y'}{R^3}; \qquad [765]$$

which give

$$\sin.\alpha.\frac{ddx'}{dt^2}-\cos.\alpha.\frac{ddy'}{dt^2}=\frac{y'.\cos.\alpha-x'.\sin.\alpha}{R^3}; \qquad [766]$$

we shall therefore have

$$0=(y'.\cos.\alpha-x'.\sin.\alpha).\left\{\frac{1}{R^3}-\frac{1}{r^3}\right\}-2.\left(\frac{d\rho}{dt}\right).\left(\frac{d\alpha}{dt}\right)-\rho.\left(\frac{dd\alpha}{dt^2}\right). \qquad [767]$$

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* (522) The differentials of the equations [762] give

$$dx=dx'+d\rho.\cos.\alpha-\rho d\alpha.\sin.\alpha; \qquad dy=dy'+d\rho.\sin.\alpha+\rho d\alpha.\cos.\alpha;$$
$$dz=d\rho.\text{tang}.\theta+\rho d\theta.(\cos.\theta)^{-2}.$$

Again taking the differentials, and putting for brevity

$$B=2d\rho.d\alpha+\rho.dd\alpha, \qquad \text{and} \qquad C=dd\rho-\rho d\alpha^2, \qquad [764a]$$

we obtain

$$ddx=ddx'+C.\cos.\alpha-B.\sin.\alpha; \qquad ddy=ddy'+C.\sin.\alpha+B.\cos.\alpha; \qquad [764b]$$

$$ddz=d^2\rho.\text{tang}.\theta+2d\rho.d\theta.(\cos.\theta)^{-2}+2\rho.d\theta^2.\sin.\theta.(\cos.\theta)^{-3}+\rho.dd\theta.(\cos.\theta)^{-2}. \qquad [764c]$$

These values of ddx, ddy, being substituted in $\sin.\alpha.ddx-\cos.\alpha.ddy$, it becomes equal to $\sin.\alpha.ddx'-\cos.\alpha.ddy'-B$. Dividing this by dt^2 we get

$$\sin.\alpha.\frac{ddx}{dt^2}-\cos.\alpha.\frac{ddy}{dt^2}=\sin.\alpha.\frac{ddx'}{dt^2}-\cos.\alpha.\frac{ddy'}{dt^2}-2.\left(\frac{d\rho}{dt}\right).\left(\frac{d\alpha}{dt}\right)-\rho.\left(\frac{dd\alpha}{dt^2}\right). \qquad [764d]$$

Again, the values of x, y, [762] give $x.\sin.\alpha-y.\cos.\alpha=x'.\sin.\alpha-y'.\cos.\alpha$, which being divided by r^3, and added to the equation [764d], the first member of the sum is nothing, by means of [763], and the second member becomes as in [764].

† (523) The earth being attracted by the sun in like manner as the comet, we shall have the equations of the earth's motion by changing in the two first equations [761], x, y, r, into x', y', R, respectively. The third equation is not used, because by hypothesis, [761′], the earth moves in the plane of x, y. Hence we obtain the equations [765]. Multiplying the first by $\sin.\alpha$, the second by $-\cos.\alpha$, and taking the sum of the products, we get [766]. Substituting this in [764] it becomes as in [767].

[767′] Let A be the longitude of the earth seen from the sun; we shall have*

[768] $$x' = R \,.\, \cos.\, A\,; \qquad\qquad y' = R \,.\, \sin.\, A\,;$$

therefore

[769] $$y' \,.\, \cos.\, \alpha - x' \,.\, \sin.\, \alpha = R \,.\, \sin.\, (A - \alpha)\,;$$

the preceding equation will thus become

[770] $$\left(\frac{d\rho}{dt}\right) = \frac{R \,.\, \sin.\, (A-\alpha)}{2 \,.\, \left(\frac{d\alpha}{dt}\right)} \,.\, \left\{\frac{1}{R^3} - \frac{1}{r^3}\right\} - \frac{\rho \,.\, \left(\frac{dd\alpha}{dt^2}\right)}{2 \,.\, \left(\frac{d\alpha}{dt}\right)}. \qquad (1)$$

We shall now investigate another value† of $\left(\frac{d\rho}{dt}\right)$. For this purpose, we shall multiply the first of the equations (k) [761], by $\tang.\, \theta \,.\, \cos.\, \alpha$; the second by $\tang.\, \theta \,.\, \sin.\, \alpha$; and from the sum of these two products shall

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* (524) In the preceding figure, page 420, we have

$$AB = AD \,.\, \cos.\, XAD\,; \qquad\qquad BD = AD \,.\, \sin.\, XAD\,;$$

which in symbols are $x' = R \,.\, \cos.\, A$; $y' = R \,.\, \sin.\, A$, [768]. The first multiplied by $-\sin.\, \alpha$, and added to the second, multiplied by $\cos.\, \alpha$, gives

$$y' \,.\, \cos.\, \alpha - x' \,.\, \sin.\, \alpha = R \,.\, (\sin.\, A \,.\, \cos.\, \alpha - \cos.\, A \,.\, \sin.\, \alpha) = R \,.\, \sin.\, (A - \alpha),$$

[22] Int. as in [769]. Substituting this in [767], we get

$$0 = R \,.\, \sin.\, (A - \alpha) \,.\, \left(\frac{1}{R^3} - \frac{1}{r^3}\right) - 2 \,.\, \left(\frac{d\rho}{dt}\right) \,.\, \left(\frac{d\alpha}{dt}\right) - \rho \,.\, \left(\frac{dd\alpha}{dt^2}\right).$$

Transposing $-2 \,.\, \left(\frac{d\rho}{dt}\right) \,.\, \left(\frac{d\alpha}{dt}\right)$, and dividing by $2 \,.\, \left(\frac{d\alpha}{dt}\right)$, we get [770].

† (525) It may be observed relative to the calculation here used, that by substituting the values of x, y, z, [762], and their differentials computed in note 522, in the equations [761], we shall obtain *three* differential equations, linear in ρ, $\frac{d\rho}{dt}$, $\frac{dd\rho}{dt^2}$. Deducing from these, in any manner *three* values of $\frac{dd\rho}{dt^2}$ and putting them equal to each other, we shall obtain *two* independent equations containing ρ and $\frac{d\rho}{dt}$; from each of which if we can find a value of $\left(\frac{d\rho}{dt}\right)$, like that in [770] or [774], and put them equal to each other, we shall obtain an equation containing ρ, without its differentials, as in [776].

subtract the third equation; we shall thus have

$$0 = \text{tang.}\,\theta . \left\{ \cos.\alpha . \frac{ddx}{dt^2} + \sin.\alpha . \frac{ddy}{dt^2} \right\} + \text{tang.}\,\theta . \frac{\{x.\cos.\alpha + y.\sin.\alpha\}}{r^3} - \frac{ddz}{dt^2} - \frac{z}{r^3} . \qquad [771]$$

This equation will become, by substituting the values of x, y, z,*

$$0 = \text{tang.}\,\theta . \left\{ \left(\frac{ddx'}{dt^2} + \frac{x'}{r^3} \right) . \cos.\alpha + \left(\frac{ddy'}{dt^2} + \frac{y'}{r^3} \right) . \sin.\alpha \right\} - \frac{2.\left(\frac{d\theta}{dt}\right).\left(\frac{d\rho}{dt}\right)}{\cos.^2\theta}$$

$$- \rho . \left\{ \frac{\left(\frac{dd\theta}{dt^2}\right)}{\cos.^2\theta} + \frac{2.\left(\frac{d\theta}{dt}\right)^2 \sin.\theta}{\cos.^3\theta} + \left(\frac{d\alpha}{dt}\right)^2 . \text{tang.}\,\theta \right\} ; \qquad [772]$$

now we have†

$$\left(\frac{ddx'}{dt^2} + \frac{x'}{r^3} \right) . \cos.\alpha + \left(\frac{ddy'}{dt^2} + \frac{y'}{r^3} \right) . \sin.\alpha = (x'.\cos.\alpha + y'.\sin.\alpha) . \left(\frac{1}{r^3} - \frac{1}{R^3} \right)$$

$$= R . \cos.(A - \alpha) . \left\{ \frac{1}{r^3} - \frac{1}{R^3} \right\} ; \qquad [773]$$

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* (526) Substituting the values of x, y, z, [762], in [771], the terms depending on x', y', which for brevity we shall call X, will agree with those in [772], and as x, y, z, enter the equation [771] under a linear form, we may find the parts of [772], independent of x', y', by substituting in [771] the parts of x, y, z, [762], independent of x', y', namely $x = \rho . \cos.\alpha$; $y = \rho . \sin.\alpha$; $z = \rho . \text{tang.}\,\theta$. These give $x . \cos.\alpha + y . \sin.\alpha = \rho$; and by [764*b*], neglecting x', y', we have $ddx = C . \cos.\alpha - B . \sin.\alpha$, $ddy = C . \sin.\alpha + B . \cos.\alpha$, hence $\cos.\alpha . ddx + \sin.\alpha . ddy = C = dd\rho - \rho\, d\alpha^2$, [764*a*]. Substituting these in [771], and then in the last term putting $z = \rho . \text{tang}\,\theta$, [762], we shall get

$$0 = X + \text{tang.}\,\theta . \left\{ \left(\frac{dd\rho}{dt^2}\right) - \rho . \left(\frac{d\alpha}{dt}\right)^2 \right\} + \text{tang.}\,\theta . \frac{\rho}{r^3} - \frac{ddz}{dt^2} - \frac{\rho.\text{tang.}\,\theta}{r^3} ,$$

and by reduction,

$$0 = X + \text{tang.}\,\theta . \left\{ \left(\frac{dd\rho}{dt^2}\right) - \rho . \left(\frac{d\alpha}{dt}\right)^2 \right\} - \frac{ddz}{dt^2} ;$$

but from [764*c*] we have

$$-\frac{ddz}{dt^2} = -\left(\frac{dd\rho}{dt^2}\right) . \text{tang.}\,\theta - 2 . \left(\frac{d\rho}{dt}\right) . \left(\frac{d\theta}{dt}\right) . \frac{1}{\cos.^2\theta} - 2\rho . \left(\frac{d\theta}{dt}\right)^2 . \frac{\sin.\theta}{\cos.^3\theta} - \left(\frac{dd\theta}{dt^2}\right) . \frac{\rho}{\cos^2\theta} ,$$

hence by substitution we get [772].

† (527) From [765] we have $\frac{ddx'}{dt^2} = -\frac{x'}{R^3}$; $\frac{ddy'}{dt^2} = -\frac{y'}{R^3}$. Substituting these in

therefore

[774]
$$\left(\frac{d\rho}{dt}\right) = -\tfrac{1}{2}\rho \,.\left\{ \frac{\left(\frac{dd\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)} + 2\,.\left(\frac{d\theta}{dt}\right).\,\text{tang.}\,\theta + \frac{\left(\frac{d\alpha}{dt}\right)^2.\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta}{\left(\frac{d\theta}{dt}\right)} \right\} + \frac{R\,.\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta\,.\,\text{cos.}\,(A-\alpha)}{2\,.\left(\frac{d\theta}{dt}\right)}\,.\left\{\frac{1}{r^3} - \frac{1}{R^3}\right\}. \qquad (2)$$

If we subtract this value of $\frac{d\rho}{dt}$ from the first [770], supposing

[775]
$$\mu' = \frac{\left(\frac{d\alpha}{dt}\right).\left(\frac{dd\theta}{dt^2}\right) - \left(\frac{d\theta}{dt}\right).\left(\frac{dd\alpha}{dt^2}\right) + 2\,.\left(\frac{d\alpha}{dt}\right).\left(\frac{d\theta}{dt}\right)^2.\,\text{tang.}\,\theta + \left(\frac{d\alpha}{dt}\right)^3.\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta}{\left(\frac{d\alpha}{dt}\right).\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta\,.\,\text{cos.}\,(A-\alpha) + \left(\frac{d\theta}{dt}\right).\,\text{sin.}\,(A-\alpha)};$$

we shall have

[776]
$$\rho = \frac{R}{\mu'}\,.\left\{\frac{1}{r^3} - \frac{1}{R^3}\right\}. \qquad (3)$$

[776'] The projected distance of the comet from the earth ρ, being always positive,
this equation shows that the distance r of the comet from the sun is *less* than
[776''] the distance R of the sun from the earth, if μ' is *positive*, but r is *greater* than
R, if μ' is *negative*;* these two distances are equal, if $\mu' = 0$.

$\left(\frac{ddx'}{dt^2} + \frac{x'}{r^3}\right).\,\text{cos}\;\alpha + \left(\frac{ddy'}{dt^2} + \frac{y'}{r^3}\right).\,\text{sin.}\,\alpha$, it becomes $(x'\,.\,\text{cos.}\,\alpha + y'\,.\,\text{sin.}\,\alpha)\,.\left(\frac{1}{r^3} - \frac{1}{R^3}\right)$, as in [773]; and by using the values of x', y', [768], we find

[774a]
$$x'\,.\,\text{cos.}\,\alpha + y'\,.\,\text{sin.}\,\alpha = R\,.\,\{\text{cos.}\,A\,.\,\text{cos.}\,\alpha + \text{sin.}\,A\,.\,\text{sin.}\,\alpha\} = R\,.\,\text{cos.}\,(A-\alpha),$$

[24] Int.; hence we obtain the last expression [773]. Substituting this in [772], then multiplying by $\frac{\text{cos.}^2\,\theta}{2\,.\left(\frac{d\theta}{dt}\right)}$, putting also $\text{cos.}\,\theta\,.\,\text{tang.}\,\theta = \text{sin.}\,\theta$, and reducing, we shall get [774].

* (528) Subtracting the equation [774] from [770], we get

$$0 = \tfrac{1}{2}\rho\,.\left\{ \frac{\left(\frac{dd\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)} - \frac{\left(\frac{dd\alpha}{dt^2}\right)}{\left(\frac{d\alpha}{dt}\right)} + 2\,.\left(\frac{d\theta}{dt}\right).\,\text{tang.}\,\theta + \frac{\left(\frac{d\alpha}{dt}\right)^2.\,\text{sin.}\,\theta\,.\,\text{cos.}\,\theta}{\left(\frac{d\theta}{dt}\right)} \right\} - R\,.\left\{\frac{1}{r^3} - \frac{1}{R^3}\right\}.\left\{ \frac{\text{sin.}\,\theta\,.\,\text{cos.}\,\theta\,.\,\text{cos.}\,(A-\alpha)}{2\,.\left(\frac{d\theta}{dt}\right)} + \frac{\text{sin.}\,(A-\alpha)}{2\,.\left(\frac{d\alpha}{dt}\right)} \right\},$$

We may, by the inspection of a celestial globe, ascertain the sign of μ', and by that means determine whether the comet or the earth is most distant from the sun. For this purpose, suppose a great circle to pass through two geocentric places of the comet, infinitely near to each other. Let the inclination of this circle to the ecliptic be γ, and λ the longitude of its ascending node ; we shall have* [776'''] [776^{iv}]

$$\text{tang}\,\gamma\,.\,\sin.\,(\alpha-\lambda)=\text{tang.}\,\theta\,; \qquad [777]$$

hence we deduce

$$d\theta\,.\,\sin.\,(\alpha-\lambda)=d\alpha\,.\,\sin.\,\theta\,.\,\cos.\,\theta\,.\,\cos.\,(\alpha-\lambda)\,; \qquad [778]$$

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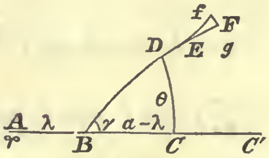
which being multiplied by $2\,.\left(\frac{d\theta}{dt}\right)\,.\left(\frac{d\alpha}{dt}\right)$, and then divided by the coefficient of ρ, gives $0=\rho-\frac{R}{\mu'}\,.\left\{\frac{1}{r^3}-\frac{1}{R^3}\right\}$, hence we get ρ, [776]. Now when $r<R$, $\frac{1}{r^3}-\frac{1}{R^3}=\frac{R^3-r^3}{r^3R^3}$ will be positive, therefore μ' must then be positive to render ρ positive. On the contrary when $r>R$, $\frac{1}{r^3}-\frac{1}{R^3}$ will be negative, and then μ' must be negative to render ρ positive. When $r=R$, the numerator of the value of ρ becomes 0, consequently, $\mu'=0$, because ρ must have a real positive value.



* (529) Let ABC be the ecliptic, D, E, two observed geocentric places of the comet, infinitely near to each other, DC the circle of latitude. Continue ED to meet the ecliptic in the ascending node B. A being the first point of aries, we shall have $AB=\lambda$, $AC=\alpha$, $BC=\alpha-\lambda$, $DC=\theta$, the angle $DBC=\gamma$; then by spherics, tang. $DC=$ tang. DBC . sin. BC, or tang. $\theta=$ tang. γ . sin. $(\alpha-\lambda)$, as in [777]. The differential of this, supposing θ, α, variable, is

$$d\alpha\,.\,\text{tang.}\,\gamma\,.\,\cos.\,(\alpha-\lambda)=\frac{d\theta}{\cos.^2\theta}, \qquad [777a]$$

and by substituting $\text{tang.}\,\gamma=\frac{\text{tang.}\,\theta}{\sin.\,(\alpha-\lambda)}$, [777], it becomes

$$d\alpha\,.\,\frac{\text{tang.}\,\theta}{\sin.\,(\alpha-\lambda)}\,.\,\cos.\,(\alpha-\lambda)=\frac{d\theta}{\cos^2\theta},$$

this being multiplied by $\cos.^2\theta\,.\,\sin.\,(\alpha-\lambda)$, becomes as in [778]. Taking the differential of [777a], we find,

$$dd\alpha\,.\,\text{tang.}\,\gamma\,.\,\cos.\,(\alpha-\lambda)-d\alpha^2\,.\,\text{tang.}\,\gamma\,.\,\sin.\,(\alpha-\lambda)=\frac{dd\theta_{,}}{\cos.^2\theta}+2\,d\theta^2\,.\,\frac{\sin.\,\theta}{\cos^3\theta}\,.$$

and by taking the differential of this, we shall have

[779] $$0=\left(\frac{d\,\alpha}{dt}\right).\left(\frac{d\,d\,\theta_{\prime}}{d\,t^2}\right)-\left(\frac{d\,\theta}{d\,t}\right).\left(\frac{d\,d\,\alpha}{d\,t^2}\right)+2.\left(\frac{d\,\alpha}{d\,t}\right).\left(\frac{d\,\theta}{d\,t}\right)^2.\,\mathrm{tang.}\,\theta+\left(\frac{d\,\alpha}{d\,t}\right)^3.\mathrm{sin.}\theta.\mathrm{cos.}\theta\,;$$

[779'] $d\,d\,\theta_{\prime}$ being the value of $d\,d\,\theta$, which would take place if the apparent motion of the comet continued in the great circle. Therefore by substituting for

[779''] $d\,\theta$ its value $\frac{d\,\alpha\,.\,\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta\,.\,\mathrm{cos.}\,(\alpha-\lambda)}{\mathrm{sin.}\,(\alpha-\lambda)}$ [778], we shall obtain the following value of μ' :*

[780] $$\mu'=\frac{\left\{\left(\frac{d\,d\,\theta}{d\,t^2}\right)-\left(\frac{d\,d\,\theta_{\prime}}{d\,t^2}\right)\right\}.\,\mathrm{sin.}\,(\alpha-\lambda)}{\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta\,.\,\mathrm{sin.}\,(A-\lambda)}\,;$$

Substituting in this for $\mathrm{tang.}\,\gamma\,.\,\mathrm{cos.}\,(\alpha-\lambda)$, and $\mathrm{tang.}\,\gamma\,.\,\mathrm{sin.}\,(\alpha-\lambda)$, their values deduced from [777*a*, 777], namely, $\frac{d\,\theta}{d\,\alpha\,.\,\mathrm{cos.}^2\,\theta}$, and $\mathrm{tang.}\,\theta$, we get

$$d\,d\,\alpha\,.\,\frac{d\,\theta}{d\,\alpha\,.\,\mathrm{cos.}^2\,\theta}-d\,\alpha^2\,.\,\mathrm{tang.}\,\theta=\frac{d\,d\,\theta_{\prime}}{\mathrm{cos.}^2\,\theta}+2\,d\,\theta^2\,.\,\frac{\mathrm{sin.}\,\theta}{\mathrm{cos.}^3\,\theta}\,,$$

which being multiplied by $\mathrm{cos.}^2\,\theta\,.\,\frac{d\,\alpha}{d\,t^3}$, putting $\frac{\mathrm{sin.}\,\theta}{\mathrm{cos.}\,\theta}=\mathrm{tang.}\,\theta$, and transposing all the terms to the second member, becomes as in [779].

* (530) Substituting $d\,\theta$, [779''], in the denominator of μ', [775], it becomes

$$\left(\frac{d\,\alpha}{d\,t}\right).\,\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta\,.\,\mathrm{cos.}\,(A-\alpha)+\left(\frac{d\,\alpha}{d\,t}\right).\,\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta\,.\,\frac{\mathrm{sin.}\,(A-\alpha)\,.\,\mathrm{cos.}\,(\alpha-\lambda)}{\mathrm{sin.}\,(\alpha-\lambda)}$$

$$=\left(\frac{d\,\alpha}{d\,t}\right).\,\frac{\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta}{\mathrm{sin.}\,(\alpha-\lambda)}\,.\left\{\mathrm{sin.}\,(\alpha-\lambda)\,.\,\mathrm{cos.}\,(A-\alpha)+\mathrm{cos.}\,(\alpha-\lambda)\,.\,\mathrm{sin.}\,(A-\alpha)\right\},$$

of which the part between the braces is, by [21] Int. equal to

$$\mathrm{sin.}\,\{(A-\alpha)+(\alpha-\lambda)\}=\mathrm{sin.}\,(A-\lambda),$$

therefore the denominator of μ' is $\left(\frac{d\,\alpha}{d\,t}\right).\,\frac{\mathrm{sin.}\,\theta\,.\,\mathrm{cos.}\,\theta\,.\,\mathrm{sin.}\,(A-\lambda)}{\mathrm{sin.}\,(\alpha-\lambda)}$. Again, by subtracting from the numerator of μ', [775], the expression [779] which is equal to nothing, it will become $\left(\frac{d\,\alpha}{d\,t}\right).\left(\frac{d\,d\,\theta}{d\,t^2}\right)-\left(\frac{d\,\alpha}{d\,t}\right).\left(\frac{d\,d\,\theta_{\prime}}{d\,t^2}\right)$. This being divided by the preceding expression of the denominator, gives, by rejecting the term $\left(\frac{d\,\alpha}{d\,t}\right)$, common to the numerator and denominator, the value of μ', [780].

The function $\frac{\sin.(\alpha-\lambda)}{\sin.\theta\,.\cos.\theta}$ is always positive ;* the value of μ' is therefore [780']
positive or negative, according as $\left(\frac{dd\theta}{dt^2}\right)-\left(\frac{dd\theta_{,}}{dt^2}\right)$ is of the same or of a [780'']
different sign from $\sin.(A-\lambda)$; now $(A-\lambda)$ is equal to the distance of the sun from the ascending node of the great circle increased by two right angles [767', 776$^{\text{iv}}$] ; hence it is easy to perceive that μ' will be positive or negative, according as the comet shall be found on the same side of the great circle on which the sun is, or on the opposite side, at the time of a third
observation, taken immediately after the two preceding observations, and [780''']
infinitely near to them.† Suppose therefore, *through two very near geocentric*

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* (531) As θ never exceeds a right angle, its cosine is always positive; and in the figure page 425, it is evident that θ remains positive, while the arch $B\,C$ increases from 0 to two right angles, during which the signs of $\sin.\theta$, $\sin.(\alpha-\lambda)$ are always positive. In the other semi-circle, where $\alpha-\lambda$ is between two and four right angles, θ is negative, and its sine, as well as the sine of $\alpha-\lambda$, is negative, consequently $\frac{\sin.(\alpha-\lambda)}{\sin.\theta\,.\cos.\theta}$ is always positive. The same result may be obtained from [777], which gives

$$\sin.(\alpha-\lambda)=\frac{\text{tang.}\,\theta}{\text{tang.}\gamma}=\frac{\sin.\theta}{\text{tang.}\gamma.\cos.\theta},$$

and this being divided by $\sin.\theta\,.\cos.\theta$, becomes $\frac{\sin.(\alpha-\lambda)}{\sin.\theta\,.\cos\theta}=\frac{1}{\text{tang.}\gamma.\cos^2\theta}$, the second member of which is evidently positive.

† (532) Supposing π to be equal to two right angles, the distance of the comet from the south pole of the ecliptic will be $\frac{1}{2}\pi+\theta$, which we shall put $=\theta'$, and write $dd\theta_{,}'$ for the value of $dd\theta_{,}$, [779'], also $dd\theta'$ for $dd\theta$, observing that θ' is always positive. By this means the expression [780] becomes

$$\mu'=\frac{\sin.(\alpha-\lambda)}{\sin.\theta\,.\cos.\theta}\cdot\frac{\left\{\left(\frac{dd\theta'}{dt^2}\right)-\left(\frac{dd\theta_{,}'}{dt^2}\right)\right\}}{\sin.(A-\lambda)}.$$

Now it is evident that if $\left(\frac{dd\theta'}{dt^2}\right)-\left(\frac{dd\theta_{,}'}{dt^2}\right)$ is *positive*, the south polar distance of the comet at the third observation, will be greater than it would if it continued to move in the great circle ; therefore, instead of moving on the arch $E\,F$, on the continuation of the great circle BE, fig. page 425, it will fall to the *northward* of it on Ef; but if $\left(\frac{dd\theta'}{dt^2}\right)-\left(\frac{dd\theta_{,}'}{dt^2}\right)$ is *negative*, the comet will fall to the *southward* of the great circle, towards g. But C' being the place of *places of the comet, a great circle to be drawn. Then if at another third observation, taken very soon after the two others, the comet deviate from the great circle towards the part of the heavens where the sun is, the comet will* [780^iv] *be situated within the earth's orbit, or nearer to the sun than the earth is: but if the deviation be to the opposite side of the great circle to that in which the sun is placed, the comet will be without the earth's orbit, or farther from the sun than the earth is. If the comet continue to move in the great circle, the comet and the earth would both be equally distant from the sun.* Thus the [780^v] various inflections of the apparent path, will enable us to estimate the variations of the distance of the comet from the sun.

Use of a Celestial Globe in judging of the distance of a Comet from the Sun.

To eliminate r from the equation (3) [776], so that this equation may contain only the unknown quantity ρ, we shall observe that we have [555]

[780^vi] $$r^2 = x^2 + y^2 + z^2\,;$$

and by substituting for $x, y, z,$ their values in ρ, α and θ [762], we shall have*

[781] $$r^2 = x'^2 + y'^2 + 2\rho\,.\,\{x'\,.\,\cos.\alpha + y'\,.\,\sin.\alpha\} + \frac{\rho^2}{\cos.^2\theta}\,;$$

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the earth in the ecliptic, we have $A\,C' = A$, [767'], hence $B\,C' = A - \lambda$, is the distance of the earth from the node B, and $A - \lambda + \pi$, is the distance of the sun from the same node. Now it is evident that when $A - \lambda$ is between 0 and π, the sun will fall to the *northward* of the great circle $B\,E$, and when $A - \lambda$ is between π and $2\,\pi$ it will fall to the *southward*. In the first case $\sin.\,(A - \lambda)$ is *positive*, in the second *negative*. Hence it evidently follows that when the comet and sun fall both on the same side of the great circle, that is, both to the northward or both to the southward, the sign of $\frac{\left(\frac{dd\theta'}{dt^2}\right) - \left(\frac{dd\theta'_{,}}{dt^2}\right)}{\sin.(A - \lambda)}$, and therefore that of μ', will be positive; but if the sun and comet fall on different sides of the great circle, that quantity will be negative.

* (533) The values [762] being substituted in [780^vi], we get

$$r^2 = (x' + \rho\,.\,\cos.\alpha)^2 + (y' + \rho\,.\,\sin.\alpha)^2 + \rho^2\,.\,\mathrm{tang.}^2\,\theta$$
$$= x'^2 + y'^2 + 2\rho\,.\,(x'\,.\,\cos.\alpha + y'\,.\,\sin.\alpha) + \rho^2\,.\,(\cos.^2\alpha + \sin.^2\alpha + \mathrm{tang.}^2\,\theta).$$

[781a] Now $\cos.^2\alpha + \sin.^2\alpha + \mathrm{tang.}^2\,\theta = 1 + \mathrm{tang.}^2\,\theta = \frac{1}{\cos.^2\theta}$, hence the preceding expression becomes as in [781].

but we have [768] $x'=R.\cos.A$; $y'=R.\sin.A$; therefore* [781']

$$r^2=\frac{\rho^2}{\cos.^2\theta}+2R\rho.\cos.(A-\alpha)+R^2.$$ [782]

If we square both sides of the equation (3) [776], after having put it under the following form,†

$$r^3.\{\mu' R^2\rho+1\}=R^3;$$ [783]

we shall have, by substituting for r^2 its value [782]

$$\left\{\frac{\rho^2}{\cos.^2\theta}+2R\rho.\cos.(A-\alpha)+R^2\right\}^3.\{\mu' R^2\rho+1\}^2=R^6; \qquad (4)$$ [784]

in which equation there is only one unknown quantity ρ, and it is of the seventh degree, because the known term of the first member being R^6, as in [784'] the second member, the whole equation becomes divisible by ρ. Having thus found ρ, we shall obtain $\left(\frac{d\rho}{dt}\right)$ by means of the equations (1) and (2), [770, 774]. By substituting, for example, in the equation (1) [770], for $\frac{1}{r^3}-\frac{1}{R^3}$, its value $\frac{\mu'\rho}{R}$, given by the equation (3) [776], we shall [784''] have

$$\left(\frac{d\rho}{dt}\right)=-\frac{\rho}{2.\left(\frac{d\alpha}{dt}\right)}.\left\{\left(\frac{dd\alpha}{dt^2}\right)+\mu'.\sin.(A-\alpha)\right\}.$$ [785]

The equation (4) [784] is often susceptible of several real positive roots. For by transposing the second member, and dividing by ρ, its last term [785'] will be‡

$$2.R^5.\cos.^6\theta.\{\mu' R^3+3.\cos.(A-\alpha)\};$$ [786]

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* (534) From [774*a*] we have $x'.\cos.\alpha+y'.\sin.\alpha=R.\cos.(A-\alpha)$; also the sum of the squares of x', y', [768], gives $x'^2+y'^2=R^2$, these being substituted in [781], we obtain [782].

† (535) This is obtained by multiplying [776] by $R^2r^3\mu'$, which makes it $\mu'\rho R^2r^3=R^3-r^3$, hence $R^3=r^3+\mu'\rho R^2r^3=r^3.\{\mu' R^2\rho+1\}$, as in [783]. The square of this is $r^6.\{\mu' R^2\rho+1\}^2=R^6$, and by substituting for r^6 the cube of r^2, [782], we obtain [784].

‡ (536) The equation [784] is supposed to be multiplied also by $\cos.^6\theta$, to avoid fractions.

therefore, as the equation in ρ is of the seventh degree, or of an uneven degree, it will have at least two real positive roots, if $\mu' R^3 + 3 . \cos. (A - \alpha)$ is *positive*; for it ought always, by the nature of the problem, to have one positive root, and it cannot then have an uneven number of positive roots.* [786'] Each real and positive value of ρ, gives a different conic section, for the orbit of the comet; we shall therefore have as many curves which satisfy the three observations, as ρ has real and positive values; and to determine the orbit of the comet, we must then have recourse to another observation. [786'']

32. The value of ρ, deduced from the equation (4) [784] would be rigorously correct, if α, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, θ, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$, were accurately known; but the approximate values of these quantities only have been found. [786'''] It is true we may approach more and more towards the exact values, by using the method before explained, and taking a greater number of observations, which has the advantage of embracing longer intervals, and compensating the one by the other for the errors of observations. But this method has the *analytical* inconvenience of using more than three observations, where no more than three are absolutely necessary. We may obviate this inconvenience in the following manner, and render the solutions as exact as may be required, using only three observations. [786iv]

For this purpose, suppose that α and θ represent the geocentric longitude and latitude of the middle observation; if we substitute in the equations (k) [761] of the preceding article, for x, y, z, their values [762] $x' + \rho . \cos. \alpha$, [786v]

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* (537) Supposing a', a'', a''', a'''', &c. to be real positive quantities. The factors of this equation depending on imaginary roots, which always enter by pairs, are of the forms $\rho \pm a' + a'' . \sqrt{-1} = 0$, $\rho \pm a' - a'' . \sqrt{-1} = 0$, whose product is

$$\rho^2 \pm 2 \rho a' + (a'^2 + a''^2) = 0.$$

Negative roots depend on factors of the form $\rho + a''' = 0$, and positive roots depend on factors of the form $\rho - a'''' = 0$; and the constant term of the proposed equation of the seventh degree, must be formed by products of the constant terms of these factors, that is by terms of the forms $a'^2 + a''^2$, $+a'''$, $-a''''$. Now in order that this final product may be equal to the positive quantity $2 R^5 . \cos.^6 \theta . \{\mu' R^3 + 3 . \cos. (A - \alpha)\}$, it is necessary that the number of the factors of the form $-a''''$ should be even, or in other words, that the number of positive roots should be even.

$y' + \rho . \sin. \alpha$, and $\rho . \mathrm{tang}. \theta$, they will give $\left(\frac{d^2\rho}{dt^2}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, and $\left(\frac{d^2\theta}{dt^2}\right)$, [786$^{\rm vi}$]
in functions of ρ, α and θ, and of their first differentials with known quantities.
If we take the differentials of these functions, we shall have $\left(\frac{d^3\rho}{dt^3}\right)$, $\left(\frac{d^3\alpha}{dt^3}\right)$, [786$^{\rm vii}$]
and $\left(\frac{d^3\theta}{dt^3}\right)$, in functions of ρ, α and θ, and of their first and second differentials. We may eliminate the second differential of ρ, by means of its value, and the first differential by means of the equation (2) [774] of the
preceding article. By continuing to take the differentials of $\left(\frac{d^3\alpha}{dt^3}\right)$, $\left(\frac{d^3\theta}{dt^3}\right)$, [786$^{\rm viii}$]
successively, and eliminating the differentials of α and θ above the second, and all the differentials of ρ, we shall have the values of $\left(\frac{d^3\alpha}{dt^3}\right)$, $\left(\frac{d^4\alpha}{dt^4}\right)$, &c.; $\left(\frac{d^3\theta}{dt^3}\right)$, $\left(\frac{d^4\theta}{dt^4}\right)$, &c., in functions of ρ, α, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, θ, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$; this being premised,

Let $\alpha_{,}$, α, α', be the three observed geocentric longitudes of the comet; [786$^{\rm ix}$]
$\theta_{,}$, θ, θ', the geocentric latitudes corresponding; i the number of days [786$^{\rm x}$]
between the first and second observation, i' the number of days between the
second and third observation; λ the arch in seconds which the earth describes [786$^{\rm xi}$]
in a day, by its mean sidereal motion; we shall have, by § 29 [754, 759],*

$$\alpha_{,} = \alpha - i . \lambda \left(\frac{d\alpha}{dt}\right) + \frac{i^2 . \lambda^2}{1.2} . \left(\frac{d^2\alpha}{dt^2}\right) - \frac{i^3 . \lambda^3}{1.2.3} . \left(\frac{d^3\alpha}{dt^3}\right) + \&c.;$$

$$\alpha' = \alpha + i' . \lambda . \left(\frac{d\alpha}{dt}\right) + \frac{i'^2 . \lambda^2}{1.2} . \left(\frac{d^2\alpha}{dt^2}\right) + \frac{i'^3 . \lambda^3}{1.2.3} . \left(\frac{d^3\alpha}{dt^3}\right) + \&c.;$$

$$\theta_{,} = \theta - i . \lambda . \left(\frac{d\theta}{dt}\right) + \frac{i^2 . \lambda^2}{1.2} . \left(\frac{d^2\theta}{dt^2}\right) - \frac{i^3 . \lambda^3}{1.2.3} . \left(\frac{d^3\theta}{dt^3}\right) + \&c.; \qquad [787]$$

$$\theta' = \theta + i' . \lambda . \left(\frac{d\theta}{dt}\right) + \frac{i'^2 . \lambda^2}{1.2} . \left(\frac{d^2\theta}{dt^2}\right) + \frac{i'^3 . \lambda^3}{1.2.3} . \left(\frac{d^3\theta}{dt^3}\right) + \&c.$$

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* (538) The two first of these equations were deduced from the first of the equations [754], putting successively $s = -i$, $s = i'$, and for $\left(\frac{d\alpha}{ds}\right)$, $\left(\frac{d^2\alpha}{ds^2}\right)$, &c. their values $\lambda . \left(\frac{d\alpha}{dt}\right)$, $\lambda^2 . \left(\frac{d^2\alpha}{dt^2}\right)$, &c. [759]. The two last equations were deduced in a similar manner from the second of the equations [754], or by changing α into θ.

If we substitute in this series, for $\left(\frac{d^3\,\alpha}{d\,t^3}\right)$, $\left(\frac{d^4\,\alpha}{d\,t^4}\right)$, &c.; $\left(\frac{d^3\,\theta}{d\,t^3}\right)$, $\left(\frac{d^4\,\theta}{d\,t^4}\right)$, &c., their values obtained by the preceding method; we shall have four equations between the five unknown quantities ρ, $\left(\frac{d\,\alpha}{d\,t}\right)$, $\left(\frac{d^2\,\alpha}{d\,t^2}\right)$, $\left(\frac{d\,\theta}{d\,t}\right)$, $\left(\frac{d^2\,\theta}{d\,t^2}\right)$. These equations will become the more correct by using a greater number of terms of the series. We shall thus have $\left(\frac{d\,\alpha}{d\,t}\right)$, $\left(\frac{d^2\,\alpha}{d\,t^2}\right)$, $\left(\frac{d\,\theta}{d\,t}\right)$, [787′] $\left(\frac{d^2\,\theta}{d\,t^2}\right)$, in functions of ρ and of known quantities; and by substituting these values in the equation (4) [784] of the preceding article, it will contain only the unknown quantity ρ. However, this method, which I have given only to show in what manner we can obtain, by approximation, the value of ρ, [787″] by using only three observations, will require in practice very laborious calculations; and it will be more accurate, as well as more simple, to use a greater number of observations, by the method of § 29.

33. When the values of ρ and $\left(\frac{d\,\rho}{d\,t}\right)$ shall be determined, we shall have those of x, y, z, $\left(\frac{d\,x}{d\,t}\right)$, $\left(\frac{d\,y}{d\,t}\right)$, and $\left(\frac{d\,z}{d\,t}\right)$, by means of the equations*

[788] $$x = R\,.\cos.A + \rho\,.\cos.\alpha\,; \qquad y = R\,.\sin.A + \rho\,.\sin.\alpha\,; \qquad z = \rho\,.\operatorname{tang}.\theta\,;$$

and of their differentials divided by $d\,t$

[789] $$\left(\frac{d\,x}{d\,t}\right) = \left(\frac{d\,R}{d\,t}\right).\cos.A - R.\left(\frac{d\,A}{d\,t}\right).\sin.A + \left(\frac{d\,\rho}{d\,t}\right).\cos.\alpha - \rho.\left(\frac{d\,\alpha}{d\,t}\right).\sin.\alpha\,;$$
$$\left(\frac{d\,y}{d\,t}\right) = \left(\frac{d\,R}{d\,t}\right).\sin.A + R.\left(\frac{d\,A}{d\,t}\right).\cos.A + \left(\frac{d\,\rho}{d\,t}\right).\sin.\alpha + \rho.\left(\frac{d\,\alpha}{d\,t}\right).\cos.\alpha\,;$$
$$\left(\frac{d\,z}{d\,t}\right) = \left(\frac{d\,\rho}{d\,t}\right).\operatorname{tang}.\theta + \frac{\rho.\left(\frac{d\,\theta}{d\,t}\right)}{\cos.^2\theta}.$$

The values of $\left(\frac{d\,A}{d\,t}\right)$, and $\left(\frac{d\,R}{d\,t}\right)$, are given by the theory of the earth's [789′] motion: to facilitate the calculation, let E be the excentricity of the earth's orbit, H the longitude of its perihelion ; we have by the nature of the elliptical motion,*

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* (539) The values of x, y, z, were derived from [762], by substituting x', y', [768]. Their differentials, divided by $d\,t$, give the equations [789], without any reduction.

$$\left(\frac{d\mathcal{A}}{dt}\right)=\frac{\sqrt{1-E^2}}{R^2}\,; \qquad R=\frac{1-E^2}{1+E\,.\cos.(\mathcal{A}-H)}\,.$$ [790]

These two equations give†

$$\left(\frac{dR}{dt}\right)=\frac{E\,.\sin.(\mathcal{A}-H)}{\sqrt{1-E^2}}\,.$$ [791]

Let R' be the radius vector of the earth, corresponding to the longitude $\mathcal{A}'$ [791′] of that planet, increased by a right angle ; we shall have‡

$$R'=\frac{1-E^2}{1-E\,.\sin.(\mathcal{A}-H)}\,;$$ [792]

hence we deduce

$$E\,.\sin.(A-H)=\frac{R'-1+E^2}{R'}\,;$$ [793]

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* (540) We have $dt=\frac{dv}{hu^2}$, [531], $u=\frac{1}{r}=\frac{\mu}{h^2}\cdot\left\{1+e\,.\cos.(v-\varpi)\right\}$, [534], $\frac{h^2}{\mu.(1-e^2)}=a$, [534′], and if in these we put $a=1$, $\mu=1$, $e=E$, $v=\mathcal{A}$, $\varpi=H$, $r=R$, they will become $dt=\frac{d\mathcal{A}}{hu^2}$, $u=\frac{1}{R}=\frac{1}{h^2}\cdot\left\{1+E\,.\cos.(\mathcal{A}-H)\right\}$, $\frac{h^2}{1-E^2}=1$, the last gives $h=\sqrt{1-E^2}$, which being substituted in the second, gives

$$\frac{1}{u}=R=\frac{1-E^2}{1+E\,.\cos.(\mathcal{A}-H)}\,,$$

as in [790]. These values of h, and $\frac{1}{u}$ being substituted in the expression of dt, it becomes $dt=\frac{d\mathcal{A}\,.R^2}{\sqrt{1-E^2}}$, hence $\left(\frac{d\mathcal{A}}{dt}\right)=\frac{\sqrt{1-E^2}}{R^2}$, as in [790].

† (541) Taking the differential of the second of the equations [790], R and $\mathcal{A}$ being variable, we shall find

$$\left(\frac{dR}{dt}\right)=\left(\frac{d\mathcal{A}}{dt}\right)\cdot\frac{(1-E^2)\,.E\,.\sin.(\mathcal{A}-H)}{\{1+E\,.\cos.(\mathcal{A}-H)\}^2}=\frac{\sqrt{1-E^2}}{R^2}\cdot\frac{(1-E^2)\,.E\,.\sin.(\mathcal{A}-H)}{\{1+E\,.\cos.(\mathcal{A}-H)\}^2}\,,$$

[790], and by substituting for the denominator its value $(1-E^2)^2$, deduced from the second of the equations [790], it becomes as in [791].

‡ (542) By writing R' for R, and $\frac{1}{2}\pi+\mathcal{A}$, for $\mathcal{A}$, [791′], in the second equation [790].

therefore*

[794] $$\left(\frac{dR}{dt}\right)=\frac{R'+E^2-1}{R'.\sqrt{1-E^2}}.$$

If we neglect the square of the excentricity of the earth's orbit, which is very small, we shall have

[795] $$\left(\frac{dA}{dt}\right)=\frac{1}{R^2}; \qquad \left(\frac{dR}{dt}\right)=R'-1;$$

the preceding values of $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, [789], will by this means become

[796] $$\left(\frac{dx}{dt}\right)=(R'-1).\cos. A-\frac{\sin. A}{R}+\left(\frac{d\rho}{dt}\right).\cos.\alpha-\rho.\left(\frac{d\alpha}{dt}\right).\sin.\alpha;$$
$$\left(\frac{dy}{dt}\right)=(R'-1).\sin. A+\frac{\cos. A}{R}+\left(\frac{d\rho}{dt}\right).\sin.\alpha+\rho.\left(\frac{d\alpha}{dt}\right).\cos.\alpha;$$

[796'] R, R', and A, being given directly by the tables of the sun, the calculation of the six quantities x, y, z, $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, $\left(\frac{dz}{dt}\right)$, will be easy, when ρ and $\left(\frac{d\rho}{dt}\right)$ shall be known. We may thence deduce the elements of the orbit of the comet, in the following manner.

To determine the direct or retrograde motion of the Comet.

The infinitely small sector which the projection of the radius vector of the comet describes on the plane of the ecliptic, during the time dt, is

[796''] $\frac{x\,dy-y\,dx}{2}$ [167'], and it is evident that this sector is *positive* if the motion of the comet is *direct*, but *negative* if the motion is *retrograde*; therefore by

[796'''] computing the quantity $x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)$, it will indicate, by its sign, the direction of the motion of the comet.

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* (543) Substituting the value of $E.\sin.(A-H)$, [793], in $\left(\frac{dR}{dt}\right)$, [791], it becomes as in [794], and by neglecting E^2, it changes into $\frac{R'-1}{R'}$, and as the numerator is of the order E, we may, by neglecting E^2, put the denominator $=1$, making $\left(\frac{dR}{dt}\right)=R'-1$, as in [795]. Neglecting E^2 in $\left(\frac{dA}{dt}\right)$, [790], it becomes as in [795]. Substituting these in [789] we obtain [796].

To determine the position of the orbit, let φ be its inclination to the
ecliptic, and *I* the longitude of the node which would be ascending, if the [796$^{\text{iv}}$]
motion of the comet was direct, we shall have*

[797]
$$z = y\,.\cos.I\,.\,\text{tang.}\,\varphi - x\,.\sin.I\,.\,\text{tang.}\,\varphi\,;$$
$$\left(\frac{dz}{dt}\right) = \left(\frac{dy}{dt}\right).\cos.I\,.\,\text{tang.}\,\varphi - \left(\frac{dx}{dt}\right).\sin.I\,.\,\text{tang.}\,\varphi.$$

These two equations give†

[798]
$$\text{tang.}\,I = \frac{y.\left(\frac{dz}{dt}\right) - z.\left(\frac{dy}{dt}\right)}{x.\left(\frac{dz}{dt}\right) - z.\left(\frac{dx}{dt}\right)}\,;$$
$$\text{tang.}\,\varphi = \frac{y.\left(\frac{dz}{dt}\right) - z.\left(\frac{dy}{dt}\right)}{\sin.I.\left\{x.\left(\frac{dy}{dt}\right) - y.\left(\frac{dx}{dt}\right)\right\}}\,.$$

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* (545) In the figure page 351, let *C* be the place of the sun, *D* that of the comet, *B* its projection on the plane of the ecliptic, *C X* the axis of *x*, in the direction of the first point of aries, *C Y* the axis of *Y*, *C X′* the line of the node, which would be the ascending node if the comet's motion be direct; then the angle $X\,C\,X' = I$, [796$^{\text{iv}}$], is called θ in [585$^{\text{iv}}$]; φ being the same in [585$^{\text{iv}}$] as in [796$^{\text{iv}}$]; therefore, to conform to the present notation, we must change θ into *I*, to obtain from *z*, [588], its value [797]. The differential of *z* being taken, and divided by *d t*, considering *x*, *y*, *z*, only as variable, gives the second of the equations [797], which was accidentally omitted in the original work.

† (546) Multiplying the first of the equations [797] by $-\left(\frac{dy}{dt}\right)$, the second by *y*, and adding the products we get

$$y.\left(\frac{dz}{dt}\right) - z.\left(\frac{dy}{dt}\right) = \sin.I.\,\text{tang.}\,\varphi.\left\{x.\left(\frac{dy}{dt}\right) - y.\left(\frac{dx}{dt}\right)\right\}. \qquad [797a]$$

Again, multiplying the first of the equations [797] by $-\left(\frac{dx}{dt}\right)$, and the second by *x*, and taking the sum of the products, we get

$$x.\left(\frac{dz}{dt}\right) - z.\left(\frac{dx}{dt}\right) = \cos.I.\,\text{tang.}\,\varphi.\left\{x.\left(\frac{dy}{dt}\right) - y.\left(\frac{dx}{dt}\right)\right\}.$$

Dividing the former by the latter, and putting tang. *I* for $\frac{\sin.I}{\cos.I}$, we obtain the first of the equations [798]; the second of these equations is the same as [797*a*] divided by the coefficient of tang. φ.

[798′] As φ ought always to be positive, and less than a right angle, this condition will determine the sign of $\sin. I$; now the tangent of I, and the sign of its sine being determined, the angle I will be wholly determined.* This angle
[798″] is the longitude of the ascending node of the orbit, if the motion be direct; but we must add to it two right angles to obtain the longitude of this node, if the motion be retrograde.† It would be more simple to consider the motion
[798‴] always as direct, making the inclination φ to vary from 0 to two right angles; for it is evident that then the retrograde motion corresponds to an inclination
[798ⁱᵛ] greater than a right angle. In this case, $\text{tang.}\,\varphi$ is of the same sign as‡ $x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)$; which determines $\sin. I$, consequently the angle I, which always expresses the longitude of the ascending node.

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* (547) When I does not exceed a right angle, $\sin. I$ and $\text{tang.}\, I$ are both positive; between one and two right angles, $\sin. I$ is positive $\text{tang.}\, I$ negative; between two and three right angles, $\sin. I$ is negative and $\text{tang.}\, I$ positive; between three and four right angles,
[797b] $\sin. I$ and $\text{tang.}\, I$ are both negative. Therefore by knowing the signs of $\text{tang.}\, I$ and $\sin. I$, we can determine the affection of I; now the first of these is determined by the first of the equations [798], and the second by the second of these equations.

† (548) If the motion be supposed *direct*, the values of dx, dy, dz, must be considered as positive, and if it be *retrograde* these differentials would be *negative*. This would make the second equation [797] become

$$-\left(\frac{dz}{dt}\right)=-\left(\frac{dy}{dt}\right).\cos. I.\,\text{tang.}\,\varphi+\left(\frac{dx}{dt}\right).\sin. I.\,\text{tang.}\,\varphi,$$

which by changing the signs of all the terms would become identical with the equation [797], from which it was derived; therefore the two equations [797] would be the same whether the motion be supposed direct or retrograde. The same must take place in the equations [798] deduced from [797]. The angle I, determined by these equations would give the place of that node, which would be *ascending* if the motion be *direct*, and if the motion be *retrograde* the numerical value of I would remain the same, because the terms of [798] would remain unaltered, but in this last case the value of I would correspond to the *descending* node, and we must add to it two right angles to obtain the longitude of the *ascending* node.

‡ (549) It was observed in [796″], that $x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)$ is *positive* when the motion is direct, in which case φ is supposed to be less than a right angle and $\text{tang.}\,\varphi$ *positive*. When the motion is retrograde, $x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)$ becomes *negative*, and as φ then

a and *ae* being the semi-transverse axis and the excentricity of the orbit [798^{v}]
[596′], we have by § 18, 19, supposing $\mu=1$,*

$$\frac{1}{a}=\frac{2}{r}-\left(\frac{dx}{dt}\right)^2-\left(\frac{dy}{dt}\right)^2-\left(\frac{dz}{dt}\right)^2;$$

$$a.(1-e^2)=2r-\frac{r^2}{a}-\left\{x.\left(\frac{dx}{dt}\right)+y.\left(\frac{dy}{dt}\right)+z.\left(\frac{dz}{dt}\right)\right\}^2.$$ [799]

The first of these equations determines the semi-transverse axis of the orbit, the second its excentricity. The sign of the function

$$x.\left(\frac{dx}{dt}\right)+y.\left(\frac{dy}{dt}\right)+z.\left(\frac{dz}{dt}\right),$$ [799′]

shows whether the comet has passed the perihelion;† for if it is *approaching towards the perihelion, this function is negative; and in the contrary case, the comet is receding from the perihelion.*

To find whether the Comet has passed the perihelion.

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exceeds a right angle, its tangent becomes *negative*. In both cases we have tang. φ of the same sign as $x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)$. The product tang. $\varphi.\left\{x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)\right\}$ is therefore always positive, and as the second of the equations [798] gives

$$\sin. I=\frac{y.\left(\frac{dz}{dt}\right)-z.\left(\frac{dy}{dt}\right)}{\text{tang.}\,\varphi.\left\{x.\left(\frac{dy}{dt}\right)-y.\left(\frac{dx}{dt}\right)\right\}},$$ [798*a*]

we shall have sin. I of the same sign as $y.\left(\frac{dz}{dt}\right)-z.\left(\frac{dy}{dt}\right)$, with this sign, and the first equation [798] we find I by the directions in note 547.

* (550) The first of these equations is the same as the last of [572]. The second is deduced from the equation [598], putting $\mu=1$, $h^2=a.(1-e^2)$, [599], and [799*a*] $r\,dr=x\,dx+y\,dy+z\,dz$, [549′].

† (551) This function $x.\left(\frac{dx}{dt}\right)+y.\left(\frac{dy}{dt}\right)+z.\left(\frac{dz}{dt}\right)$ is by the last note equal to $r.\left(\frac{dr}{dt}\right)$, and as r is positive it must have the same sign as $\left(\frac{dr}{dt}\right)$, which must, from the nature of the perihelion, where r is a minimum, be negative before passing the perihelion, positive after passing it. The value of the function $x.\left(\frac{dx}{dt}\right)+y.\left(\frac{dy}{dt}\right)+z.\left(\frac{dz}{dt}\right)$ is found by using the values of x, y, z, [788]; $\left(\frac{dx}{dt}\right)$, $\left(\frac{dy}{dt}\right)$, [796], and $\left(\frac{dz}{dt}\right)$, [789].

Let T be the interval of time between the epoch and the passage of the
[799″] comet through the perihelion; the two first of the equations (f) § 20 [606],
will give, by observing that μ having been put equal to unity [798ᵛ], makes
$n=a^{-\frac{3}{2}}$ [605′],*

[800] $$r=a\,.\,(1-e\,.\,\cos.\,u)\,; \qquad T=a^{\frac{3}{2}}\,.\,(u-e\,.\,\sin.\,u).$$

The first of these equations gives the angle u, the second the time T. This
time added to the epoch, if the comet is approaching towards the perihelion,
but subtracted from the epoch if the comet is receding from the perihelion,
[800′] will give the instant of its passage through this point. The values of x and
y will determine the angle which the projection of the radius vector r makes
with the axis of x; and since we know the angle I made by this axis and
[800″] the line of nodes, we shall have the angle which this last line makes with
the projection of r; hence we may deduce, by means of the inclination of
the orbit φ, the angle formed by the line of nodes and the radius r.† But
the angle u being known, we shall have, by means of the third of the
[800‴] equations (f) § 20 [606], the angle v which this radius makes with the
line of apsides; hence we shall have the angle included between the lines of
apsides and nodes; consequently the position of the perihelion, and all the
elements of the orbit, will be determined.

34. These elements are given by what precedes, in functions of ρ, $\left(\frac{d\rho}{dt}\right)$,
[800ⁱᵛ] and known quantities; and as $\left(\frac{d\rho}{dt}\right)$ is given in ρ by § 31 [770], the
elements of the orbit will be functions of ρ and known quantities. If one

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* (552) The first of the equations [606], putting $\mu=1$, $n=a^{-\frac{3}{2}}$, [605′, 798ᵛ], and multiplying by $a^{\frac{3}{2}}$, gives t or $T=a^{\frac{3}{2}}\,.\,(u-e\,.\,\sin.\,u)$, as in [800]. In the original, sin. u was printed cos. u, by a typographical error.

† (553) The angle formed by the line of nodes and the projection of the radius vector, may be considered as measured by the arch $D\,C$ in the figure page 379, the angle $B\,D\,C$ being φ, the arch $B\,D$ will be the measure of the angle formed by the radius and the line of nodes, and by spherics we shall have $\text{cotang.}\,B\,D=\cos.\,\varphi\,.\,\text{cotang.}\,D\,C$.

of them be given,* we should have another equation, by means of which we might determine ρ ; this equation would have a common divisor with the equation (4) § 31 [784], and if we seek this divisor by the usual methods, we should obtain an equation of the first degree in ρ ; we should also have an [800$^{\text{v}}$] equation of condition between the quantities given by the observations, and this equation would be that which ought to take place, in order that the given element may appertain to the orbit of the comet.

We shall now apply this principle to the case of nature. For this purpose we shall observe that the orbits of comets are very excentric ellipses, which [800$^{\text{vi}}$] nearly coincide with a parabola, in the part in which these bodies are visible ; we may therefore suppose, without sensible error, $a=\infty$, [800$^{\text{vii}}$] consequently $\frac{1}{a}=0$; the expression of $\frac{1}{a}$ of the preceding article [799], will in this case become

$$0=\frac{2}{r}-\frac{(d\,x^2+d\,y^2+d\,z^2)}{d\,t^2}. \qquad [800^{\text{viii}}]$$

If we substitute for, $\left(\frac{d\,x}{dt}\right)$, $\left(\frac{d\,y}{dt}\right)$, and $\left(\frac{d\,z}{dt}\right)$, their values found in the same article [796, 789], we shall have, after making the necessary reductions, [800$^{\text{ix}}$] and neglecting the square of $R'-1$,†

$$\begin{aligned}0=\left(\frac{d\,\rho}{d\,t}\right)^2+\rho^2.\left(\frac{d\,\alpha}{dt}\right)^2&+\left\{\left(\frac{d\,\rho}{d\,t}\right).\,\text{tang.}\,\theta+\frac{\rho.\left(\frac{d\,\theta}{d\,t}\right)}{\cos.^2\theta}\right\}^2\\&+2.\left(\frac{d\,\rho}{d\,t}\right).\left\{(R'-1).\cos.(A-\alpha)-\frac{\sin.(A-\alpha)}{R}\right\} \qquad (5)\\&+2\rho.\left(\frac{d\,\alpha}{dt}\right).\left\{(R'-1).\sin.(A-\alpha)+\frac{\cos.(A-\alpha)}{R}\right\}+\frac{1}{R^2}-\frac{2}{r}\,;\end{aligned} \qquad [801]$$

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* (553a) That is, if one of the elements be given. This is supposed to be the case in calculating the orbit of a comet, in which a is supposed to be infinite, as in [800$^{\text{vii}}$], from which is deduced the equation in ρ, [805].

† (554) Changing the signs of [800$^{\text{viii}}$], and substituting for $\left(\frac{d\,z}{d\,t}\right)^2$ its value in [789], we shall have, $0=\left\{\left(\frac{d\,\rho}{d\,t}\right).\,\text{tang.}\,\theta+\frac{\rho.\left(\frac{d\,\theta}{d\,t}\right)}{\cos.^2\theta}\right\}^2+\left(\frac{d\,x}{d\,t}\right)^2+\left(\frac{d\,y}{d\,t}\right)^2-\frac{2}{r}$, in which are [801$a$]

by substituting in this equation, for $\left(\frac{d\rho}{dt}\right)$, its value [785],

[802]
$$-\frac{\rho}{2\cdot\left(\frac{d\alpha}{dt}\right)}\cdot\left\{\left(\frac{dd\alpha}{dt^2}\right)+\mu'\cdot\sin.(A-\alpha)\right\},$$

---

to be substituted $\left(\frac{dx}{dt}\right)^2$, $\left(\frac{dy}{dt}\right)^2$, [796]. In making this substitution we shall, for brevity,

[801b] put $(R'-1)\cdot\cos.A-\frac{\sin.A}{R}=D,$ $\qquad (R'-1)\cdot\sin.A+\frac{\cos.A}{R}=D',$

$\left(\frac{d\rho}{dt}\right)\cdot\cos.\alpha-\rho\cdot\left(\frac{d\alpha}{dt}\right)\cdot\sin.\alpha=E,$ $\qquad \left(\frac{d\rho}{dt}\right)\cdot\sin.\alpha+\rho\cdot\left(\frac{d\alpha}{dt}\right)\cdot\cos.\alpha=E',$ by which means the formulas [796] will become, $\left(\frac{dx}{dt}\right)=D+E,$ $\left(\frac{dy}{dt}\right)=D'+E',$ and the sum of their squares is

[801c]
$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(D^2+D'^2)+(E^2+E'^2)+2DE+2D'E'.$$

Now the values of D, D', E, E', evidently give $D^2+D'^2=(R'-1)^2+\frac{1}{R^2}=\frac{1}{R^2}$, neglecting the square of $R'-1$, [800^ix], and $E^2+E'^2=\left(\frac{d\rho}{dt}\right)^2+\rho^2\cdot\left(\frac{d\alpha}{dt}\right)^2$. In finding the value of $2DE+2D'E'$, we shall connect together the terms multiplied by $2\cdot\left(\frac{d\rho}{dt}\right)$, and in another group those multiplied by $2\rho\cdot\left(\frac{d\alpha}{dt}\right)$, and we shall have

$$2DE+2D'E'=2\cdot\left(\frac{d\rho}{dt}\right)\cdot\left\{\begin{array}{l}(R'-1)\cdot(\cos.A\cdot\cos.\alpha+\sin.A.\sin.\alpha)\\ -\frac{1}{R}\cdot(\sin.A\cdot\cos.\alpha-\cos.A\cdot\sin.\alpha)\end{array}\right\}$$

$$+2\rho.\left(\frac{d\alpha}{dt}\right)\cdot\left\{(R'-1).(\sin.A.\cos.\alpha-\cos.A\cdot\sin.\alpha)+\frac{1}{R}\cdot(\cos.A.\cos.\alpha+\sin.A.\sin.\alpha)\right\}$$

$$=2\cdot\left(\frac{d\rho}{dt}\right)\cdot\left\{(R'-1)\cdot\cos.(A-\alpha)-\frac{1}{R}\cdot\sin.(A-\alpha)\right\}$$

$$+2\rho\cdot\left(\frac{d\alpha}{dt}\right)\cdot\left\{(R'-1)\cdot\sin.(A-\alpha)+\frac{1}{R}\cdot\cos.(A-\alpha)\right\},$$

[24, 22] Int. These being substituted in [801c] we get

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{d\rho}{dt}\right)^2+\rho^2\cdot\left(\frac{d\alpha}{dt}\right)^2+\frac{1}{R^2}+2\cdot\left(\frac{d\rho}{dt}\right)\cdot\left\{(R'-1)\cdot\cos.(A-\alpha)-\frac{\sin.(A-\alpha)}{R}\right\}$$

$$+2\rho\cdot\left(\frac{d\alpha}{dt}\right)\cdot\left\{(R'-1)\cdot\sin.(A-\alpha)+\frac{\cos.(A-\alpha)}{R}\right\},$$

hence [801a] becomes as in [801].

found in § 31; and then putting

$$4.\left(\frac{d\alpha}{dt}\right)^2.B=4.\left(\frac{d\alpha}{dt}\right)^4+\left\{\left(\frac{dd\alpha}{dt^2}\right)+\mu'.\sin.(A-\alpha)\right\}^2$$

$$+\left\{\text{tang.}\,\theta.\left(\frac{dd\alpha}{dt^2}\right)+\mu'.\text{tang.}\,\theta.\sin.(A-\alpha)-\frac{2.\left(\frac{d\alpha}{dt}\right).\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2;$$

[803]

$$C=\frac{\left(\frac{dd\alpha}{dt^2}\right)+\mu'.\sin.(A-\alpha)}{\left(\frac{d\alpha}{dt}\right)}.\left\{\frac{\sin.(A-\alpha)}{R}-(R'-1).\cos.(A-\alpha)\right\}$$

$$+2.\left(\frac{d\alpha}{dt}\right).\left\{(R'-1).\sin.(A-\alpha)+\frac{\cos.(A-\alpha)}{R}\right\};$$

we shall have*

$$0=B.\rho^2+C.\rho+\frac{1}{R^2}-\frac{2}{r};$$ [804]

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* (555) Put for brevity $\left(\frac{dd\alpha}{dt^2}\right)+\mu'.\sin.(A-\alpha)=F$, and we shall get from [802], $\left(\frac{d\rho}{dt}\right)=-\frac{\rho F}{2.\left(\frac{d\alpha}{dt}\right)}$. Substituting this in [801], we find

$$0=\frac{\rho^2F^2}{4.\left(\frac{d\alpha}{dt}\right)^2}+\rho^2.\left(\frac{d\alpha}{dt}\right)^2+\left\{-\frac{\rho F}{2.\left(\frac{d\alpha}{dt}\right)}.\text{tang.}\,\theta+\frac{\rho.\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2$$

$$-\frac{\rho F}{\left(\frac{d\alpha}{dt}\right)}.\left\{(R'-1).\cos.(A-\alpha)-\frac{\sin.(A-\alpha)}{R}\right\}$$

$$+2\rho.\left(\frac{d\alpha}{dt}\right).\left\{(R'-1).\sin.(A-\alpha)+\frac{\cos.(A-\alpha)}{R}\right\}+\frac{1}{R^2}-\frac{2}{r};$$

and by arranging according to the powers of ρ,

$$0=\rho^2.\left\{\left(\frac{d\alpha}{dt}\right)^2+\frac{F^2}{4.\left(\frac{d\alpha}{dt}\right)^2}+\frac{1}{4.\left(\frac{d\alpha}{dt}\right)^2}.\left(F.\text{tang.}\,\theta-\frac{2.\left(\frac{d\theta}{dt}\right).\left(\frac{d\alpha}{dt}\right)}{\cos.^2\theta}\right)^2\right\}$$

$$+\rho.\left\{\begin{array}{l}\frac{F}{\left(\frac{d\alpha}{dt}\right)}.\left\{-(R'-1).\cos.(A-\alpha)+\frac{\sin.(A-\alpha)}{R}\right\}\\+2.\left(\frac{d\alpha}{dt}\right).\left\{(R'-1).\sin.(A-\alpha)+\frac{\cos.(A-\alpha)}{R}\right\}\end{array}\right\}+\frac{1}{R^2}-\frac{2}{r},$$

consequently

[805] $$r^2 \cdot \left\{ B \cdot \rho^2 + C \cdot \rho + \frac{1}{R^2} \right\}^2 = 4 \,;$$

this equation is only of the sixth degree, and in this respect it is more simple [805′] than the equation (4) § 31 [784]; but it is restricted to the parabola, whereas the equation (4) [784] extends to every kind of conic section.

35. We find from the preceding analysis, that the determination of the parabolic orbit of a comet leads to more equations than there are unknown [805″] quantities;* we may, by combining these equations in different manners, form several different methods of computing these orbits. We shall examine those from which we ought to expect the greatest precision in the results, and which are the least affected by the errors of the observations.

It is chiefly in the values of the second differentials $\left(\frac{dd\alpha}{dt^2}\right)$, and $\left(\frac{dd\theta}{dt^2}\right)$, that these errors have a sensible influence. In fact, to determine them, it is [805‴] necessary to take the second differences† of the geocentric longitudes and latitudes of the comet, observed in a short interval of time; now these differences being less than the first differences, the errors of observation will be a greater aliquot part of these second differences; moreover, the formulas of § 29 [758], which determine, by combining the observations, the values of α, θ, $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{dd\alpha}{dt^2}\right)$, $\left(\frac{dd\theta}{dt^2}\right)$, give with greater precision the [805iv] four first of these quantities, than the two last; it is therefore advantageous

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in which the coefficient of ρ^2 is equal to B, [803], and that of ρ is equal to C, [803], hence the preceding equation becomes $0 = B\rho^2 + C\rho + \frac{1}{R^2} - \frac{2}{r}$, as in [804]. Transposing $\frac{2}{r}$, multiplying by r, and squaring both sides, we obtain [805].

[*] (556) Thus the *four* independent equations [782, 770, 774, 801], which compose the equations [806], are given to find the three unknown quantities ρ, $\left(\frac{d\rho}{dt}\right)$, and r, being *one* more than is absolutely requisite.

[†] (557) The word here translated "second" was in the original printed "finies" instead of "secondes."

to depend as little as possible on the second differences of α and θ; and as we cannot reject them both, at the same time, the method which uses only the greatest, must give the most accurate results; this being premised,

We shall resume the equations of § 31 and 34, [782, 770, 774, 801]

$$r^2 = \frac{\rho^2}{\cos.^2\theta} + 2R.\rho.\cos.(A-\alpha) + R^2;$$

$$\left(\frac{d\rho}{dt}\right) = \frac{R.\sin.(A-\alpha)}{2.\left(\frac{d\alpha}{dt}\right)}.\left\{\frac{1}{R^3} - \frac{1}{r^3}\right\} - \frac{\rho.\left(\frac{dd\alpha}{dt^2}\right)}{2.\left(\frac{d\alpha}{dt}\right)}; \qquad (L)$$

Fundamental equations for computing the orbit of a comet.

$$\left(\frac{d\rho}{dt}\right) = -\tfrac{1}{2}\rho.\left\{\frac{\left(\frac{dd\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)} + 2.\left(\frac{d\theta}{dt}\right).\text{tang.}\theta + \frac{\left(\frac{d\alpha}{dt}\right)^2.\sin.\theta.\cos.\theta}{\left(\frac{d\theta}{dt}\right)}\right\}$$

$$+\frac{R.\sin.\theta.\cos.\theta.\cos.(A-\alpha)}{2.\left(\frac{d\theta}{dt}\right)}.\left\{\frac{1}{r^3} - \frac{1}{R^3}\right\}; \qquad [806]$$

$$0 = \left(\frac{d\rho}{dt}\right)^2 + \rho^2.\left(\frac{d\alpha}{dt}\right)^2 + \left\{\left(\frac{d\rho}{dt}\right).\text{tang.}\theta + \frac{\rho.\left(\frac{d\theta}{dt}\right)}{\cos.^2\theta}\right\}^2$$

$$+2.\left(\frac{d\rho}{dt}\right).\left\{(R'-1).\cos.(A-\alpha) - \frac{\sin.(A-\alpha)}{R}\right\}$$

$$+2\rho.\left(\frac{d\alpha}{dt}\right).\left\{(R'-1).\sin.(A-\alpha) + \frac{\cos.(A-\alpha)}{R}\right\} + \frac{1}{R^2} - \frac{2}{r}.$$

If we would reject $\left(\frac{dd\theta}{dt^2}\right)$, we must use the first, second and fourth of these equations;* by eliminating $\left(\frac{d\rho}{dt}\right)$ from the last, by means of the second, we shall obtain an equation, which being cleared from fractions, will

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* (558) It is to be observed that by neglecting one of the equations [806], the resulting equation in ρ or r is of a higher order; for instead of being of the sixth degree in ρ, as in [805], it becomes of the sixteenth, as in [806']. Upon further consideration of the subject the author finally concluded, not to reject wholly either of these equations, but to combine two of them together, in a manner which he supposed would probably lead to the most accurate result. We shall hereafter, [815a, &c.], speak of this method, which is particularly treated of by the author in Book XV, §5. [806a]

[806'] contain a term multiplied by $r^6 \rho^2$, and other terms affected with even and odd powers of ρ and r. If we place on one side of the equation all the terms containing even powers of r, and on the other all the terms containing the odd powers, and then square both sides to obtain only even powers of r, the term multiplied by $r^6 \rho^2$, will produce one multiplied by $r^{12} \rho^4$; and by
[806''] substituting the value of r^2 given by the first of the equations (L) [806], we shall finally obtain an equation of the sixteenth degree in ρ. But instead of forming this equation, to resolve it afterwards, it will be more simple to satisfy the three preceding equations by trials.

If we would reject $\left(\frac{d\,d\,\alpha}{d\,t^2}\right)$, we must use the first, third and fourth of the
[806'''] equations (L) [806]. These three equations lead to a final equation of the sixteenth degree in ρ; which equations may be easily satisfied by trials.

The two preceding methods appear to me to be the most accurate that we can use in finding the parabolic orbits of comets; it is even absolutely necessary to have recourse to them if the motion of the comet in longitude
[806iv] or in latitude is insensible or very small, in order that the errors of the observations may not alter sensibly the second differential; in this case we must reject that one of the equations (L) [806] which contains that second differential. But although in these methods we use only three of the equations, the fourth will be useful to determine, among all the real and
[806v] positive values of ρ which satisfy the system of the three other equations, that value which ought to be assumed.

36. The elements of the orbit of a comet, determined in the preceding manner, would be exact if the values of α, θ, and of their first and second
[806vi] differentials, were rigorously correct; for we have taken into consideration, in a very simple manner, the excentricity of the earth's orbit, by means of the radius vector R' of the earth, corresponding to its true anomaly increased by a right angle; we have only neglected the square of this excentricity, as being so small a fraction that its neglect could not sensibly affect the result.
[806vii] But θ, α, and their differentials, are always liable to some error, on account of the imperfection of the observations, and also by reason of the errors arising from the approximate method of computing their differentials. It is therefore necessary to correct the elements, by means of three distant observations, which may be done by a very great variety of methods; for if

we know very nearly two quantities relative to the motion of a comet, as,
for example, the radius vector at each of two observations, or the position of [806^{viii}]
the node and the inclination of the orbit; by calculating the observations first
with these quantities, then with other quantities which vary a little from
them; the law of the differences between the results, will easily give the
corrections to be applied to those quantities. But among all the combinations,
two by two, of the quantities relative to the motion of comets, there is one [806^{ix}]
which furnishes the most simple calculation, and which, for that reason,
deserves particular attention; it being of importance, in so complicated a
problem, to spare the calculator all unnecessary labor. The two elements
which appear to me to have this advantage, are the perihelion distance, and
the time of passing the perihelion; they are not only easily found from the
values of ρ and $\left(\frac{d\rho}{dt}\right)$; but may be very easily corrected by other observations, [806^{x}]
without being obliged, at each variation which is made in these two elements,
to determine all the other corresponding elements of the orbit. [806^{xi}]

We shall resume the equation found in § 19 [598, 599]*

$$a\,.\,(1-e^2)=2r-\frac{r^2}{a}-\frac{r^2\,.\,dr^2}{dt^2}; \qquad [807]$$

$a\,.\,(1-e^2)$ is the semi-parameter [383′, 377″] of the conic section of which [807′]
a is the semi-transverse axis, and ae the excentricity; in the parabola, where
a is infinite, and e equal to unity, $a\,.\,(1-e^2)$ is the double of the perihelion [807″]
distance; naming this distance D, the preceding equation becomes,
relatively to this curve,†

$$D=r-\tfrac{1}{2}\,.\left(\frac{r\,dr}{dt}\right)^2. \qquad [808]$$

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* (559) This is like the second of the equations [799], putting

$$xdx+ydy+zdz=r\,dr, \quad [799a].$$

† (560) Since $D=a\,.\,(1-e)$, [681″], we have

$$a\,.\,(1-e^2)=a\,.\,(1-e)\,.\,(1+e)=D\,.\,(1+e),$$

and in a parabola, where $e=1$, [378b], it becomes $2\,D$. This being substituted in [807], observing that when $a=\infty$; $\frac{r^2}{a}=0$, it becomes $2\,D=2r-\frac{r^2dr^2}{dt^2}$. Dividing this by

$\frac{r\,d\,r}{d\,t}$ is equal to $\frac{\frac{1}{2}d\,.\,r^2}{d\,t}$. Substituting for r^2, its value

$$\frac{\rho^2}{\cos.^2\theta}+2\,R\,\rho\,.\,\cos.\,(A-\alpha)+R^2,$$

and instead of $\left(\frac{d\,R}{d\,t}\right)$, and $\left(\frac{d\,A}{d\,t}\right)$, their values found in § 33, we shall have, by putting for brevity P equal to the last member of the following expression of $\frac{r\,d\,r}{d\,t}$,

[809]
$$\begin{aligned}P=\frac{r\,d\,r}{d\,t}=\frac{\rho}{\cos.^2\theta}\,.\left\{\left(\frac{d\,\rho}{d\,t}\right)+\rho\,.\left(\frac{d\,\theta}{d\,t}\right).\,\text{tang.}\,\theta\right\}&+R\,.\left(\frac{d\,\rho}{d\,t}\right).\cos.\,(A-\alpha)\\ &+\rho\,.\left\{(R'-1)\,.\cos.\,(A-\alpha)-\frac{\sin.\,(A-\alpha)}{R}\right\}\\ &+\rho\,.R\,.\left(\frac{d\,\alpha}{d\,t}\right).\,\sin.\,(A-\alpha)+R\,.\,(R'-1)\,;\end{aligned}$$

[809′] if P be negative, the radius vector r would be decreasing, consequently the comet would tend towards the perihelion; but it would be receding from the perihelion if P be positive.* We thus have

[810]
$$D=r-\tfrac{1}{2}\,P^2\,;$$

the angular distance v of the comet from its perihelion, is found by the polar equation of the parabola [691]

[811]
$$\cos.^2\tfrac{1}{2}\,v=\frac{D}{r}\,;$$

[811′] lastly, we shall have the time employed in describing the angle v, by the table of the motion of comets. This time, added to the time of the epoch,

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2 we get D, [808], which may be written $D=r-\frac{1}{2}\,.\left(\frac{\frac{1}{2}d\,.\,r^2}{d\,t}\right)^2$. Now half the differential of the first of the equations [806], is

$$\begin{aligned}\tfrac{1}{2}\,d\,.\,r^2=\frac{\rho\,d\,\rho}{\cos.^2\theta}+\frac{\rho^2\,.\,\sin.\,\theta}{\cos.^3\theta}\,.\,d\,\theta+R\,d\,\rho\,.\cos.\,(A-\alpha)+\rho\,.\,d\,R\,.\cos.\,(A-\alpha)&\\ +R\,\rho\,.\,(d\,\alpha-d\,A)\,.\,\sin.\,(A-\alpha)+R\,d\,R.&\end{aligned}$$

Dividing this by $d\,t$ and substituting the values of $\left(\frac{d\,A}{d\,t}\right)$, $\left(\frac{d\,R}{d\,t}\right)$, [795], we obtain [809].

* (561) This is conformable to what is shown in note 551. The expression P [809], substituted in [808], gives [810].

if P be negative, or subtracted from the time of the epoch, if P be positive, will give the instant that the comet passes the perihelion.

37. Collecting together these various results, we shall have the following method of computing the parabolic orbit of a comet.

GENERAL METHOD FOR COMPUTING THE ORBIT OF A COMET.

This method will be divided into two parts; in the first we shall give the method of obtaining very nearly the perihelion distance of the comet, and the instant of passing the perihelion; in the second we shall determine accurately all the other elements of the orbit, supposing the former to be known very nearly. [811'']

APPROXIMATE COMPUTATION OF THE PERIHELION DISTANCE OF A COMET, AND THE INSTANT OF ITS PASSING THE PERIHELION.

We must select three, four, or five, &c., observations of the comet, as nearly equidistant from each other as possible. With three observations we may embrace an interval of 30° [27^{d} of the sexagesimal division]; with five observations, an interval of 36° or 40° [$32^{d}\,24^{m}$ to 36^{d} of the sexagesimal division], and in like manner for a greater number; but it is always necessary that the interval should be increased with the increase of the number of observations, in order to diminish the effect of the errors of the observations. This being premised, [811''']

Rules for computing the orbit of a comet.

Let β, β', β'', &c., be the successive geocentric longitudes of the comet; $\gamma, \gamma', \gamma''$, &c., the corresponding latitudes, these latitudes being supposed *positive* if *north*, but *negative* if *south*. We must divide the difference $\beta' - \beta$, by the number of days elapsed between the first and second observation; in like manner we must divide the difference $\beta'' - \beta'$, by the number of days elapsed between the second and third observation; we must also divide the difference $\beta''' - \beta''$, by the number of days elapsed between the third and fourth observation; and so on for the others. Let these quotients be $\delta\beta, \delta\beta', \delta\beta''$, &c.* [811^iv] [811^v]

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* (563) This and the following part of the article correspond to §29, [755,756,758,759]. The intervals between the observations are to be expressed in days and decimal parts of a day.

We must divide the difference $\delta\beta' - \delta\beta$, by the number of days interval
[811vi] between the first and third observation; in like manner we must divide the
difference $\delta\beta'' - \delta\beta'$ by the number of days between the second and fourth
observation; the difference $\delta\beta''' - \delta\beta''$ by the number of days between the
third and fifth observation; and so on for the rest. Let these quotients be
$\delta^2\beta$, $\delta^2\beta'$, $\delta^2\beta''$, &c.

We must divide the difference $\delta^2\beta' - \delta^2\beta$ by the number of days between
[811vii] the first and fourth observation; in like manner we must divide $\delta^2\beta'' - \delta^2\beta'$
by the number of days between the second and fifth; and so on. Let these
quotients be $\delta^3\beta$, $\delta^3\beta'$, &c. We must proceed in the same manner till we
[811viii] obtain $\delta^{n-1}\beta$, n being the number of observations used.

This being done, we must take an epoch, which is equidistant, or nearly
[811ix] so, from the two extreme observations, and putting i, i', i'', i''', &c., for the
number of days it precedes each observation, i, i', i'', &c., being supposed
negative in observations preceding the epoch; the longitude of the comet,
after a small number of days, denoted by z, counted from the epoch, will be
expressed by the following formula:

[812]
$$\begin{aligned}&\beta - i.\delta\beta + ii'.\delta^2\beta - ii'i''.\delta^3\beta + \&c. \qquad (p)\\ &+z.\{\delta\beta - (i+i').\delta^2\beta + (ii'+ii''+i'i'').\delta^3\beta - (ii'i''+ii'i'''+ii''i'''+i'i''i''').\delta^4\beta + \&c.\}\\ &+z^2.\{\delta^2\beta - (i+i'+i'').\delta^3\beta + (ii'+ii''+ii'''+i'i''+i'i'''+i''i''').\delta^4\beta - \&c.\}\end{aligned}$$

The coefficients of $-\delta\beta$, $+\delta^2\beta$, $-\delta^3\beta$, &c., in the part independent of
[812'] z, are, *First*, the number i; *Second*, the product of the two numbers i and
i'; *Third*, the product of the three numbers i, i', i'', &c.

The coefficients of $-\delta^2\beta$, $+\delta^3\beta$, $-\delta^4\beta$, &c., in the part multiplied by
[812''] z, are, *First*, the sum of the two numbers i and i'; *Second*, the sum of the
products, two by two, of the three numbers i, i', i''; *Third*, the sum of
the products, three by three, of the four numbers i, i', i'', i''', &c.

The coefficients of $-\delta^3\beta$, $+\delta^4\beta$, $-\delta^5\beta$, &c., in the part multiplied
by z^2, are, *First*, the sum of the three numbers, i, i', i''; *Second*, the sum
[812'''] of the products, two by two, of the four numbers i, i' i'', i''', ; *Third*,
the sum of the products, three by three, of the five numbers i, i', i'', i''',
i'''', &c.

Instead of forming these products, it is as easy to develop the function

[813] $\beta + (z-i).\delta\beta + (z-i).(z-i').\delta^2\beta + (z-i).(z-i').(z-i'').\delta^3\beta + \&c.,$

rejecting the powers of z, above the square; which will give the preceding formula [812].

If we perform a similar operation upon the observed geocentric latitudes of the comet, its geocentric latitude in z days after the epoch, will be expressed by the formula (p) [812], changing in it β into γ. Let us call the formula thus changed (q) [813′]. This being premised, [813′]

α will be the part independent of z in the formula (p) [812], θ will be the part independent of z in the formula (q) [813′]. [813″]

Reducing into seconds the coefficient of z in the formula (p) [812], and subtracting from the tabular logarithm of this number of seconds, the logarithm* 4,0394622 [or 3,5500072 sex.], we shall have the logarithm of a number that we shall denote by a. [813‴]

Reducing into seconds the coefficient of z^2 in the same formula, and subtracting from the logarithm of that number of seconds the logarithm† 1,9740144 [or 1,4845594 sex.], we shall have the logarithm of a number that we shall denote by b. [813^{iv}]

By reducing in like manner into seconds the coefficients of z and z^2, in the formula (q) [813′], and subtracting from the logarithms of these numbers the logarithms 4,0394622,* and 1,9740144† respectively [or 3,5500072 and [813^{v}]

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* (564) If we use the common sexagesimal division of the quadrant into 90^d, or 324000^s, the logarithm must be 3,5500072 as is observed in [759c]. The values a, h being respectively equal to $\left(\frac{d\,\alpha}{d\,t}\right)$, $\left(\frac{d\,\theta}{d\,t}\right)$.

† (565) The coefficient of z^2 in the function [812] is the same as $\frac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,s^2}\right)$, [758], and it is shown, in [759c], that by subtracting from the logarithm of $\frac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,s^2}\right)$, in seconds, the quantity 2,2750444, for the centesimal division, or 1,7855894, for the sexagesimal division, we shall obtain the log. of $\frac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,t^2}\right) = \text{log.}\ \frac{1}{2}\,b$, [813′]. Adding to this the log. of 2 or 0,3010300, we shall get the log. of b. Moreover, since

$$2{,}2750444 - 0{,}3010300 = 1{,}9740144,$$

and

$$1{,}7855894 - 0{,}3010300 = 1{,}4845594,$$

we may obtain the log. of b, by subtracting from log. $\frac{1}{2}\cdot\left(\frac{d^2\alpha}{d\,s^2}\right)$ in seconds, the number 1,9740144, if centesimal seconds are used, or 1,4845594, if sexagesimal seconds are used.

1,4845594 in sexagesimals], we shall have the logarithms of two numbers which we shall call h and l.

The accuracy of this method depends on the precision of the values of a, b, [813vi] h, l; and as the computation of these quantities is very simple, we must select and increase the number of observations, so as to ascertain them with all the exactness that the observations will allow of. It is evident that a, b, [813vii] h, l, represent the quantities $\left(\frac{d\alpha}{dt}\right)$, $\left(\frac{d^2\alpha}{dt^2}\right)$, $\left(\frac{d\theta}{dt}\right)$, $\left(\frac{d^2\theta}{dt^2}\right)$; which, for greater simplicity, have been expressed by the preceding letters.

If the number of observations be odd, we may fix the epoch at the instant [813viii] of the middle observation; and then we may dispense with the calculation of the parts independent of z, in the two preceding formulas; for it is evident [813ix] that these parts would then be equal to the longitude and latitude of the middle observation respectively.

Having thus determined the values of α, a, b, θ, h, and l, we must find the [813x] longitude of the sun, at the time of the epoch; let E be this longitude, R the corresponding distance of the earth from the sun, and R' the distance corresponding to E increased by a right angle; we must then form the following equations:*

[814]
$$r^2 = \frac{x^2}{\cos.^2\theta} - 2Rx.\cos.(E-\alpha) + R^2; \qquad (1)$$

[815]
$$y = \frac{R.\sin.(E-\alpha)}{2a}.\left\{\frac{1}{r^3} - \frac{1}{R^3}\right\} - \frac{bx}{2a}; \qquad (2)$$

[816]
$$\begin{aligned} y = -x.&\left\{h.\text{tang.}\,\theta + \frac{l}{2h} + \frac{a^2.\sin.\theta.\cos.\theta}{2h}\right\} \\ &+ \frac{R.\sin.\theta.\cos.\theta}{2h}.\cos.(E-\alpha).\left\{\frac{1}{R^3} - \frac{1}{r^3}\right\}; \end{aligned} \qquad (3)$$

Equations for computing the first approximation to the elements of the orbit of a comet.

[817]
$$\begin{aligned} 0 = y^2 + a^2x^2 + \left(y.\text{tang.}\,\theta + \frac{hx}{\cos.^2\theta}\right)^2 + 2y.&\left\{\frac{\sin.(E-\alpha)}{R} - (R'-1).\cos.(E-\alpha)\right\} \\ -2ax.&\left\{(R'-1).\sin.(E-\alpha) + \frac{\cos.(E-\alpha)}{R}\right\} + \frac{1}{R^2} - \frac{2}{r}. \end{aligned} \qquad (4)$$

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* (566) These are the same as the equations [806]. Putting $\rho = x$, $\left(\frac{d\rho}{dt}\right) = y$, [814a] $\left(\frac{d\alpha}{dt}\right) = a$, $\left(\frac{dd\alpha}{dt^2}\right) = b$, $\left(\frac{d\theta}{dt}\right) = h$, $\left(\frac{dd\theta}{dt^2}\right) = l$, $A = E + \pi$, [813x, 767′], π being

To deduce from these equations the values of the unknown quantities x, y, and r, we must consider whether b, independent of its sign, be greater or less than l. In the former case, we must use the equations [814, 815, 817], and [817′]

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two right angles. The quantity x being the distance of the comet from the earth, projected on the plane of the ecliptic, r the distance of the comet from the sun. The remarks above given relative to the equations to be used, are conformable to what was observed immediately following the formulas [806].

We have already observed [806a], that the author modified this calculation, in Book XV, § 5, by changing the manner of computing the quantities $a, b, h, l,$ and connecting together the two equations [815, 816]. In this new method, the quantities $a, b, h, l,$ are computed in the following manner, by combining only three observations, instead of using a greater number, as in the formulas [754—758]. Let the geocentric longitudes of the comet corresponding to these three observations be $\alpha_{,}$, α, α'; the geocentric latitudes $\theta_{,}$, θ, θ', respectively. Then fixing the epoch at the time of the middle observation, and putting i for the interval in days and decimals of a day, between the first and second observations, also i' for the interval between the second and third observations; the general expression of the longitude corresponding to s days, after the epoch, will be of the form $\alpha + s\,a + \frac{1}{2}s^2.b$, [757], and that of the latitude will be $\theta + s.h + \frac{1}{2}s^2.l$. If we now put $s = -i$, they will become $\alpha_{,}$, $\theta_{,}$, respectively; and if $s = i'$, they will become α', and θ', respectively; hence we shall obtain these four equations, in which $a, b, h, l, \alpha, \theta$, &c., are expressed in seconds,

Improved method of computation.

[815a]

$$\alpha - \alpha_{,} = i\,a - \tfrac{1}{2}i^2.b\,; \qquad \theta - \theta_{,} = i\,h - \tfrac{1}{2}i^2.l\,;$$
$$\alpha' - \alpha = i'a + \tfrac{1}{2}i'^2.b; \qquad \theta' - \theta = i'h + \tfrac{1}{2}i'^2.l\,; \qquad [815b]$$

The values of $a, b, h, l,$ being found from these equations, in sexagesimal seconds, we must from the logarithms of $a, h,$ subtract the logarithm 3,5500072, [759‴, 814a], and from the logarithms of $b, l,$ in seconds, subtract the logarithm 1,7855894, and we shall obtain the logarithms of the values of $a, b, h, l,$ to be used in the formulas [815l', m, n]. [815c]

With the same epoch and the same middle observation α, θ, we may use another extreme observation, $\alpha_{,,}$, $\theta_{,,}$, made before the epoch, and another α'', θ'', after the epoch, and by means of the intervals corresponding to these observations we can compute other equations similar to [815b], which may also be used in finding $a, b, h, l,$ so that it is not necessary to confine the calculation to three observations, since the triple combinations of observations may be augmented at pleasure. Any number of these equations may then be connected together, to determine the values of $a, b, h, l,$ in such manner as shall be judged most advantageous. [815d]

The method recommended by the author for the combination of such equations, is derived from the principle of making the sum of the squares of the errors a minimum, which principle will hereafter be more fully explained. In the present case all the equations [815e]

form a first hypothesis for x, by supposing it, for example, to be equal to
unity; and then compute, by means of the equations [814, 815], the values
[817''] of r and y. Substituting these in the equation [817], if it become nothing,

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containing a, b, are to be combined together. *First*, by multiplying each of the equations by
[815f] the coefficient of a, in that equation, and taking the sum of these products for one of the
final equations, to be used in computing a, b. *Second*, by multiplying each of these equations
[815g] by the coefficient of b, always noticing the sign of this coefficient, and taking the sum of the
products for the second final equation. From these two equations are to be computed the
values of a, b. In like manner from the equations in h and l, two final equations are to be
found, for the determination of h and l. It may also be observed that if we denote, as in
[815h] [754v], by i, i', i'', &c., the number of days and parts of a day, which the several observations
follow the epoch, considering these numbers as negative if they precede the epoch, noticing
the signs and putting

[815i] $A = i^2 + i'^2 + i''^2 + \&c.$; $B = i^3 + i'^3 + i''^3 + \&c.$; $C = i^4 + i'^4 + i''^4 + \&c.$;

the terms depending on a, b, in the two final equations, will be $Aa + \frac{1}{2}Bb$, and
[815k] $\frac{1}{2}Ba + \frac{1}{4}Cb$, respectively, and the similar terms in the equations depending on h, l, will
be $Ah + \frac{1}{2}Bl$, and $\frac{1}{2}Bh + \frac{1}{4}Cl$, which may be very expeditiously calculated,
when the numbers are large, by means of Barlow's excellent table of the powers of
numbers.

These final equations become very simple, when every positive term of the series i, i', &c. [815h], is accompanied by a negative one of equal value, because in this case the quantity B, [815i] will vanish, and the terms depending on a, b, in the final equations [815k), will be reduced to Aa and $\frac{1}{4}Cb$. As an example of this method, we shall take the four following equations, in which the series i, i', i'', &c. is represented by -4, -2, 2, 4, respectively, the epoch being taken at the middle time between the extreme observations.

$$0 = 4a - 8b - 23,$$
$$0 = 2a - 2b - 15,$$
[815l] $$0 = 2a + 2b - 23,$$
$$0 = 4a + 8b - 55.$$

Multiplying these equations by the coefficients of a, namely, 4, 2, 2, 4, and adding the products, we get the first final equation $0 = 40a - 388$, hence $a = 9,7$. Again, multiplying the same equations by the coefficients of b, namely, -8, -2, 2, 8, and adding the products, we get the second final equation, $0 = 136.b - 272$, hence $b = 2$. These values of a, b, being substituted in the second members of the equations [815l], they become $-0,2$, 0,4, 0,4, $-0,2$, instead of being nothing. The sum of the squares of these errors is 0,40, and no values of a, b can be found which will make this sum less, as will be seen when we shall explain the method of the least squares.

it will prove that the value of x was rightly assumed; if the result be
negative, we must increase the value of x; but it must be diminished if the [817''']
result be positive.* We shall thus obtain, by a few essays, the values of x,

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Instead of the four equations [814—817], the author finally adopted the three following

$$r^2 = \frac{x^2}{\cos.^2\theta} - 2Rx.\cos.(E-a) + R^2, \qquad [815l']$$

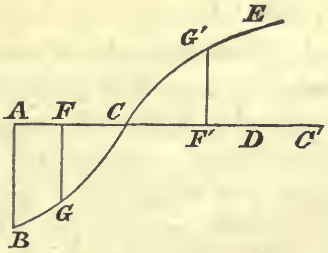
$$y = \left\{ \begin{array}{l} \frac{\{a.\sin.(E-\alpha) - h.\sin.\theta.\cos.\theta.\cos.(E-\alpha)\}}{2.(a^2+h^2)} . R . \left(\frac{1}{r^3} - \frac{1}{R^3}\right) \\ -x.\frac{\{h^3.\mathrm{tang}.\theta + \frac{1}{2}ab + \frac{1}{2}hl + \frac{1}{2}a^2h.\sin.\theta.\cos.\theta\}}{a^2+h^2} \end{array} \right\}, \qquad [815m]$$

Final equation adopted by the author.

$$0 = \frac{y^2}{\cos.^2\theta} + a^2x^2 + \frac{h^2x^2}{\cos.^4\theta} + \frac{2hyx.\mathrm{tang}.\theta}{\cos.^2\theta}$$
$$+2y.\left\{\frac{\sin.(E-\alpha)}{R} - (R'-1).\cos.(E-\alpha)\right\} \qquad [815n]$$
$$-2ax.\left\{(R'-1).\sin.(E-\alpha) + \frac{\cos.(E-\alpha)}{R}\right\} + \frac{1}{R^2} - \frac{2}{r},$$

of which the first is the same as [814]; the second is found by multiplying [815], by $\frac{a^2}{a^2+h^2}$, and [816] by $\frac{h^2}{a^2+h^2}$, and taking the sum of the products; the third is the same as [817], connecting together the terms depending on y^2, and putting in its coefficient $\frac{1}{\cos.^2\theta}$ for $1+\mathrm{tang}.^2\theta$. After substituting in these equations the values of a, b, h, l, we may from them compute the values of x, y, r, and then the other elements in the manner pointed out in [817'—820']. We shall hereafter give a numerical example of this method.

* (567) Put Y equal to the second member of the expression [817], and upon the line AD, taken as the axis of x, erect the ordinates AB, FG, $F'G'$, &c. representing the values of Y, which correspond to $x=0$, $x=AF$, $x=AF'$, &c. respectively; the positive ordinates $F'G'$, &c. being taken above the axis; the negative ones AB, FG, &c. below. Through the extremities of these co-ordinates draw the regular curve $BGCG'E$, and it will intersect the axis at least once, at C, from the nature of the question. This is also evident from the consideration that when $x=0$, we shall have $r=R$, [814], $y=0$, [815], hence Y, [817], will become $\frac{1}{R^2}-\frac{2}{R}$, as R, [790], is nearly equal to unity, this will become $Y=-1$, nearly; so that when $x=0$, Y will be negative. On the contrary, when $x=\infty$, we shall have $r=\infty$, [814], $y=\pm\infty$, [815], hence $Y=\infty$, [817], because the three first terms of [817], depending on the



y and r. But as the unknown quantity may be susceptible of several real and positive values, we must select that which satisfies, either accurately, or very nearly, the equation [816). [817$^{\text{iv}}$]

In the second case, that is when $l > b$, we must use the equations [814, 816, 817], and then the equation [815] will serve for verification. [817$^{\text{v}}$]

Having thus the values of x, y, r, we must compute the quantity [809]*

$$\begin{aligned} P = {} & \frac{x}{\cos.^2\theta} . \{y + hx \,.\, \text{tang.}\,\theta\} - Ry \,.\, \cos.\,(E-\alpha) \\ & + x \,.\, \left\{ \frac{\sin.\,(E-\alpha)}{R} - (R'-1) \,.\, \cos.\,(E-\alpha) \right\} - Rax \,.\, \sin.\,(E-\alpha) \\ & + R \,.\, (R'-1). \end{aligned} \qquad [818]$$

The perihelion distance of the comet D, will be [810]

$$D = r - \tfrac{1}{2} P^2 ; \qquad [819]$$

the cosine of half the true anomaly v will be given by the equation [811]

$$\cos.^2 \tfrac{1}{2} v = \frac{D}{r} ; \qquad [820]$$

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squares of x, y, &c. are infinitely greater than the others, and are all positive; therefore, when $x = \infty$, Y will be positive and infinite. Now, without examining into the nature of this curve, we find that for every value of x, from 0 to ∞, there is a real value of Y, positive or negative, the negative value taking place when $x = 0$, the positive when $x = \infty$, this could not be, unless the curve crossed the axis at some point C, between these extreme values of x.

If the assumed value of x in the first hypothesis [817''] be AF, corresponding to the *negative* ordinate FG, it is evident that by *increasing* the value of x, we shall finally obtain a value AC, in which the ordinate Y is nothing, corresponding to the next following point C, where the curve cuts the axis; but if the value of x, selected, should be AF', corresponding to the *positive* ordinate $F'G'$, by *decreasing* the value of x, we should obtain the next immediately *preceding* point C, where the curve crosses the axis. The same rule would apply with a curve of this kind, which should cross the axis in more than one point; it being evident, from a little consideration, that if the assumed value of x corresponds to a *negative* ordinate Y, we must *increase* x to obtain the *following* point of crossing the axis; but if the assumed value of x corresponds to a *positive* ordinate Y, we must *decrease* x to obtain the *preceding* point of crossing the axis.

* (568) The formulas [818, 819, 820], are the same as [809, 810, 811], substituting the values [813$^{\text{vii}}$].

and we may deduce, from the table of the motion of comets [693″], the
time employed in describing the angle v. To obtain the time of passing the
perihelion, we must add this time to the epoch if P be negative, but subtract [820′]
it if P be positive; because in the first case, the comet approaches the
perihelion [809′], in the second case it recedes from it.

Having thus obtained, nearly, the perihelion distance of the comet, and
the time of its passing the perihelion, we may correct these elements by the [820″]
following method, which has the advantage of being independent of the
knowledge of the approximate values of the other elements of the orbit.

ACCURATE DETERMINATION OF THE ELEMENTS OF THE ORBIT, WHEN WE KNOW NEARLY THE PERIHELION DISTANCE OF THE COMET, AND THE TIME OF PASSING THE PERIHELION.

Select three distant observations of the comet, and by means of the
perihelion distance, and the time of passing the perihelion, obtained by the [820‴]
preceding method compute three anomalies of the comet, and the three
radii vectores corresponding to the times of the three observations. Let v,
v', v'', be these anomalies, those preceding the perihelion being supposed
negative, and r, r', r'', being the corresponding radii vectores; $v'-v$, $v''-v$, $[820^{iv}]$
will be the angles contained between r', r, and r'', r; put U for the first of
these angles, and U' for the second, so that

$$U=v'-v\,; \qquad\qquad U'=v''-v. \qquad [820^{v}]$$

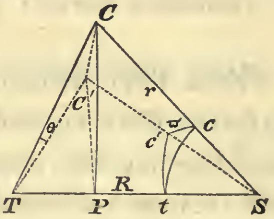
Let α, α', α'', be the three observed geocentric longitudes of the comet, referred
to a *fixed equinox*; θ, θ', θ'', the three geocentric latitudes, the *southern* latitudes
being supposed *negative*; β, β', β'', the three corresponding heliocentric $[820^{vi}]$
longitudes, and ϖ, ϖ', ϖ'', the three heliocentric latitudes; E, E', E'', the
three corresponding longitudes of the sun, and R, R', R'', its distances from $[820^{vii}]$
the centre of the earth.

Suppose the letter S denotes the place of the centre of the sun, T that of
the earth, C the centre of the comet, and C' its projection upon the plane of $[820^{viii}]$
the ecliptic. The angle STC' will be the difference of the geocentric
longitudes of the sun and comet; and by adding the logarithm cosine of this
angle to the logarithm cosine of the geocentric latitude of the comet, we shall

[820ix] have the logarithm cosine of the angle STC;* therefore we shall have, in the triangle STC, the side ST or R; the side SC or r, and the angle STC: we shall then have, by trigonometry, the angle CST.† We may obtain the heliocentric latitude of the comet ϖ, by means of the equation‡

[821]
$$\sin.\varpi = \frac{\sin.\theta \,.\, \sin.\, CST}{\sin.\, CTS}.$$

The angle TSC' is the base of a rectangular spherical triangle, whose hypotenuse is the angle TSC, and side ϖ; and from the two last, we may easily compute the angle TSC', and then find the heliocentric longitude of the comet β.§



* (569) In the annexed figure are marked the places of the sun, earth and comet, as directed above; the lines CP, $C'P$ are drawn perpendicular to TS. Then in the rectangular triangles TPC, TPC', we have for TP the expression $TC \,.\, \cos.\, STC = TC' \,.\, \cos.\, STC'$; and in the rectangular triangle $TC'C$ we have

$$TC' = TC \,.\, \cos.\, CTC' = TC \,.\, \cos.\theta,$$

substituting this in the preceding equation, and dividing by TC, we get

$$\cos.\, STC = \cos.\, STC' \,.\, \cos.\theta,$$

as in [820ix], which is also easily obtained by spherics.

† (570) For $SC : ST :: \sin.\, STC : \sin.\, SCT$; then the angle

$$CST = 180^d - STC - SCT.$$

[820a] It may be observed that the angle SCT being found by its sine, has two values, as is observed in [826′]. This might cause some embarrassment, when the angle SCT is nearly a right angle, and this is to be avoided as in note 575.

‡ (571) In the rectangular plane triangles $TC'C$, $SC'C$, we have $CC' = TC \,.\, \sin.\, CTC' = TC \,.\, \sin.\theta$, and $CC' = SC \,.\, \sin.\, CSC' = SC \,.\, \sin.\varpi$. Hence $TC \,.\, \sin.\theta = SC \,.\, \sin.\varpi$. Now in the triangle STC we have $TC : SC :: \sin.\, CST : \sin.\, CTS$, which being substituted, we get

$$\sin.\, CST \,.\, \sin.\theta = \sin.\, CTS \,.\, \sin.\varpi,$$

hence we obtain $\sin.\varpi$, as in [821].

§ (572) With the centre S, and radius unity, suppose a spherical surface to be described, intersecting the lines ST, SC', SC, in the points t, c', c, respectively, and forming the rectangular spherical triangle $t\,c'\,c$. Then the arches tc', tc, cc', are of the same

In the same manner we shall have ϖ', β', ϖ'', β''; and the values of β, β', β'', will show whether the motion be direct or retrograde. [821']

If we suppose the two arcs of latitude ϖ, ϖ', to be continued to meet in the pole of the ecliptic, they will make there an angle equal to $\beta'-\beta$; and in [821''] the spherical triangle formed by this angle, and the sides $\frac{\pi}{2}-\varpi$, $\frac{\pi}{2}-\varpi'$, π being the semi-circumference of the circle, the side opposite to the angle $\beta'-\beta$ will be the angle formed at the sun, by the two radii vectores r, r'. [821'''] This angle may be found by spherical trigonometry, or by the following formula:*

$$\sin.^2 \tfrac{1}{2} V = \cos.^2 \tfrac{1}{2} (\varpi + \varpi') - \cos.^2 \tfrac{1}{2} (\beta' - \beta) \,.\, \cos.\varpi \,.\, \cos.\varpi' ; \qquad [822]$$

in which V represents this angle. Now if we suppose A to be found from the tables by means of the following equation,

$$\sin.^2 A = \cos.^2 \tfrac{1}{2} (\beta' - \beta) \,.\, \cos.\varpi \,.\, \cos.\varpi' , \qquad [822']$$

we shall have†

$$\sin.^2 \tfrac{1}{2} V = \cos. (\tfrac{1}{2}\varpi + \tfrac{1}{2}\varpi' + A) \,.\, \cos. (\tfrac{1}{2}\varpi + \tfrac{1}{2}\varpi' - A). \qquad [823]$$

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number of degrees as the angles TSC', CST, CSC', respectively, of which the two last are given, and we may obtain the first by the usual rule of spherics

$$\cos. t c' = \frac{\cos. tc}{\cos. cc'}, \quad \text{or} \quad \cos. TSC' = \frac{\cos. CST}{\cos. \varpi}.$$

* (573) If A, B, C, be the sides of a spherical triangle, and c the angle opposite to the side C, we shall have $\cos. C = \cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B \,.\, \cos. c$, [172i]. Putting $c=\beta'-\beta$, $A=\frac{1}{2}\pi-\varpi$, $B=\frac{1}{2}\pi-\varpi'$, $\frac{1}{2}\pi$ being a right angle, and $C=V$, it will become $\cos. V = \sin.\varpi \,.\, \sin.\varpi' + \cos.\varpi \,.\, \cos.\varpi' \,.\, \cos.(\beta'-\beta)$. Now by [1, 6] Int. $\cos. V = 1 - 2 \,.\, \sin.^2 \frac{1}{2} V$, $\cos.(\beta'-\beta) = 2 \,.\, \cos.^2 \frac{1}{2} (\beta'-\beta) - 1$, hence by substitution,

$$\begin{aligned} 1 - 2 \,.\, \sin.^2 \tfrac{1}{2} V &= 2 \,.\, \cos.^2 \tfrac{1}{2} (\beta'-\beta) \,.\, \cos.\varpi \,.\, \cos.\varpi' - \cos.\varpi \,.\, \cos.\varpi' + \sin.\varpi \,.\, \sin.\varpi' \\ &= 2 \,.\, \cos.^2 \tfrac{1}{2} (\beta'-\beta) \,.\, \cos.\varpi \,.\, \cos.\varpi' - \cos.(\varpi+\varpi') \\ &= 2 \,.\, \cos.^2 \tfrac{1}{2} (\beta'-\beta) \,.\, \cos.\varpi \,.\, \cos.\varpi' - 2 \,.\, \cos.^2 \tfrac{1}{2} (\varpi+\varpi') + 1, \end{aligned}$$

as appears by [23, 6] Int.; hence, by rejecting the term 1 from each member, and dividing by -2, we obtain [822].

† (574) From [20] Int. $\cos.(B+A) \,.\, \cos.(B-A) = \frac{1}{2} \cos. 2B + \frac{1}{2} \cos. 2A$, and by [6, 1] Int. we get $\frac{1}{2} \cos. 2B = \cos.^2 B - \frac{1}{2}$, $\frac{1}{2} \cos. 2A = \frac{1}{2} - \sin.^2 A$, hence $\cos.(B+A) \,.\, \cos.(B-A) = \cos.^2 B - \sin.^2 A$. Putting now $B=\frac{1}{2}\varpi+\frac{1}{2}\varpi'$, and $\sin.^2 A$, as in [822'], the second member will become equal to the value of $\sin.^2 \frac{1}{2} V$, [822], and the first member will therefore represent $\sin.^2 \frac{1}{2} V$, as in [823].

If we likewise put V' for the angle comprised between the two radii vectores r and r'', we shall have

[824] $$\sin.^2 \tfrac{1}{2} V' = \cos. (\tfrac{1}{2}\varpi + \tfrac{1}{2}\varpi'' + A') \,.\, \cos. (\tfrac{1}{2}\varpi + \tfrac{1}{2}\varpi'' - A').$$

[824'] A' being what A [823] becomes by changing ϖ' and β' into ϖ'' and β''.

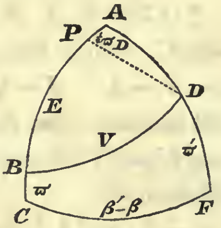
Now if the perihelion distance of the comet, and the time of passing the perihelion, were exactly known, and the observations were rigorously correct, we should have [820$^{\text{v}}$]

[825] $$V = U \,; \qquad V' = U' \,;$$

but as this very rarely happens, we shall suppose

[826] $$m = U - V \,; \qquad m' = U' - V'.$$

We may observe that the calculation of the triangle STC, gives for the angle CST, two different values [820a]; in general the nature of the motion [826'] of the comet will show which ought to be used, especially if the angles are very different;* for then the one will place the comet farther from the earth than the other; and it will be easy to judge, by the apparent motion of the comet at the time of observation, which ought to be selected. But if there is any uncertainty in this respect, we may avoid it, by choosing that value [826''] which renders V and V' nearly equal to U and U' respectively.



This method of finding V has however no advantage over the common method used in spherical trigonometry. To prove this, let B, D, be the geocentric places of the comet at the first and second observations; A the pole of the ecliptic CF; ABC, ADF, circles of latitude. Draw DP perpendicular to AB. Then $BC = \varpi$, $DF = \varpi'$, $AB = \frac{1}{2}\pi - \varpi$, $AD = \frac{1}{2}\pi - \varpi'$, $BD = V$, $BAD = \beta' - \beta$, $CP = D$, $BP = E$, $AP = \frac{1}{2}\pi - D$. By spherics we have

$$\text{tang.}\, AP = \text{tang.}\, AD \,.\, \cos.\, PAD, \qquad \text{or,} \qquad \text{cotang.}\, D = \text{cotang.}\, \varpi' \,.\, \cos. (\beta' - \beta).$$

Noticing the sign of D in the same manner as those of ϖ, ϖ'. Then $BP = CP - BC$, gives $E = D - \varpi$, and by spherics

$$\cos.\, BD = \frac{\cos.\, AD \,.\, \cos.\, BP}{\cos.\, AP}, \qquad \text{or,} \qquad \cos.\, V = \frac{\sin.\, \varpi' \,.\, \cos.\, E}{\sin.\, D}.$$

This requires less labour than the former method, but rather more attention to the signs.

* (575) If however the angle SCT should be very near a right angle, the observation might be changed for another a day or two earlier or later.

We must then form another hypothesis, in which the time of passing the perihelion is to be retained, while the perihelion distance is varied by a small [826‴] quantity, as for example a fiftieth part; and we must compute in this hypothesis the values of $U-V$, and $U'-V'$. Then put [826ⁱᵛ]

$$n=U-V; \qquad n'=U'-V'. \qquad [827]$$

Lastly we must form a third hypothesis, in which the same perihelion distance is used as in the first hypothesis, while the time of passing the perihelion is varied half a day, or a day, more or less. We must find, in this [827′] hypothesis, the values of $U-V$, and $U'-V'$. Then put

$$p=U-V; \qquad p'=U'-V'. \qquad [828]$$

This being supposed, if we put u for the number by which we must multiply the supposed variation in the perihelion distance, to obtain its true value; [828′] and t the number by which we ought to multiply the supposed variation in the time of passing the perihelion, to obtain the true time; we shall have the two following equations:*

$$\begin{aligned}(m-n)\,.\,u+(m-p)\,.\,t&=m\,;\\(m'-n')\,.\,u+(m'-p')\,.\,t&=m'\,;\end{aligned} \qquad [829]$$

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* (576) Suppose the time of passing the perihelion, and the perihelion distance to be respectively, in the first hypothesis, T, D; in the second T, $D+\delta$; in the third $T+\tau$, D; the true values being $T+t\tau$, $D+u\delta$. By the equations [826, 827, 828], the angle $U-V$, which ought to be nothing, was in the first, second and third hypotheses, m, n, and p, respectively; consequently the increment δ in the perihelion distance produced an increase of this angle from m to n, the variation being $n-m$, and if the variation of D, instead of being δ, were $u\delta$, the variation of the angle $U-V$, or m would be nearly $u\,.\,(n-m)$, because these variations, when small, are proportional to the increments. Again, by increasing the time of passing the perihelion by τ, the angle $U-V$, or m is changed into p, increasing by $p-m$, therefore if the time of passing the perihelion were increased by $t\tau$, the angle m would be augmented by $t\,.\,(p-m)$; hence it appears that by increasing the perihelion distance by $u\delta$, and the time of passing the perihelion by $t\tau$, the angle m will be increased by the sum of the two quantities $u\,.\,(n-m)+t\,.\,(p-m)$; consequently when the perihelion distance is $D+u\delta$, and the time of passing the perihelion is $T+t\tau$, the angle $U-V$ will become $m+u\,.\,(n-m)+t\,.\,(p-m)$; and since, by hypothesis, this corresponds to the true orbit, the angle $U-V$ must then be nothing; hence

$$m+u\,.\,(n-m)+t\,.\,(p-m)=0, \qquad \text{or} \qquad (m-n)\,.\,u+(m-p)\,.\,t=m,$$

which is the first of the equations [829]; the second is obtained in exactly the same manner

[829] from which we may find the values of u and t, and thence the corrected perihelion distance; also the true time of the comet's passing the perihelion.

In making the preceding corrections, it is supposed that the elements found by the first approximation are so nearly exact that the errors may be

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from the values of the angle $U'-V'$, and it may also be deduced from the first by accenting m, n, p.

It may be observed that this method of correcting the assumed elements may be generally used, in similar cases, making those alterations which the nature of the case may require. Thus, if instead of the perihelion distance and the time of passing the perihelion, we assume, as Newton has done, in Prop. 42, Lib. 3, Princip. the inclination of the orbit to the ecliptic and the longitude of the node, the resulting equations for correcting these quantities ought to be similar to those in [829]. It is however a fact that in all the editions of the Principia which I have seen, these equations are given inaccurately; and an attempt has been made by Le Seur and Jacquier, in the commentary annexed to their edition, to prove these rules to be correct; and the same has also been done by Emerson in his "Short Commentary on Sir Isaac Newton's Principia, &c." Now if Newton's rules are correct, the equations [829], must be erroneous, because they are both founded on the same principles. I have therefore thought it necessary to enter into some explanation of the true rules which ought to be used in Newton's method, to prevent any embarrassment from the incongruity of the two methods as they now appear. Newton formed, in the same manner as above, three hypotheses. In the first, the inclination of the orbit was put $=I$, and the longitude of the node $=K$; in the second, these quantities were put equal to I and $K+P$; in the third, $I+Q$ and K; the true values being supposed $I+nQ$, and $K+mP$. In each of these three hypotheses, he calculates the ratio of the areas described by the radius vector between the first and second observation, and between the second and third, and denotes them by $\frac{G}{1}$, $\frac{g}{1}$, $\frac{\gamma}{1}$, or simply by G, g, γ, respectively; also the times of describing the areas from the first to the third observation, which are denoted by T, t, τ, respectively. Hence by comparing the results of the first hypothesis with those of the second and third, the increment P, in the longitude of the node, makes G increase by $g-G$, and T increase by $t-T$, therefore the increment mP in the longitude will make these increments $m.(g-G)$ and $m.(t-T)$ respectively; and by comparing the first and third hypotheses, we find that the increment Q in the inclination causes G and T to increase by $\gamma-G$ and $\tau-T$, respectively; hence, by proportion, the increment nQ will cause the increments $n.(\gamma-G)$, $n.(\tau-T)$, in these quantities. These increments applied to G and T, give the true values of these quantities corresponding to the inclination $I+nQ$, and longitude $K+mP$, namely, the proportion of the areas will be $G+m.(g-G)+n.(\gamma-G)$, and the time of description $T+m.(t-T)+n.(\tau-T)$. Now the areas are

Newton's method of correcting the orbit of a Comet.

considered as infinitely small. But if the second approximation do not appear [829′] to be sufficient, we may have recourse to a third, using the corrected elements like those of the first hypothesis, but making the variations less. It is even sufficient to compute, by these corrected elements, the values of $U-V$ and [829‴] $U'-V'$; for by denoting them by M and M', we may substitute them for m and m', in the second members of the two preceding equations [829]; we shall thus have two other equations which will give the values of u and t, corresponding to these last elements.

Having thus obtained the perihelion distance and the time of passing the perihelion, we may thence compute the other elements, in the following manner.

Let j be the longitude of the node which would be ascending if the motion [829^{iv}] of the comet was direct, and φ the inclination of the orbit; we shall have, by comparing the first and last observation,*

$$\text{tang.}\, j = \frac{\text{tang.}\, \varpi \,.\, \sin.\, \beta'' - \text{tang.}\, \varpi'' .\, \sin.\, \beta}{\text{tang.}\, \varpi \,.\, \cos.\, \beta'' - \text{tang.}\, \varpi'' .\, \cos.\, \beta}\,;$$

$$\text{tang.}\, \varphi = \frac{\text{tang.}\, \varpi''}{\sin.\, (\beta''-j)}\,. \qquad [830]$$

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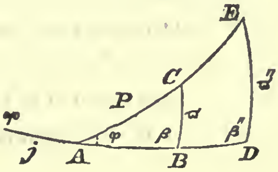
proportional to the times of description, which are known from observation, and by putting the ratio of the time elapsed between the first and second observation, to that between the first and third equal to $C:1$, and the whole observed time from the first to the third observation $=S$, we shall have $C=G+m\,.\,(g-G)+n\,.\,(\gamma-G)$, and $S=T+m\,.\,(t-T)+n\,.\,(\tau-T)$, which by transposition become

$$G-C=m\,.\,(G-g)+n\,.\,(G-\gamma)\,; \qquad T-S=m\,.\,(T-t)+n\,.\,(T-\tau),$$

hence m and n may be found. The equations given by Newton are

$$2\,G-2\,C=m\,.\,(G-g)+n\,.\,(G-\gamma), \qquad 2\,T-2\,S=m\,.\,(T-t)+n\,.\,(T-\tau),$$

which make m and n twice their real values.



* (577) Let ♈ $A\,B\,D$ be the ecliptic, $A\,C\,E$ the orbit of the comet, ♈ the first point of aries, A the ascending node, the motion of the comet being supposed direct, C the place of the comet at the first observation, and E its place at the last observation, $C\,B$, $E\,D$, arcs of latitude. Then ♈$A=j$, ♈$B=\beta$, ♈$D=\beta''$, $B\,C=\varpi$, $D\,E=\varpi''$, the

As we can compare the three preceding equations two by two, it will be most accurate to select those which give, to the formulas [830], the greatest numerators and denominators.

Tang. j may either appertain to the angle j or $\pi + j$, j being the least
[830′] positive angle corresponding to that expression; to determine which of these must be used, we shall observe that φ is positive and less than a right angle;
[830″] therefore $\sin.(\beta'' - j)$ ought to have the same sign as $\text{tang.}\,\varpi''$.* This condition will determine the angle j, which will correspond to the ascending node, if the motion of the comet be direct; but if this motion be retrograde,
[830‴] we must increase the angle j by two right angles, to obtain the position of this node.†

[830a] angle $BAC = DAE = \varphi$, and by spherics $\text{tang.}\,\varphi = \frac{\text{tang.}\,BC}{\sin.\,AB} = \frac{\text{tang.}\,DE}{\sin\,AD}$, hence $\text{tang.}\,BC \,.\, \sin.\,AD = \text{tang.}\,DE \,.\, \sin.\,AB$, or, $\text{tang.}\,\varpi \,.\, \sin.(\beta'' - j) = \text{tang.}\,\varpi'' \,.\, \sin.(\beta - j)$. Putting, for $\sin.(\beta'' - j)$, $\sin.(\beta - j)$, their values, [22] Int., we get

$$\text{tang.}\,\varpi \,.\, \{\sin.\,\beta'' \,.\, \cos.\,j - \cos.\,\beta'' \,.\, \sin.\,j\} = \text{tang.}\,\varpi'' \,.\, \{\sin.\,\beta \,.\, \cos.\,j - \cos.\,\beta \,.\, \sin.\,j\},$$

[830b] dividing by $\cos.\,j$, and putting $\frac{\sin.\,j}{\cos.\,j} = \text{tang.}\,j$, it becomes

$$\text{tang.}\,\varpi \,.\, \{\sin.\,\beta'' - \cos.\,\beta'' \,.\, \text{tang.}\,j\} = \text{tang.}\,\varpi'' \,.\, \{\sin.\,\beta - \cos.\,\beta \,.\, \text{tang.}\,j\},$$

hence we easily obtain tang. j, as in the first of the equations [830]. The second of these equations is the same as the second of the expressions of tang. φ, [830a].

* (578) Having $\text{tang.}\,\varphi = \frac{\text{tang.}\,\varpi''}{\sin.(\beta'' - j)}$, [830], and φ not exceeding a right angle, its tangent must always be positive, consequently, $\text{tang.}\,\varpi''$ and $\sin.\,\beta'' - j$ must always have the same sign. Hence, if ϖ'' be positive $\beta'' - j$ must be less than two right angles, but if ϖ'' be negative $\beta'' - j$ must exceed two right angles, hence the affection of j may be determined.

† (579) If the comet, instead of moving from C towards E, moved from E towards C, the first observation would correspond to β'', ϖ'', the last to β, ϖ, and the expression of tang. j, being found in the same manner as in note 577, from the equation [830b] would be identical with the expression of tang. j, [830]; therefore in both cases tang. j must be equal and of the same sign; but it is evident that when the motion is from E towards C, or retrograde, A must be the descending node, and the angle j must correspond to that node, and to obtain the longitude of the ascending node, we must increase j by two right angles.

The hypotenuse of the spherical triangle, whose sides are $\beta''-j$ and ϖ'', $[830^{\text{iv}}]$ is the distance of the comet from its ascending node at the time of the third observation;* and the difference between v'' and this hypotenuse, is the interval between the node and perihelion, counted on the orbit.

If we wish to obtain the greatest degree of accuracy, in the theory of a comet, we must combine together all the best observations, which may be done in the following manner. Mark the letters m, n, p, with one accent $[830^{\text{v}}]$ for the second observation, two accents for the third observation, &c., all of them being compared with the first observation, we shall have these equations,†

$$\begin{aligned}(m-n)\,.\,u+(m-p)\,.\,t&=m\,;\\(m'-n')\,.\,u+(m'-p')\,.\,t&=m'\,;\\(m''-n'')\,.\,u+(m''-p'')\,.\,t&=m''\,;\\\&\text{c.}&\end{aligned}\qquad[831]$$

Combining these equations in the most advantageous manner to determine u and t, we shall obtain the corrections of the perihelion distance, and the time of passing the perihelion, resulting from the whole of these observations. Hence we may deduce the values of β, β', β'', &c., ϖ, ϖ', ϖ'', &c., and we shall have*

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* (580) In the spherical triangle ADE, of the figure, page 461, we have the base $AD=\beta''-j$, and the perpendicular $DE=\varpi''$, to find by spherics, the hypotenuse AE. Then P being the place of the perihelion, we have $PE=v''$, the difference between this and AE is equal to AP, the distance of the node from the perihelion.

† (581) These equations are exactly similar to those in [829], and require no farther explanation. It may however be observed, that although this method is simple, it is attended with the inconvenience, that any error in the first observation affects *all* the equations; and if the second, third, &c. observations are very near to the first, and the described arcs very [831*a*] small, the resulting equations may be considerably affected by this circumstance; moreover, when the second, third, &c. observations are very near to the first, there appears to be as much propriety in combining them with the subsequent observations, as there is in using only the first observation. This difficulty may be obviated by computing each observation separately, with small changes in the elements of the orbit, in the manner which will be more fully explained in note 591.

[832]
$$\text{tang.}\,j=\frac{\text{tang.}\,\varpi\,.\,\{\text{sin.}\,\beta'+\text{sin.}\,\beta''+\&\text{c.}\}-\text{sin.}\,\beta\,.\,\{\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.}\}}{\text{tang.}\,\varpi\,.\,\{\text{cos.}\,\beta'+\text{cos.}\,\beta''+\&\text{c.}\}-\text{cos.}\,\beta\,.\,\{\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.}\}}\,;$$

$$\text{tang.}\,\varphi=\frac{\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.}}{\text{sin.}\,(\beta'-j)+\text{sin.}\,(\beta''-j)+\&\text{c.}}\,.$$

38. There is a case, which however very rarely occurs, in which the orbit
of a comet may be determined rigorously in a simple manner; this happens
[832′] when the comet has been observed in both nodes. The right line drawn
through these two observed positions then passes through the sun's centre,
and coincides with the line of nodes. The length of this line is ascertained
[832″] by the time elapsed between the two observations; putting T for this time
reduced to decimals of a day, and denoting by c the proposed right line, we

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* (582) The equation [830b], for computing tang. j, using the second observation β', ϖ', instead of the third β'', ϖ'', becomes

$$\text{tang.}\,\varpi\,.\,\{\text{sin.}\,\beta'-\text{cos.}\,\beta'\,.\,\text{tang.}\,j\}=\text{tang.}\,\varpi'\,.\,\{\text{sin.}\,\beta-\text{cos.}\,\beta\,.\,\text{tang.}\,j\},$$

and in a similar way, by using β''', ϖ''', instead of β'', ϖ'', we obtain

$$\text{tang.}\,\varpi\,.\,\{\text{sin.}\,\beta'''-\text{cos.}\,\beta'''\,.\,\text{tang.}\,j\}=\text{tang.}\,\varpi'''\,.\,\{\text{sin.}\,\beta-\text{cos.}\,\beta\,.\,\text{tang.}\,j\},$$

and other observations give similar expressions. By adding all these equations together we obtain the following,

$$\begin{aligned}&\text{tang.}\,\varpi\,.\,\{\text{sin.}\,\beta'+\text{sin.}\,\beta''+\&\text{c.}\}-\text{tang.}\,\varpi\,.\,\text{tang.}\,j\,.\,\{\text{cos.}\,\beta'+\text{cos.}\,\beta''+\&\text{c.}\}\\&=\text{sin.}\,\beta\,.\,\{\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.}\}-\text{cos.}\,\beta\,.\,\text{tang.}\,j\,.\,\{\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.}\},\end{aligned}$$

which gives for tang. j the expression [832]. Again, the second equation [830], becomes by using the second observation, β', ϖ', instead of the third β'', ϖ'',

$$\text{tang.}\,\varpi'=\text{tang.}\,\varphi\,.\,\text{sin.}\,(\beta'-j),$$

and in a similar manner,

$$\text{tang.}\,\varpi''=\text{tang.}\,\varphi\,.\,\text{tang.}\,(\beta''-j)\,;\qquad \text{tang.}\,\varpi'''=\text{tang.}\,\varphi\,.\,\text{tang.}\,(\beta'''-j),\ \&\text{c.}$$

These equations being added together, which makes each of them enter into the determination of φ, we get,

$$(\text{tang.}\,\varpi'+\text{tang.}\,\varpi''+\&\text{c.})=\text{tang.}\,\varphi\,.\,\{\text{sin.}\,(\beta'-j)+\text{sin.}\,(\beta''-j)+\&\text{c.}\},$$

hence the second of the equations [832] is easily obtained. This method of combining the equations to find tang. j and tang. φ, is somewhat arbitrary, since the first observation is connected with all the others, an arrangement which may sometimes not be conducive to the attainment of the most accurate result.

shall have by § 27,*

$$c = \tfrac{1}{2} \cdot \left(\frac{T^2}{(9^d688724)^2} \right)^{\frac{1}{3}}. \qquad [833]$$

Now let β be the heliocentric longitude of the comet at the time of the first observation, r its radius vector, ρ its distance from the earth, and α its geocentric longitude; also R the radius vector of the earth, and E the corresponding longitude of the sun at the same instant, we shall have† [833′]

$$r \,.\, \sin.\, \beta = \rho \,.\, \sin.\, \alpha - R \,.\, \sin.\, E\,;$$
$$r \,.\, \cos.\, \beta = \rho \,.\, \cos.\, \alpha - R \,.\, \cos.\, E. \qquad [834]$$

$\pi + \beta$ will be the heliocentric longitude of the comet at the second observation; and if we accent the quantities r, α, ρ, R, E, corresponding to this time, we shall have [834′]

$$r' \,.\, \sin.\, \beta = R' \,.\, \sin.\, E' - \rho' \,.\, \sin.\, \alpha'\,;$$
$$r' \,.\, \cos.\, \beta = R' \,.\, \cos.\, E' - \rho' \,.\, \cos.\, \alpha'. \qquad [835]$$

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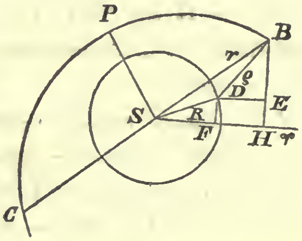
[*] (583) In this case $r + r'$, [750] is evidently equal to c, because r, r', fall on the line of nodes, and that equation, [750], becomes $\mathrm{T} = \frac{T}{12\pi} \cdot (2\,c)^{\frac{3}{2}}$; and by [750′] we have $\frac{T}{12\pi} = 9^d{,}688724$, substituting this we get c [833].



† (584) These values are easily found by means of the adjoined figure, in which S is the centre of the sun, B the place of the comet situated in the ecliptic, and D that of the earth at the first observation, C the place of the comet, at the second observation, SH the line drawn from the sun through the first point of aries, from which the angles α, β, &c. are counted; DF, BEH are perpendicular to SH, and DE parallel to SH. Then $HSB = \beta$, $SB = r$, $SD = R$, $BD = \rho$, the angle $HSD =$ longitude of the earth $= \pi + E$, and the angle EDB equal to the geocentric longitude of the comet α. Then in the triangles BSH, BDE, DSF, we have

$$BH = SB \,.\, \sin.\, HSB = r \,.\, \sin.\, \beta\,; \qquad SH = SB \,.\, \cos.\, HSB = r \,.\, \cos.\, \beta\,;$$
$$BE = BD \,.\, \sin.\, EDB = \rho \,.\, \sin.\, \alpha\,; \qquad DE = BD \,.\, \cos.\, EDB = \rho \,.\, \cos.\, \alpha\,;$$
$$DF = SD \,.\, \sin.\, HSD = R \,.\, \sin.\, (\pi + E) = -R \,.\, \sin.\, E\,;$$
$$SF = SD \,.\, \cos.\, HSD = R \,.\, \cos.\, (\pi + E) = -R \,.\, \cos.\, E.$$

Substituting these in $BH = BE + DF$, $SH = DE + SF$, they will become as in [834]. When the comet is at C, its heliocentric longitude is evidently $\pi + \beta$; substituting this and changing ρ, r, &c. into ρ', r', &c. in [834], they will become, by changing the signs of all the terms, the same as [835].

These four equations give*

[836] $$\text{tang.}\,\beta=\frac{\rho\,.\,\text{sin.}\,\alpha-R\,.\,\text{sin.}\,E}{\rho\,.\,\text{cos.}\,\alpha-R\,.\,\text{cos.}\,E}=\frac{\rho'\,.\,\text{sin.}\,\alpha'-R'\,.\,\text{sin.}\,E'}{\rho'\,.\,\text{cos.}\,\alpha'-R'\,.\,\text{cos.}\,E'};$$

hence we deduce

[837] $$\rho'=\frac{RR'\,.\,\text{sin.}\,(E-E')-R'\rho\,.\,\text{sin.}\,(\alpha-E')}{\rho\,.\,\text{sin.}\,(\alpha'-\alpha)-R\,.\,\text{sin.}\,(\alpha'-E)}.$$

We then have†

[838] $$\begin{aligned}(r+r')\,.\,\text{sin.}\,\beta&=\rho\,.\,\text{sin.}\,\alpha-\rho'\,.\,\text{sin.}\,\alpha'-R\,.\,\text{sin.}\,E+R'\,.\,\text{sin.}\,E'\,;\\(r+r')\,.\,\text{cos.}\,\beta&=\rho\,.\,\text{cos.}\,\alpha-\rho'\,.\,\text{cos.}\,\alpha'-R\,.\,\text{cos.}\,E+R'\,.\,\text{cos.}\,E'.\end{aligned}$$

By adding together the squares of these equations, and substituting c for $r+r'$, we shall have‡

[839] $$\begin{aligned}c^2=R^2&-2RR'\,.\,\text{cos.}\,(E-E')+R'^2\\&+2\rho\,.\{R'\,.\,\text{cos.}\,(\alpha-E')-R\,.\,\text{cos.}\,(\alpha-E)\}\\&+2\rho'\,.\{R\,.\,\text{cos.}\,(\alpha'-E)-R'\,.\,\text{cos.}\,(\alpha'-E')\}\\&+\rho^2-2\rho\rho'\,.\,\text{cos.}\,(\alpha'-\alpha)+\rho'^2.\end{aligned}$$

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* (585) These two values of tang. β are found by dividing the first equations by the second in [834, 835], respectively. Putting these two expressions of tang. β equal to each other, and multiplying by the denominators, we get

$$\{\rho\,.\,\text{sin.}\,\alpha-R\,.\,\text{sin.}\,E\}\,.\{\rho'\,.\,\text{cos.}\,\alpha'-R'\,.\,\text{cos.}\,E'\}=\{\rho'\,.\,\text{sin.}\,\alpha'-R'\,.\,\text{sin.}\,E'\}\,.\{\rho\,.\,\text{cos.}\,\alpha-R\,.\,\text{cos.}\,E\};$$

performing the multiplications, and connecting together the coefficients of $\rho\rho'$, $R\rho'$, RR', $R'\rho$, it becomes

$$\begin{aligned}&RR'\,.\,(\text{sin.}\,E\,.\,\text{cos.}\,E'-\text{sin.}\,E'\,.\,\text{cos.}\,E)-R'\rho\,.\,(\text{sin.}\,\alpha\,.\,\text{cos.}\,E'-\text{cos.}\,\alpha\,.\,\text{sin.}\,E'\}\\&=\rho\rho'\,.\,(\text{sin.}\,\alpha'\,.\,\text{cos.}\,\alpha-\text{cos.}\,\alpha'\,.\,\text{sin.}\,\alpha)-R\rho'\,.\,(\text{sin.}\,\alpha'\,.\,\text{cos.}\,E-\text{cos.}\,\alpha'\,.\,\text{sin.}\,E),\end{aligned}$$

which being reduced, by [22] Int. changes into

$$RR'\,.\,\text{sin.}\,(E-E')-R'\rho\,.\,\text{sin.}\,(\alpha-E')=\rho\rho'\,.\,\text{sin.}\,(\alpha'-\alpha)-R\rho'\,.\,\text{sin.}\,(\alpha'-E).$$

Dividing this by $\rho\,.\,\text{sin.}\,(\alpha'-\alpha)-R\,.\,\text{sin.}\,(\alpha'-E)$, we obtain ρ' [837].

† (586) The first of the equations [834, 835] being added together, we get the first of [838]; and the second of the equations [834, 835] being added together, we get the second of [838].

‡ (586a) Putting c for $r+r'$ in [838] and taking the sum of the squares of both equations, the first member will become $c^2\,.\,(\text{sin.}^2\,\beta+\text{cos.}^2\,\beta)=c^2$, as in [839]. In squaring the second member of [838], there will be two species of terms, the one composed of the squares of ρ, $-\rho'$, $-R$, R', and the other of the double of the products of these quantities two by two. The former will, in the square of the first equation, produce terms of

Substituting in this equation the value of ρ' in terms of ρ given in [837], the result will be an equation in ρ of the fourth degree, which may be solved by the usual methods; but it will be easier to assume for ρ any value at pleasure, and to compute the corresponding value of ρ' [837]; then we must substitute these values of ρ, ρ', in the preceding equation [839], and see if they satisfy it. By a few trials, in this manner, we may determine ρ and ρ' with accuracy. [839′]

By means of these quantities we shall obtain β, r, r'. Put D for the perihelion distance, v for the angle included by the line D and the radius vector r; $\pi - v$ will be the angle formed by the lines D and r'; we shall then have, by § 23,* [839″]

$$r = \frac{D}{\cos.^2 \tfrac{1}{2} v}\,; \qquad r' = \frac{D}{\sin.^2 \tfrac{1}{2} v}\,; \qquad [840]$$

which give

$$\text{tang.}^2 \tfrac{1}{2} v = \frac{r}{r'}\,; \qquad D = \frac{r\,r'}{r + r'}\,. \qquad [841]$$

We shall therefore have the anomaly of the comet v, at the time of the first observation, and the perihelion distance D; thence it is easy to deduce the

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the form $(\rho \,.\, \sin.\, \alpha)^2$, which will be accompanied, in the square of the second equation, by a term of the form $(\rho \,.\, \cos.\, \alpha)^2$, the sum of these two terms will be ρ^2, and the other similar terms will produce ρ'^2, R^2, R'^2, [839]. In the second species of terms the double product depending on ρ, $-\rho'$, will produce in the sum of the squares of the two equations [838], the quantity $-2\,\rho\,\rho' \,.\, (\cos.\, \alpha' \,.\, \cos.\, \alpha + \sin.\, \alpha' \,.\, \sin.\, \alpha)$, which, by [24] Int., is $= -2\,\rho\,\rho' \,.\, \cos.\,(\alpha' - \alpha)$, as in [839]. In like manner we obtain the terms depending on $-2\rho R$, $2\rho R'$, $2\rho' R$, $-2\rho' R'$, $-2RR'$.

* (587) Let SP, in the figure page 465, be the perihelion distance, we shall have $BSP = v$, hence $CSP = \pi - v$. The first equation [691] gives $r = \frac{D}{\cos.^2 \frac{1}{2} v}$, and $r' = \frac{D}{\cos.^2 \frac{1}{2} . (\pi - v)} = \frac{D}{\sin.^2 \frac{1}{2} v}$, as in [840]. Dividing the value of r by that of r', we get $\frac{r}{r'} = \frac{\sin.^2 \frac{1}{2} v}{\cos.^2 \frac{1}{2} v} = \text{tang.}^2 \tfrac{1}{2} v$, [841]. The sum and product of r, r', are

$$r + r' = D \,.\, \frac{\sin.^2 \frac{1}{2} v + \cos.^2 \frac{1}{2} v}{\sin.^2 \frac{1}{2} v \,.\, \cos.^2 \frac{1}{2} v} = \frac{D}{\sin^2 \frac{1}{2} v \,.\, \cos.^2 \frac{1}{2} v}\,, \quad \text{and} \quad r\,r' = \frac{D^2}{\sin.^2 \frac{1}{2} v \,.\, \cos.^2 \frac{1}{2} v}\,,$$

dividing the latter by the former we get D, [841].

[841′] position of the perihelion, and the time of the comet's passing that point. Thus, of the five elements of the orbit, four will be known; namely, the perihelion distance, the position of the perihelion, the time of passing the perihelion, and the position of the node; the only element which remains to be investigated is the inclination of the orbit; and for this purpose it will [841″] be necessary to recur to a third observation, which will also serve to determine which of the real and positive roots of the equation in ρ is to be used.

39. The hypothesis of the parabolic motion of comets is not perfectly correct; the probability of it is even extremely small, considering the infinite [841‴] number of cases producing an elliptical or hyperbolic motion, in comparison with those producing a parabolic. Besides, a comet moving either in a parabola or hyperbola, would be visible but once; hence we may suppose, with great probability, that the comets describing these curves, if there be any, have disappeared a long time since; so that those we now observe, are such as move in returning or oval curves, which, at greater or less intervals of time, come back to the regions of space near the sun. We may, by [841^{iv}] the following method, determine within a few years the duration of the revolution, when we have a great number of very accurate observations, before and after passing the perihelion.

For this purpose, suppose we have four or a greater number of good observations, including all that part of the orbit in which the comet was visible, and that we have found by the preceding method the parabola [841^{v}] which nearly satisfies these observations. Let v, v', v'', v''', &c., be the corresponding anomalies; and r, r', r'', r''', &c., the radii vectores. Put also

$$v'-v=U\,; \qquad v''-v=U'\,; \qquad v'''-v=U''\,; \qquad \&c.\,; \tag{842}$$

this being supposed, we must calculate by the preceding method, with the parabola already found, the values of U, U', U'', &c., V, V', V'', &c.; then put

$$m=U-V\,; \quad m'=U'-V'\,; \quad m''=U''-V''\,; \quad m'''=U'''-V'''\,; \quad \&c. \tag{843}$$

We must then vary by a very small quantity, the perihelion distance in the parabola; suppose in this hypothesis

$$n=U-V\,; \quad n'=U'-V'\,; \quad n''=U''-V''\,; \quad n'''=U'''-V'''\,; \quad \&c. \tag{844}$$

We must then form a third hypothesis, in which we must preserve the same perihelion distance as in the first, and vary the time of passing the perihelion by a very small quantity; then putting

$$p=U-V; \quad p'=U'-V'; \quad p''=U''-V''; \quad p'''=U'''-V'''; \quad \&c. \qquad [845]$$

Lastly, with the perihelion distance and the time of passing the perihelion of the first hypothesis, we must compute the angle v and the radius vector r, supposing the orbit to be elliptical, and the difference $1-e$ between its excentricity and unity to be a very small quantity, for example $\frac{1}{50}$. To obtain [845′]
the angle v, in this hypothesis, it is sufficient, by § 23 [697], to add to the anomaly v, computed in the parabola of the first hypothesis, a small angle, whose sine is

$$\tfrac{1}{10}\,.\,(1-e)\,.\,\text{tang.}\,\tfrac{1}{2}v\,.\{4-3\,.\,\cos.^2\tfrac{1}{2}v-6\,.\,\cos.^4\tfrac{1}{2}v\}. \qquad [846]$$

Substituting, in the equation*

$$r=\frac{D}{\cos.^2\frac{1}{2}v}\,.\left\{1-\frac{(1-e)}{2}\,.\,\text{tang.}^2\tfrac{1}{2}v\right\}, \qquad [847]$$

for v, the anomaly calculated in this ellipsis, we shall have the radius vector r corresponding. We must compute in the same manner, v', r', v'', r'', v''', r''', &c.; hence we may deduce the values of U, U', U'', &c.; and by § 37, [847′]
those of V, V', V'', &c. Suppose in this case

$$q=U-V; \quad q'=U'-V'; \quad q''=U''-V''; \quad q'''=U'''-V'''; \quad \&c. \qquad [848]$$

Lastly, let u be the number by which we ought to multiply the supposed variation in the perihelion distance, to obtain its true value; t the number by which we ought to multiply the supposed variation in the time of passing the [848′]
perihelion to obtain the true time; and s the number by which we ought to multiply the supposed value of $1-e$, to have its true quantity; we shall form the equations†

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* (589) This is the same as the equation [683], neglecting α^2, which reduces it to

$$r=\frac{D}{\cos.^2\frac{1}{2}v}\,.\left\{1-\tfrac{1}{2}\alpha\,.\,\text{tang.}^2\tfrac{1}{2}v\right\},$$

and substituting $1-e$ for α, [681′].

† (590) The values of [843, 844, 845], are precisely like those in [826, 827, 828]. Those of q, q', &c. [848] depend on the same principles, and it is evident that the equations [849] are found like those in [829].

[849]

$$
\begin{aligned}
&(m-n)\,.\,u+(m-p)\,.\,t+(m-q)\,.\,s=m\,;\\
&(m'-n')\,.\,u+(m'-p')\,.\,t+(m'-q')\,.\,s=m'\,;\\
&(m''-n'')\,.\,u+(m''-p'')\,.\,t+(m''-q'')\,.\,s=m''\,;\\
&(m'''-n''')\,.\,u+(m'''-p''')\,.\,t+(m'''-q''')\,.\,s=m'''\,;\\
&\&c.
\end{aligned}
$$

By means of these equations we may determine the values of u, t, s; hence we may deduce the perihelion distance, the true time of passing the perihelion, and the correct value of $1-e$. Let D be the perihelion distance, a the
[849′] semi-transverse axis of the orbit; we shall have $a=\frac{D}{1-e}$ [681″]; the time of a sidereal revolution of the comet will be expressed by a number of
[849″] sidereal years equal to* $a^{\frac{3}{2}}$, or $\left(\frac{D}{1-e}\right)^{\frac{3}{2}}$, the mean distance of the sun

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* (591) From [705] we have $T^2:T'^2::a^3:a'^3$, and by putting $T'=1$ year, $a'=$ the unity of distance, or the mean distance of the earth from the sun, we find $T^2=a^3$, or $T=a^{\frac{3}{2}}$.

The system of equations [849] is liable to the same objections that were made to the system [831] in note 581. To obviate this the following method may be used.

Let the approximate elements of the orbit be the perihelion distance D, the time of passing the perihelion T, the longitude of the perihelion counted upon the orbit of the comet P, the longitude of the ascending node of the orbit N, the inclination of the orbit of the ecliptic I, the excentricity expressed in parts of the mean distance of the comet from the sun E, this last element being omitted when the comet is supposed to move in a parabolic orbit. With these elements, we must, for a *first* operation, calculate the geocentric longitude and latitude of the comet at the time of any observation. The same calculation must be repeated in six successive operations, varying *one* of the elements at each operation, by some small quantity, while the others remain unaltered. In the *second* operation, the distance D must be changed into $D+d$, d being a very small part of D; T must be changed into
[849a] $T+t$ in the *third* operation, t being a fraction of a day; P into $P+p$, in a *fourth*; N into $N+n$, in a *fifth*; I into $I+i$, in the *sixth*; p, n, i being small arcs or parts of a degree. Lastly, if the ellipticity of the orbit be taken into consideration, we must for a *seventh* operation, change E into $E+e$, e denoting a very small increment of the
[849b] excentricity E. Then representing the longitudes, or latitudes, computed in these successive
[849c] operations, by L', L'', L''', L'''', L^{v}, L^{vi}, L^{vii}, and the corresponding observed longitude or latitude by L, and supposing the true elements to be $D+d\,\delta$, $T+t\,\tau$, $P+p\,\varpi$, $N+n\,\nu$, $I+i\,\iota$, $E+e\,\varepsilon$, each observed longitude or latitude will furnish an equation

from the earth being taken for unity. We shall then have, by § 37 [832], the inclination of the orbit φ, and the position of the node j.

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of this form, which was computed upon the same principles as those in [829, 849], explained in note 576.

$$0 = (L - L') + (L' - L'') \, . \, \delta + (L' - L''') \, . \, \tau + (L' - L'''') \, . \, \pi + (L' - L^{\mathrm{v}}) \, . \, \nu + (L' - L^{\mathrm{vi}}) \, . \, \iota + (L' - L^{\mathrm{vii}}) \, . \, \varepsilon, \qquad [849d]$$

so that n observations of the comet will produce $2\,n$ equations, each of which will be independent of the others. Gauss, in his invaluable work, *Theoria Motus Corporum Coelestium*, has given many differential formulas, by means of which the variations of the geocentric longitudes and latitudes of the comet, corresponding to small variations in the elements of the orbit, may be computed without the trouble of repeating the whole calculation of the longitude and latitude, at every operation; and, by this means, the equations of the form [849*d*], may be found with much less labour than by a direct operation. Bessel, in his excellent work on the comet of 1807, entitled *Untersuchungen über die scheinbare und wahre Bahn des im Jahre* 1807 *erschienenen grossen Kometen*, gives several of the formulas of Gauss, with additional ones of his own, for the purpose of abridging such calculations. Both these works deserve the careful perusal of any one who wishes for full information on this subject. If there are only six of the equations of the form [849*d*], they will be just sufficient to obtain the unknown quantities δ, τ, &c., and thence the corrections of the elements. If the observations of the comet were accurate, the orbit a perfect ellipsis, and the variations of the elements infinitely small, all these equations, however great the number might be, would be satisfied, by using these corrected elements; but the imperfections of the observations, and the finite nature of these variations, with other causes, generally prevent this from taking place; and the second member of any one of the equations, instead of [849*e*] vanishing, becomes in general equal to a small quantity c', which may be considered as the correction, or error of the particular observation, from which the equation was derived; so that by putting for brevity $L - L' = A'$, $L' - L'' = B'$, $L' - L''' = C'$, $L' - L'''' = D'$, $L' - L^{\mathrm{v}} = E'$, $L' - L^{\mathrm{vi}} = F'$, $L' - L^{\mathrm{vii}} = G'$, the preceding equation will become, [849*f*]

$$c' = A' + B'\,\delta + C'\,\tau + D'\,\pi + E'\,\nu + F'\,\iota + G'\,\varepsilon. \qquad [849g]$$

If we have more than six of these equations, we must combine them together, so as to make the sum of the squares of the errors c' a minimum. Before using this method it will often be conducive to the accuracy of the result, to examine carefully the observations, and if any of them are considered to be more imperfect than the rest, as might frequently be the case with the observations made just before the time of the disappearance of the comet, when it is very faint; such observations may be made to have less influence, on the final result of the calculation, by multiplying the equation, computed as above, by some fraction, less than unity, [849*h*] as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., or by rejecting it wholly, if it shall be found to differ very much from the rest. Moreover, we ought to multiply the equations derived from the observed longitude by

However great the accuracy of the observations may be, they will always leave a degree of uncertainty on the time of revolution of a

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the cosine of the corresponding latitude of the comet, in order to reduce the difference of longitude $L-L'$, to the parallel of latitude of the star, so that it may correspond to the actual arc, described by the comet in the heavens. This will appear, by referring to the figure in page 216, supposing $E\,Q\,q$ to represent the ecliptic, P its pole, a the observed place of the comet, and B its computed place, at the first operation; then we shall have $L'-L$ equal to the arch $Q\,q$, and the arch $a\,b$ or $A\,B$, corresponding to the actual change of place of the comet, in its parallel of latitude, will evidently be nearly equal to

$$Q\,q\,.\,\cos.\,Q\,A=(L'-L)\,.\,\cos.\,\text{lat.}$$

In this way, by accenting the letters A', B', C', &c. for the successive equations, we shall obtain $2\,n$ equations of the following form,

[849i]
$$\begin{array}{l}
c'=A'+B'\,\delta+C'\,\tau+D'\,\varpi+E'\,\nu+F'\,\iota+G'\,\varepsilon, \\
c''=A''+B''\,\delta+C''\,\tau+D''\,\varpi+E''\,\nu+F''\,\iota+G''\,\varepsilon, \\
c'''=A'''+B'''\,\delta+C'''\,\tau+D'''\,\varpi+E'''\,\nu+F'''\,\iota+G'''\,\varepsilon, \\
\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
c^{(2n)}=A^{(2n)}+B^{(2n)}\,\delta+C^{(2n)}\,\tau+D^{(2n)}\,\varpi+E^{(2n)}\,\nu+F^{(2n)}\,\iota+G^{(2n)}\,\varepsilon,
\end{array}$$

which must be combined together so as to make $c'^2+c''^2+c'''^2\ldots+c^{(2n)2}$ a minimum in the following manner. *First*, Multiply each of the equations [849i] by the coefficient of δ in that equation, and take the sum of all these products for the first final equation. In [849k] other words, the first equation is to be multiplied by B', the second by B'', the third by B''', &c., always noticing the signs of these terms, so that if $B'=-3$, the factor of the first equation must be -3. *Second*, in like manner multiply the same equations [849i] by C', C'', &c., the coefficients of τ, corresponding to each equation; and take the sum of these products, for the second final equation. *Third*, Multiply the equations [849i] by D', D'', &c., and take the sum of the products for a third equation. Proceeding in the same manner, with the coefficients of ν, ι, ε, we shall obtain three other equations, making in all six final [849l] equations, from which the values of δ, τ, ϖ, ν, ι, ε may be computed, by the usual methods. In the parabolic orbit ε must be neglected, and the number of final equations will be reduced to five.

Principle of the least squares.

It is very easy to prove, that the method just given corresponds to the minimum value of $c'^2+c''^2+$&c. For if we denote all the terms of the second members of the equations [849i], independent of δ, by M', M'', &c. we shall have,

[849m]
$$c'=B'\,\delta+M', \qquad c''=B''\,\delta+M'', \qquad c'''=B'''\,\delta+M''', \qquad \text{\&c.}$$

hence

[849n]
$$c'^2+c''^2+\text{\&c.}=(B'\,\delta+M')^2+(B''\,\delta+M'')^2+\text{\&c.}$$

The minimum of this quantity, supposing δ to be variable, is found in the usual manner, by putting its differential relative to δ, equal to nothing. This differential being divided by $2\,d\,\delta$

comet. The most exact method of determining this time, is by comparing the observations in two successive revolutions ; but this method is not [849′′′]

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becomes

$$B'.(B'\delta+M')+B''.(B''\delta+M'')+\&c.=0.$$ [849o]

which is exactly the same as in the rule, given in [849k], for finding the first final equation. The demonstration, relative to the other unknown quantities, is made in exactly the same manner, and it holds good, whatever may be the number of these quantities.

In case the number of observations is very great, this method would be too laborious, if the calculation were made separately for each observation. This difficulty is avoided by dividing the observations into five or six groups, comprising the observations of several [849p] successive days, and using only the middle day of each group, correcting the observed longitude on that day for the mean error of all the observations of the group, to which it corresponds, as they were computed from the original elements, in the terms A', A'', &c. of the equations [849i]. Bessel, in his work on the comet of 1807, combined 70 observations in six different sets, each furnishing one equation for the longitude, and one for the latitude, [849q] and the twelve equations thus obtained were reduced to six, by the method of the least squares [849i—l]. The variations of the elements used by him, in his last calculations, after he had obtained the elements to a great degree of accuracy, were $d=0{,}0001$, $t=0^{day}{,}005$, $p=10^{s}$, $n=10^{s}$, $i=10^{s}$, $e=0{,}0001$. In the calculation of the elements of the orbit of the comet of 1811, in the third volume of the memoirs of the American Academy of Arts and Sciences, I used $d=0{,}004$, $t=0^{day}{,}05$, $p=10^{m}$, $n=-10^{m}$, $i=10^{m}$.

In strictness, the observed longitudes and latitudes of the comet should be corrected, for the perturbations caused by the attraction of the planets, before insertion in the equations [849i]. If the comet should pass near to any one of the larger planets, it would be absolutely necessary to notice this circumstance, and it is always conducive to accuracy to do it. La Place has given a method for this purpose, in Book IX, § 1—13. The same [849r] subject is also treated of by Bessel, in the above mentioned work, where he has given many useful formulas, with their application to the comet of 1807. In vol. xxiv of the Memoirs of the Royal Academy of Sciences of Turin, is an elaborate article, by Baron Damoiseau, on Halley's comet of 1759, in which he computes the disturbing forces of the planets from 1759 to 1835, and fixes the time of passing the perihelion on November 16, 1835. Any one who wishes to know, in detail, the methods of making such calculations, would do well to refer to these works of Bessel and Damoiseau.

It has been observed, both by Gauss and Le Gendre, that the rule for taking the mean of any number n of observations, follows as a simple result from this general method of the least squares. For if a', a'', a''', &c. represent several observed values of an unknown quantity x, practicable, until the comet in the course of time, shall return back again towards its perihelion.

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the sum of the squares of the errors will be $(x-a')^2+(x-a'')^2+(x-a''')^2+\&c.$ Its minimum is found by taking the differential relative to x and putting it equal to nothing. This differential being divided by $2\,dx$, becomes

[849s] $$(x-a')+(x-a'')+(x-a''')+\&c.=0,$$

hence $nx=a'+a''+a'''+\&c.$ and $x=\frac{a'+a''+a'''+\&c.}{n}$. In like manner, if the rectangular co-ordinates of a point in space be x, y, z, and by one observation they be a', b', c'; by another a'', b'', c'', &c.; then the square of the distance of the point x, y, z, from the point a', b', c', will be $(x-a')^2+(y-b')^2+(z-c')^2$, [12]. This represents the square of the error of the first observation; and by a similar calculation, that of the second is $(x-a'')^2+(y-b'')^2+(z-c'')^2+\&c.$ The sum of all these, using the

[849t] symbol Σ of finite integrals, as in page 9, is $\Sigma.(x-a')^2+\Sigma.(y-b')^2+\Sigma.(z-c')^2$, which is to be a minimum; therefore its differential, taken successively, relative to x, y, z, is to be put equal to nothing. Its differential relative to x is $2\,dx.\Sigma.(x-a')=0$, hence $\Sigma.(x-a')=0$, and $\Sigma x=\Sigma a'$, and if the number of points be n, Σx will be the same as nx, hence $nx=\Sigma a'$, and the differentials relative to y and z, will give similar

[849u] expressions, hence we get $x=\frac{\Sigma a'}{n}$, $y=\frac{\Sigma b'}{n}$, $z=\frac{\Sigma c'}{n}$; these formulas are the same as those in [154], for finding the common centre of gravity of n equal masses, situated at given points, each mass being represented by $m=1$; therefore we find that the common centre of gravity of any body has this general property, pointed out by Le Gendre. *If we divide the mass of a body into very small equal particles, considered as points, the sum of the squares of the distances of the particles from the centre of gravity, will be a minimum.* [849v]

Property of the centre of gravity.

CHAPTER V.

GENERAL METHODS FOR FINDING THE MOTIONS OF THE HEAVENLY BODIES, BY SUCCESSIVE APPROXIMATIONS.

40. In the first approximation of the motions of the heavenly bodies, we have only considered the principal forces which act on them, and have thence deduced the laws of the elliptical motion. In the following researches we shall notice the forces which disturb this motion. The effect of these forces is to add some small terms to the differential equations of the elliptical [849iv] motion, of which we have already given the finite integrals: we must now determine, by successive approximations, the integrals of the same equations, increased by the terms arising from the effect of these disturbing forces. The following is a general method for obtaining such integrals by successive approximations, whatever be the number and the degree of the differential equations proposed to be integrated.

Suppose we have, between n variable quantities y, y', y'', &c., and the [849v] variable quantity t, whose element dt is considered as constant, n differential equations

$$0=\frac{d^i y}{dt^i}+P+\alpha . Q ;$$

$$0=\frac{d^i y'}{dt^i}+P'+\alpha . Q' ; \qquad [850]$$

&c.

P, Q, P', Q', &c., being functions of t, y, y', &c., and of their differentials as far as the order $i-1$ inclusively, and α being a very small constant [850'] coefficient, which, in the theory of the heavenly motions, is of the order of the disturbing forces. Suppose also that we have the finite integrals of these equations, when Q, Q', &c., are nothing. Then taking their differentials

$i-1$ times in succession, they will form, with their differentials, $i\,n$ equations, by means of which we may find, by elimination, the arbitrary quantities c, c', c'', &c., in functions of t, y, y', y'', &c., and of their differentials as far as the [850''] order $i-1$. Denoting therefore these functions by V, V', V'', &c., we shall have

[851] $$c=V; \qquad c'=V'; \qquad c''=V''; \qquad \text{\&c.}$$

These equations are the $i\,n$ integrals of the order $i-1$, which the differential equations ought to have, and which, by the elimination of the differentials of the variable quantities, give their finite integrals.

If we take the differentials of the preceding equations [851] of the order $i-1$, we shall have

[852] $$0=d\,V; \qquad 0=d\,V'; \qquad 0=d\,V''; \qquad \text{\&c.}$$

Now it is evident, that these equations being differentials of the order i, without arbitrary constant quantities, they must be the sums of the following equations:

[853] $$0=\frac{d^i\,y}{d\,t^i}+P; \qquad 0=\frac{d^i\,y'}{d\,t^i}+P'; \qquad \text{\&c.};$$

multiplied respectively by such factors as will render the sums exact [853'] differentials;* putting therefore $F\,d\,t$, $F'\,d\,t$, &c., for the factors to be used in forming the equation $0=dV$; also $H\,dt$, $H'\,dt$, &c., for the factors to be used in forming $0=dV'$, and in like manner for the rest; we [853″] shall have

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* (592) By hypothesis [850''], V, V', &c., contain no differentials of a higher order than $i-1$. The differentials of these quantities cannot, therefore, contain any differentials of a higher order than i, and the terms of this order must appear of the first degree, or under [852a] a *linear form*. For if V contain a term of the form $A\,.\,(d^{i-1}\,y)^m$, its differential $d\,V$ would contain the term $A\,m\,.\,(d^{i-1}\,y)^{m-1}\,.\,d^i\,y$, which is of the first degree, as it respects $d^i\,y$. Again, the integration of the equations [853] has produced the $i\,n$ equations of the form [851], containing the $i\,n$ constant quantities c, c', c'', &c. If we now take the differential of these last equations, we shall get the $i\,n$ equations [852], which are of the same order i as the equations [853], from which they were derived; but they will contain none of the constant quantities c, c', &c., which were introduced by the integrations. The equations [852] must therefore be deducible from the equations [853], by the common rules of elimination in algebra. But we have just shown [852a], that $d\,V$, $d\,V'$, &c., in [852], contain $d^i\,y$, $d\,y'$, &c. only under a *linear* form, as they are in [853]. Therefore, if we multiply the equations [853] by factors F, F', &c., H, H', &c., of an order not exceeding $i-1$, and add them, as directed above, the sums will be of the same forms as

$$dV=F.dt.\left\{\frac{d^iy}{dt^i}+P\right\}+F'.dt.\left\{\frac{d^iy'}{dt^i}+P'\right\}+\&c.;$$
$$dV'=H.dt.\left\{\frac{d^iy}{dt^i}+P\right\}+H'.dt.\left\{\frac{d^iy'}{dt^i}+P'\right\}+\&c.;\qquad[854]$$

&c.

F, F', &c., H, H', &c., are functions of t, y, y', y'', &c., and their differentials [854′] as far as the order $i-1$: it is easy to determine them, when V, V', &c., are known; for F is evidently the coefficient of $\frac{d^iy}{dt^i}$, in the differential of V; F' is the coefficient of $\frac{d^iy'}{dt^i}$, in the same differential, and so on for the [854″] others. Likewise H, H', &c., are the coefficients of $\frac{d^iy}{dt^i}$, $\frac{d^iy'}{dt^i}$, &c., in [854‴] the differential of V'; and since the functions V, V', &c., are supposed to be known, if we take their differentials relative to $\frac{d^{i-1}y}{dt^{i-1}}$, $\frac{d^{i-1}y'}{dt^{i-1}}$, &c., only, [854$^{\text{iv}}$] we shall have the factors by which we ought to multiply the differential equations,

$$0=\frac{d^iy}{dt^i}+P;\qquad 0=\frac{d^iy'}{dt^i}+P';\qquad \&c.;\qquad[855]$$

to obtain exact differentials. This being premised,

We shall now resume the differential equations [850],

$$0=\frac{d^iy}{dt^i}+P+\alpha.Q;\qquad 0=\frac{d^iy'}{dt^i}+P'+\alpha.Q';\qquad \&c.\qquad[856]$$

If we multiply the first by Fdt, the second by $F'dt$, and so on for the rest; we shall have, by adding these products,*

$$0=dV+\alpha dt.\{FQ+F'Q'+\&c.\};\qquad[857]$$

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the equations [852], and they may be rendered identical by using the appropriate values of F, F', &c., requisite to make these sums exact differentials of the form $dV=0$, $dV'=0$, &c. [852].

* (593) Substituting also in the sums, dV, dV', &c., instead of the equivalent expressions, given in the second members of the equations [854].

in like manner we shall have

[858] $$0 = d\,V' + \alpha\, d\,t.\{H\,Q + H'\,Q' + \&c.\};$$
$$\&c.;$$

hence by integration

[859] $$c - \alpha.\int d\,t.\{F\,Q + F'\,Q' + \&c.\} = V;$$
$$c - \alpha.\int d\,t.\{H Q + H' Q' + \&c.\} = V';$$
$$\&c.;$$

thus we shall have, $i\,n$ differential equations, which will be of the same form as in the case where Q, Q', &c. are nothing, [851], with this difference only, that the arbitrary quantities c, c', c'', &c., ought to be changed into

[860] $$c - \alpha.\int dt.\{FQ + F'\,Q' + \&c.\}; \qquad c' - \alpha.\int dt.\{HQ + H'\,Q' + \&c.\}; \qquad \&c.$$

Now if in the hypothesis of Q, Q', &c., being equal to nothing, we eliminate from the $i\,n$ integrals [851] of the order $i-1$, the differentials of the variable quantities y, y', y'', &c., we shall have the n finite integrals of the proposed equations; we shall therefore have these same integrals, when Q, Q', &c., do not vanish, by changing in the first integrals, c, c', &c., into

[860'] $$c - \alpha.\int dt.\{FQ + F'\,Q' + \&c.\}; \qquad c' - \alpha.\int dt.\{HQ + H'\,Q' + \&c.\}; \qquad \&c.$$

41. If the differentials

[861] $$d\,t.\{FQ + F'\,Q' + \&c.\}, \qquad d\,t.\{H\,Q + H'\,Q' + \&c.\}, \qquad \&c.,$$

be exact, we shall have, by the preceding method, the finite integrals of the proposed differential equations [856]; but this takes place only in some particular cases, of which the most extensive and interesting is that in which the equations are linear. Let us therefore suppose P, P', &c., to be linear functions of y, y', &c., and of their differentials as far as the order $i-1$ [861'] inclusively, without any term independent of those variable quantities; and we shall, in the first place, consider the case in which Q, Q', &c., are nothing. The differential equations being linear, their successive integrals will also be [861''] linear, so that $c = V$, $c' = V'$, &c., being the in integrals of the order $i-1$, of the linear differential equations,

[862] $$0 = \frac{d^i y}{d t^i} + P; \qquad 0 = \frac{d^i y'}{d t^i} + P'; \qquad \&c.;$$

V, V', &c., may be supposed linear functions of y, y', &c., and of their differentials as far as the order $i-1$. To prove this, suppose in the expressions of y, y', &c., the arbitrary constant quantity c to be equal to a determinate quantity, augmented by an indeterminate constant quantity δc; the arbitrary constant quantity c' to be equal to a determinate quantity, increased by the indeterminate $\delta c'$, &c. Reducing these expressions into series, arranged according to the powers and products of δc, $\delta c'$, &c., we shall have, by the formulas of § 21,*

$$\begin{aligned} y &= Y + \delta c \,.\left(\frac{dY}{dc}\right) + \delta c' \,.\left(\frac{dY}{dc'}\right) + \&c. \\ &\quad + \frac{\delta c^2}{1.2}\,.\left(\frac{d^2 Y}{dc^2}\right) + \&c.; \\ y' &= Y' + \delta c \,.\left(\frac{dY'}{dc}\right) + \delta c' \,.\left(\frac{dY'}{dc'}\right) + \&c.; \\ &\quad + \frac{\delta c^2}{1.2}\,.\left(\frac{d^2 Y'}{dc^2}\right) + \&c.; \\ &\&c.; \end{aligned} \qquad [863]$$

Y, Y', $\left(\frac{dY}{dc}\right)$, &c., being functions of t without arbitrary constant quantities. Substituting these values in the proposed differential equations, it is evident that δc, $\delta c'$, &c., being indeterminate, the coefficients of the first powers of each of them ought to be nothing in all these equations; now these equations being linear, we shall evidently have the terms affected by the first powers of δc, $\delta c'$, &c., by substituting $\left(\frac{dY}{dc}\right).\delta c + \left(\frac{dY}{dc'}\right).\delta c' + \&c.$ [863']
for y; $\left(\frac{dY'}{dc}\right).\delta c + \left(\frac{dY'}{dc'}\right).\delta c' + \&c.$ for y';† &c. These expressions

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* (595) The formulas here referred to are [607—612]. The general expression of $q_{n,n',n''\&c.}$, [612], corresponds to the coefficient of $\delta c^{n}\,.\,\delta c'^{n'}\,.\,\delta c''^{n''}$ &c. [863], putting δc, $\delta c'$, $\delta c''$, &c. for α, α', α'', &c., respectively.

† (596) Since δc, $\delta c'$, &c. are arbitrary, we might put $\delta c = 0$, $\delta c' = 0$, in [863], and the resulting values of y, y', &c., namely, $y = Y$, $y' = Y'$, &c., would satisfy the [863a] proposed equations [862]; the same equations being likewise satisfied, by substituting the whole values of y, y', &c. [863]. If we now suppose for a moment, that δc, $\delta c'$, &c. are infinitely small, we may neglect the powers and products of δc, $\delta c'$, &c., in [863], and put simply,

of y, y', &c., therefore satisfy separately the proposed differential equations; and as they contain the $i\,n$ arbitrary quantities δc, $\delta c'$, &c., they are the complete integrals of them. We thus see that the arbitrary constant quantities exist in a linear form in the expressions of y', y', &c., consequently [863″] also in their differentials; hence it is easy to conclude that the variable quantities y, y', &c., and their differentials, may be supposed to exist under a linear form, in the successive integrals of the proposed differentials.

$$y=Y+\delta c.\left(\frac{dY}{dc}\right)+\delta c'.\left(\frac{dY}{dc'}\right)+\&c. \qquad y'=Y'+\delta c.\left(\frac{dY'}{dc}\right)+\delta c'.\left(\frac{dY'}{dc'}\right)+\&c.,$$

and as the proposed equations [862] are linear, and are satisfied by putting $y=Y$, $y'=Y'$, &c., [863*a*], they must, from the nature of linear equations, be also satisfied, by putting for y, y', &c. the differences of their two preceding values respectively, that is by putting

[863*b*] $$y=\left(\frac{dY}{dc}\right).\delta c+\left(\frac{dY}{dc'}\right).\delta c'+\&c.; \qquad y'=\left(\frac{dY'}{dc}\right).\delta c+\left(\frac{dY'}{dc'}\right).\delta c'+\&c.$$

Again, as the proposed equations [862], are linear, in y, y', &c., containing no constant term [861″], independent of y, y', &c., the last values of y, y', &c. [863*b*], will also satisfy the equations, if they are all multiplied by a very great constant quantity C, making

$$y=C.\left\{\delta c.\left(\frac{dY}{dc}\right)+\delta c'.\left(\frac{dY}{dc'}\right)+\&c.\right\}; \quad y'=C.\left\{\delta c.\left(\frac{dY'}{dc}\right)+\delta c'.\left(\frac{dY'}{dc'}\right)+\&c.\right\}; \&c.$$

Hence it is evident, that for the quantities $C\,\delta c$, $C\,\delta c'$, &c., we may put arbitrary finite quantities e, e', e'', &c.; making

[863*c*] $$y=e.\left(\frac{dY}{dc}\right)+e'.\left(\frac{dY}{dc'}\right)+\&c.; \qquad y'=e.\left(\frac{dY'}{dc}\right)+e'.\left(\frac{dY'}{dc'}\right)+\&c.;$$

in which the arbitrary constant quantities e, e', &c., are under a linear form, as in [863′], where δc, $\delta c'$, &c., are used for e, e', &c.

Now having n quantities y, y', y'', &c., expressed in these functions of t, [863*c*], and the $i\,n$ indeterminate constant quantities e, e', e'', &c. If we take successively, the differentials of these expressions, as far as the order $i-1$, we shall obtain $i\,n$ equations, which will be linear in e, e', e'', &c.; and they will also be linear in y, $d\,y$, $d^2\,y$, &c., y', $d\,y'$, &c.; and [863*d*] by eliminating all the arbitrary constant quantities except e, we shall find $e=$ linear function of y, y', &c., and their differentials as far as the order $i-1$. In like manner by eliminating all these constant quantities except e', we shall find e' equal to a similar function of y, y', &c., and their differentials. In this manner we shall have $i\,n$ equations, which will give the values of e, e', e'', &c., in linear functions of y, y', &c., and their differentials, as far as the order $i-1$. These correspond to $c=V$, $c'=V'$, &c., [861″].

Hence it follows, that F, F', &c., being the coefficients of $\frac{d^i y}{dt^i}$, $\frac{d^i y'}{dt^i}$, &c., in the differential of V; H, H', &c., being the coefficients of the same differentials in the differential of V', and so on for others; these quantities will be functions of the single variable quantity t.* Therefore, if we suppose [863‴] Q, Q', &c., to be functions of t only, the differentials

$$dt.\{FQ+F'Q'+\&c.\}, \qquad dt.\{HQ+H'Q'+\&c.\}, \qquad \&c., \qquad [863^{iv}]$$

will be exact.

From this we obtain a simple *method of finding the integrals of any number n of linear differential equations of the order i, containing any terms* αQ, $\alpha Q'$, &c., *functions of the single variable quantity t, when we know how to integrate the* [863ᵛ] *same equations in the case where these terms vanish*; for in this case, if we take the differential of these n finite integrals $i-1$ times in succession, we shall have in equations, which will give, by elimination, the values of the in arbitrary quantities c, c', &c., in functions of t, y, y', &c., and of the differentials of these variable quantities as far as the order $i-1$. We may thus form the in equations [861″], $c=V$, $c'=V'$, &c.; this being supposed, F, F', &c., will be the coefficients of $\frac{d^{i-1} y}{dt^{i-1}}$, $\frac{d^{i-1} y'}{dt^{i-1}}$, &c., in V, [854″]; [863ᵛⁱ] H, H', &c., will be the coefficients of the same differentials in V' [854″], and so on for the others; we shall therefore obtain the finite integrals of the linear differential equations

$$0=\frac{d^i y}{dt^i}+P+\alpha Q; \qquad 0=\frac{d^i y'}{dt^i}+P'+\alpha Q'; \qquad \&c.; \qquad [864]$$

by changing in the finite integrals of these equations, deprived of their last terms, αQ, $\alpha Q'$, &c., the arbitrary quantities c, c', &c., into [860]

$$\begin{array}{l} c-\alpha.\int dt.\{FQ+F'Q'+\&c.\}; \\ c'-\alpha.\int dt.\{HQ+H'Q'+\&c.\}; \qquad \&c. \end{array} \qquad [864']$$

We shall now, for an example, consider the equation

$$0=\frac{d^2 y}{dt^2}+a^2 y+\alpha.Q. \qquad [865]$$

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* (598) They cannot contain y, y', &c., or their differentials, because of the *linear* form of the equations [861″].

The finite integral of the equation*

[865′] $$0 = \frac{d^2 y}{d t^2} + a^2 y,$$

is

[866] $$y = \frac{c}{a} \,.\, \sin.\, a t + \frac{c'}{a} \,.\, \cos.\, a t\,;$$

c and c' being arbitrary constant quantities. Taking the differential of this equation, we have

[867] $$\frac{d y}{d t} = c \,.\, \cos.\, a t - c' \,.\, \sin.\, a t.$$

If we combine this differential with the integral itself, we shall obtain the two following equations of the first order,†

[868] $$c = a y \,.\, \sin.\, a t + \frac{d y}{d t} \,.\, \cos.\, a t\,;$$
$$c' = a y \,.\, \cos.\, a t - \frac{d y}{d t} \,.\, \sin.\, a t\,;$$

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* (600) This equation is very much used throughout this work. Its integral [866] is easily proved to be correct; for by taking the first differential of [866] it becomes as in [867], and the differential of this divided by $-dt$ is $-\frac{d d y}{d t^2} = c a \,.\, \sin.\, a t + c' a \,.\, \cos.\, a t$, in which the second member is equal to the value of $a^2 y$ deduced from [866]; hence $-\frac{d^2 y}{d t^2} = a^2 y$, and $0 = \frac{d^2 y}{d t^2} + a^2 y$. Its integral $y = \frac{c}{a} \,.\, \sin.\, a t + \frac{c'}{a} \,.\, \cos.\, a t$, may be put under either of the following forms,

[864a] $$y = b \,.\, \sin.\, (a t + \varphi),$$
$$y = b \,.\, \cos.\, (a t + \varphi),$$

b and φ being arbitrary constant quantities. For by using $\sin.\, (a t + \varphi)$, [21] Int., it changes into $y = b \,.\, \cos.\, \varphi \,.\, \sin.\, a t + b \,.\, \sin.\, \varphi \,.\, \cos.\, a t$, which becomes identical with [866], by putting $\frac{c}{a} = b \,.\, \cos.\, \varphi$, $\frac{c'}{a} = b \,.\, \sin.\, \varphi$. The second form of y, [864a], by developing $\cos.\, (a t + \varphi)$, as in [23] Int., becomes

$$y = b \,.\, \cos.\, \varphi \,.\, \cos.\, a t - b \,.\, \sin.\, \varphi \,.\, \sin.\, a t,$$

which also changes into [866] by putting $\frac{c}{a} = -b \,.\, \sin.\, \varphi$, $\frac{c'}{a} = b \,.\, \cos.\, \varphi$.

† (601) Multiply [866] by $a \,.\, \sin.\, a t$, and [867] by $\cos.\, a t$; the sum of these products gives c [868]. Again, multiply [866] by $a \,.\, \cos.\, a t$, and [867] by $-\sin.\, a t$, the sum of these products gives c' [868].

hence we shall have, in this case,*

$$F = \cos. a t\,; \qquad\qquad H = -\sin. a t\,;$$ [869]

and the complete integral of the proposed equation, will be

$$y = \frac{c}{a} . \sin. at + \frac{c'}{a} . \cos. at - \frac{\alpha . \sin. a t}{a} . \int Q dt . \cos. at + \frac{\alpha . \cos. a t}{a} . \int Q\, d\, t . \sin. a\, t.$$ [870]

Hence it is easy to conclude, that if Q is composed of terms of the form

$$K . \overset{\sin.}{\underset{\cos.}{}} (m\, t + \varepsilon),$$ [870′]

each of its terms will produce, in the value of y, the corresponding term†

$$\frac{\alpha K}{m^2 - a^2} . \overset{\sin.}{\underset{\cos.}{}} (m\, t + \varepsilon).$$ [871]

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* (602) Comparing [865] with [853] we find $i = 2$, and by the rules [863^{vi}], F and H are the coefficients of $\frac{d y}{d t}$ in the expressions $c = V$, $c' = V'$, [851], which correspond to [868], hence $F = \cos. a\, t$, $H = -\sin. a\, t$, as in [869]; and the expressions [864′] become $c - \alpha . \int Q\, d\, t . \cos. a\, t$, $c' + \alpha . \int Q\, d\, t . \sin. a\, t$, These being substituted for c, c', in [866] give the complete integral of [865],

$$y = \frac{\sin. a\, t}{a} . \left\{ c - \alpha . \int Q\, d\, t . \cos. a\, t \right\} + \frac{\cos. a\, t}{a} . \left\{ c' + \alpha . \int Q\, d\, t . \sin. a\, t \right\},$$

as in [870]. Moreover, the two first terms $\frac{c}{a} . \sin. a\, t + \frac{c'}{a} . \cos. a\, t$, may be put under the form $b . \sin. (a\, t + \varphi)$, or $b . \cos. (a\, t + \varphi)$, as was observed in [864a]; therefore the complete integral of

$$\frac{d^2 y}{d t^2} + a^2 y + \alpha\, Q = 0,$$ [865a]

is

$$y = b . \overset{\sin.}{\underset{\cos.}{}} (a\, t + \varphi) - \frac{\alpha . \sin. a\, t}{a} . \int Q\, d\, t . \cos. a\, t + \frac{\alpha . \cos. a\, t}{a} . \int Q\, d\, t . \sin. a\, t.$$ [865b]

† (603) If $Q = K . \sin. (m\, t + \varepsilon)$, the expression $-\frac{\alpha . \sin. a\, t}{a} . \int Q\, d\, t . \cos. a\, t$ becomes $-\frac{K \alpha . \sin. a\, t}{a} . \int d\, t . \cos. a\, t . \sin. (m\, t + \varepsilon)$, which by [18] Int.

$$= -\frac{K \alpha . \sin. a\, t}{2 a} . \int d\, t . \{\sin. [(m + a) . t + \varepsilon] + \sin. [(m - a) . t + \varepsilon]\}$$

$$= -\frac{K \alpha . \sin. a\, t}{2 a} . \left\{ \frac{-\cos. [(m + a) . t + \varepsilon]}{m + a} - \frac{\cos. [(m - a) . t + \varepsilon]}{m - a} \right\},$$

If m is equal to a, the term

[871'] $$K\,.\,{}^{\text{sin.}}_{\text{cos.}}\,(m\,t+\varepsilon),$$

will produce in y,* *First*, the term $-\frac{\alpha K}{4\,a^2}\,.\,{}^{\text{sin.}}_{\text{cos.}}\,(a\,t+\varepsilon)$, which, being

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no constant quantity being added to the integral, because it already contains two, c and c'. Now by [19] Int.

$$2\,.\,\text{sin.}\,a\,t\,.\,\text{cos.}\,[(m+a)\,.\,t+\varepsilon]=\text{sin.}\,[(m+2\,a)\,.\,t+\varepsilon]-\text{sin.}\,(m\,t+\varepsilon),$$

and $2\,.\,\text{sin.}\,a\,t\,.\,\text{cos.}\,[(m-a)\,.\,t+\varepsilon]=\text{sin.}\,(m\,t+\varepsilon)-\text{sin.}\,[(m-2\,a)\,.\,t+\varepsilon]$,

substituting these in the preceding expression it becomes

[871a] $$-\frac{K\alpha\,.\,\text{sin.}\,a\,t}{a}\,.\,\textstyle\int d\,t\,.\,\text{cos.}\,a\,t\,.\,\text{sin.}\,(mt+\varepsilon)=\frac{K\,\alpha}{4\,a\,.\,(m+a)}\,.\left\{\text{sin.}[(m+2a)\,.\,t+\varepsilon]-\text{sin.}\,(m\,t+\varepsilon)\right\}$$
$$+\frac{K\alpha}{4\,a\,.(m-a)}\,.\left\{\text{sin.}\,(m\,t+\varepsilon)-\text{sin.}\,[(m-2\,a)\,.\,t+\varepsilon]\right\},$$

and by writing $\frac{1}{2}\pi+a\,t$ for $a\,t$, $\frac{1}{2}\pi$ being a right angle, we get

[871b] $$\frac{K\alpha\,.\,\text{cos.}\,a\,t}{a}\,.\,\textstyle\int d\,t\,.\,\text{sin.}\,a\,t\,.\,\text{sin.}\,(m\,t+\varepsilon)=\frac{K\alpha}{4\,a\,.\,(m+a)}\,.\left\{-\text{sin.}[(m+2a)\,.\,t+\varepsilon]-\text{sin.}(mt+\varepsilon)\right\}$$
$$+\frac{K\alpha}{4\,a\,.(m-a)}\,.\left\{\text{sin.}\,(m\,t+\varepsilon)+\text{sin.}\,[(m-2\,a)\,.\,t+\varepsilon]\right\}.$$

The sum of the expressions [871a, b], gives the value of

$$\frac{-\alpha\,.\,\text{sin.}\,a\,t}{a}\,.\,\textstyle\int Q\,d\,t\,.\,\text{cos.}\,a\,t+\frac{\alpha\,.\,\text{cos.}\,a\,t}{a}\,.\,\textstyle\int Q\,d\,t\,.\,\text{sin.}\,a\,t,$$

arising from $K\,.\,\text{sin.}\,(m\,t+\varepsilon)$, namely,

$$\frac{-2\,\alpha\,K}{4\,a\,.(m+a)}\,.\,\text{sin.}\,(m\,t+\varepsilon)+\frac{2\,\alpha\,K}{4\,a\,.(m-a)}\,.\,\text{sin.}\,(m\,t+\varepsilon),$$

and by reduction it becomes $\frac{\alpha\,K}{m^2-a^2}\,.\,\text{sin.}\,(m\,t+\varepsilon)$, which is like the first form [871]. If in this we write $m\,t+\frac{1}{2}\pi$ for $m\,t$, the term of Q, [870'], $K\,.\,\text{sin.}\,(m\,t+\varepsilon)$ will become $K\,.\,\text{cos.}\,(m\,t+\varepsilon)$, and the preceding result $\frac{\alpha K}{m^2-a^2}\,.\,\text{sin.}\,(mt+\varepsilon)$ will become $\frac{\alpha K}{m^2-a^2}\,.\,\text{cos.}\,(m\,t+\varepsilon)$, which is the second form [871].

* (604) When $Q=K\,.\,\text{sin.}\,(a\,t+\varepsilon)$, the term

$$-\frac{\alpha\,.\,\text{sin.}\,a\,t}{a}\,.\,\textstyle\int Q\,d\,t\,.\,\text{cos.}\,a\,t=-\frac{\alpha\,K\,.\,\text{sin.}\,a\,t}{2\,a}\,.\,\textstyle\int d\,t\,.\,\{\text{sin.}\,(2\,a\,t+\varepsilon)+\text{sin.}\,\varepsilon\}$$
$$=-\frac{\alpha\,K\,.\,\text{sin.}\,a\,t}{2\,a}\,.\left\{\frac{-\text{cos.}\,(2\,a\,t+\varepsilon)}{2\,a}+t\,.\,\text{sin.}\,\varepsilon\right\},$$

comprised in the two terms $\frac{c}{a}\,.\,\sin.\,a\,t+\frac{c'}{a}\,.\,\cos.\,a\,t$, may be neglected; *Second*, the term

$$\pm\frac{\alpha K t}{2a}\,.\,\begin{matrix}\cos.\\ \sin.\end{matrix}(a\,t+\varepsilon)\,; \qquad [871'']$$

the sign + taking place, if the term of the expression of Q be a sine, and the sign — if it be a cosine. Thus we see how the arch t is produced out of the signs of *sine* and *cosine*, in the values of y, y', &c., by successive integrations, although the differential equations do not contain them under this form. It is evident that this will occur whenever the functions $F\,Q$, $F'\,Q'$, &c., $H\,Q$, $H'\,Q'$, &c., contain constant terms. [871''']

42. If the differentials $d\,t\,.\{F\,Q+\text{\&c.}\}$, $d\,t\,.\{H\,Q+\text{\&c.}\}$, &c., be not exact, the preceding analysis will not give the rigorous integrals; but it furnishes a simple method of obtaining the integrals by approximation, when α is very small, if we have the values of y, y', &c., in the case of α being

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and

$$\frac{\alpha.\cos.a\,t}{a}\,.\int Q\,d\,t.\sin.a\,t=\frac{K\alpha.\cos.a\,t}{a}\,.\int d\,t\,.\,\{\sin.\,a\,t\,.\,\sin.\,(a\,t+\varepsilon)\}$$

$$=\frac{K\alpha.\cos.a\,t}{2a}\,.\int d\,t\,.\,\{\cos.\,\varepsilon-\cos.\,(2\,a\,t+\varepsilon)\}=\frac{K\alpha.\cos.a\,t}{2a}\,.\left\{t\,.\cos.\,\varepsilon-\frac{\sin.\,(2a\,t+\varepsilon)}{2a}\right\}.$$

Hence we have

$$-\frac{\alpha.\sin.a\,t}{a}\,.\int Q\,d\,t\,.\cos.\,a\,t+\frac{\alpha.\cos.a\,t}{a}\,.\int Q\,d\,t\,.\sin.\,a\,t$$

$$=-\frac{\alpha K}{4a^2}\,.\left\{-\sin.\,a\,t\,.\cos.\,(2\,a\,t+\varepsilon)+\cos.\,a\,t\,.\sin.\,(2\,a\,t+\varepsilon)\right\}$$

$$+\frac{\alpha K t}{2a}\,.\left\{-\sin.\,a\,t\,.\sin.\,\varepsilon+\cos.\,a\,t\,.\cos.\,\varepsilon\right\};$$

and by [22] Int. the coefficient of $\frac{-\alpha K}{4\,a^2}$ is $=\sin.\,[(2\,a\,t+\varepsilon)-a\,t]=\sin.\,(a\,t+\varepsilon)$, and the other coefficient $-\sin.\,a\,t\,.\sin.\,\varepsilon+\cos.\,a\,t\,.\cos.\,\varepsilon=\cos.\,(a\,t+\varepsilon)$, hence the preceding expression becomes $-\frac{\alpha K}{4a^2}\,.\sin.\,(a\,t+\varepsilon)+\frac{\alpha K t}{2a}\,.\cos.\,(a\,t+\varepsilon)$. These are the terms produced by $Q=K\,.\sin.\,(m\,t+\varepsilon)$, and by writing $\frac{1}{2}\pi+\varepsilon$ for ε we obtain those arising from $Q=K\,.\cos.\,(m\,t+\varepsilon)$, namely,

$$-\frac{\alpha K}{4a^2}\,.\cos.\,(a\,t+\varepsilon)-\frac{\alpha K t}{2a}\,.\cos.\,(a\,t+\varepsilon),$$

as in [871', 871''].

nothing. Taking the differentials of these values $i-1$ times successively, we shall form the differential equations of the order $i-1$,*

[872] $$c=V; \qquad c'=V'; \qquad \&c.$$

The coefficients of $\frac{d^i y}{dt^i}$, $\frac{d^i y'}{dt^i}$, in the differentials of V, V', &c., being the values of F, F', &c., H, H', &c., we must substitute them in the differential functions [864′]

[873] $$dt.(FQ+F'Q'+\&c.); \qquad dt.(HQ+H'Q'+\&c.); \qquad \&c.$$

Then, in these functions, we must substitute, for y, y', &c., their first approximate values; which will render these differentials functions of t, and [873′] of the arbitrary quantities c, c', &c. Let $T\,dt$, $T'\,dt$, &c., be these functions. If in the first approximate values of y, y', &c., we change the [873″] arbitrary quantities c, c', &c., into $c-\alpha.\int T\,dt$, $c'-\alpha.\int T'\,dt$, &c., we shall have the second approximate values of those quantities.†

We must then substitute these second values, in the differential functions [873],

[874] $$dt.(FQ+\&c.); \qquad dt.(HQ+\&c.); \qquad \&c.$$

Now it is evident, that these functions are then what $T\,dt$, $T'\,dt$, &c., [874′] become, by changing the arbitrary quantities c, c', &c., into $c-\alpha.\int T\,dt$, $c'-\alpha.\int T'\,dt$, &c. Therefore let $T_{,}$, $T'_{,}$, &c., be what T, T', &c., become by these changes, we shall have the third approximate values of y, y', &c., [874″] by changing in the first values, c, c', &c., into $c-\alpha.\int T_{,}dt$, $c'-\alpha.\int T'_{,}dt$, &c., respectively.

[874‴] In like manner, put $T_{,,}$, $T'_{,,}$, &c., for the values of T, T', &c., when c, c', &c., are changed into $c-\alpha.\int T_{,}dt$, $c'-\alpha.\int T'_{,}dt$, &c., we shall have the fourth approximate value of y, y', &c., by changing, in the first approximate values of these quantities c, c', &c., into $c-\alpha.\int T_{,,}dt$, [874iv] $c'-\alpha.\int T'_{,,}dt$, &c., and so on for farther approximations.

We shall hereafter see that the determination of the motions of the heavenly bodies depends almost always on differential equations of the form

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* (605) These equations are formed in the manner explained in [850″, 851].

† (606) This method evidently follows from what is said immediately after the equation [859] or [864].

$$0=\frac{d^2 y}{d t^2}+a^2 . y+\alpha . Q, \qquad [875]$$

Q being a rational and integral function of y, and of sines and cosines of angles, increasing in proportion to the time represented by t. The following is the most easy method of finding the integral of this equation.

We must first suppose α nothing, and we shall have by the preceding article [875′]
[866] the first value of y.

Substitute this value in Q, which will thus become a rational and integral
function of sines and cosines of angles proportional to t. Then finding the [875″]
integral [870] of the differential equation, we shall have a second value of y,
exact in terms of the order α inclusively.

This last value being substituted in Q, and the integral [870] of the
differential equation being found again, will give the third approximate value [875‴]
of y, and so on for others.

This manner of finding, by approximation, the integrals of the differential
equations of the motions of the heavenly bodies, although the most simple of
any, has however the inconvenience of producing, in the values of y', y', &c., [875^{iv}]
arcs of a circle without the signs of *sine* and *cosine*, even in those cases
where these arcs do not exist in the correct values of those integrals. For it
is easy to perceive, that if these values contain sines or cosines of angles of
the order αt, these sines or cosines must be expressed in the form of series,
in the approximate values, found by the preceding method, since these
quantities are arranged according to the powers of α.* This development of

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* (607) For an example of this method, suppose in the equation [865], $Q=(2a+\alpha).y$, [876a]
and it will become $\frac{ddy}{dt^2}+(a+\alpha)^2.y=0$, which is of the same form as [865′], changing
a into $a+\alpha$, and its complete integral [864a] is $y=b.\sin.\{(a+\alpha).t+\varphi\}$, which
being developed by [21] Int. is $y=b.\{\sin.(at+\varphi).\cos.\alpha t+\cos.(at+\varphi).\sin.\alpha t\}$,
and if for $\cos.\alpha t$, $\sin.\alpha t$, we substitute their values in series, [43, 44] Int. it will
become

$$y=b.\left\{1-\frac{(\alpha t)^2}{1.2}+\frac{(\alpha t)^4}{1.2.3.4}-\&c.\right\}.\sin.(at+\varphi)+b.\left\{\alpha t-\frac{(\alpha t)^3}{1.2.3}+\&c.\right\}.\cos.(at+\varphi);$$

the sines and cosines of angles, of the order αt, ceases to be exact, when, in
[875v] the course of time, the arc αt becomes considerable; and for this reason, the approximate values of y, y', &c., cannot be extended to an unlimited time. Now it is important to obtain these values in such forms as will include past and future ages. This is done by reducing the arcs of a circle, comprised in the approximate values, to the functions which produced them, by their development in series. This is a delicate and interesting problem of analysis. The following is a general and very simple method of solving it.

43. We shall consider the differential equation of the order i

[876] $$0=\frac{d^i y}{d t^i}+P+\alpha Q;$$

[876'] α being very small, and P, Q, being algebraical functions of y, $\frac{dy}{dt} \ldots \frac{d^{i-1}y}{dt^{i-1}}$; and of the sines and cosines of angles increasing in proportion to the time t.* We shall suppose that we have the complete integral of this differential

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and by arranging according to the powers of α,

[876b] $$y=b\,.\,\sin.(at+\varphi)+\alpha\,.\,bt\,.\,\cos.(at+\varphi)-\frac{\alpha^2\,.\,bt^2}{2}\,.\,\sin.(at+\varphi)-\frac{\alpha^3\,.\,bt^3}{6}\,.\,\cos.(at+\varphi)+\&c.$$

and it is under this last form that the integral will appear, when computed by the above method. For the purpose of illustrating this calculation, we shall compute some terms of the series, which would be found from putting in [875], $Q=(2a+\alpha)\,.\,y$, following nearly the method there pointed out. In the first place, putting $\alpha=0$, the equation becomes $0=\frac{d^2y}{dt^2}+a^2y$, hence $y=b\,.\,\sin.(at+\varphi)$, [864a]. Substituting this in Q, [876a], it becomes $Q=(2a+\alpha)b\sin.(at+\varphi)$, which being compared with [870'] gives $K=(2a+\alpha)\,.\,b$, $m=a$, $\varepsilon=\varphi$, and the term of y resulting in [871''], is $\frac{\alpha\,.\,(2a+\alpha)}{2a}\,.\,bt\,.\,\cos.(at+\varphi)$, or by neglecting α^2, $\alpha bt\,.\,\cos.(at+\varphi)$, so that the second
[876c] value of y is $y=b\,.\,\sin.(at+\varphi)+\alpha bt\,.\,\cos.(at+\varphi)$, which agree with the two first terms of [876b]. Substituting this in Q, [876a], it becomes

$$(2a+\alpha)\,.\,b\,.\,\sin.(at+\varphi)+2a\,.\,\alpha bt\,.\,\cos.(at+\varphi),$$

neglecting terms of the order α^2. This may be substituted for Q, in [865b], and by this process we may obtain successively as many terms as we please of the series [876b].

* (609) By this is meant that the *first* power only of t is included under the signs of *cosine* and *sine*, the second, third, &c., powers t^2, t^3, &c., being excluded.

equation, in the case of $\alpha = 0$, and that the value of y given by this integral, does not contain the arc t, without the signs of *sine* and *cosine*; we shall [876''] also suppose, that by integrating this equation, by the preceding method of approximation, when α is finite, we shall have

$$y = X + t \, . \, Y + t^2 \, . \, Z + t^3 \, . \, S + \&c. ; \qquad [877]$$

X, Y, Z, &c., being periodical functions of t, containing i arbitrary quantities c, c', c'', &c.; the powers of t in this expression of y, increasing infinitely in the successive approximations. It is evident that these coefficients decrease [877'] with greater rapidity the smaller the quantity α is taken.* In the theory of the motions of the heavenly bodies, α expresses the order of the disturbing forces, in comparison with the principal forces acting on them.

If we substitute the preceding value of y, in the function $\frac{d^i y}{dt^i} + P + \alpha \, Q$, [876]; it will become of this form, $k + k' t + k'' t^2 + \&c.$; k, k', k'', &c., [877''] being periodical functions of t; but by hypothesis, the value of y satisfies the differential equation [876],

$$0 = \frac{d^i y}{dt^i} + P + \alpha \, . \, Q ; \qquad [878]$$

we ought therefore to have identically,

$$0 = k + k' t + k'' t^2 + \&c. \qquad [879]$$

If k, k', k'', &c., do not vanish, this equation would give, by inverting the series, the arc t in functions of sines and cosines of angles proportional to t;† supposing therefore α to be infinitely small, we should have t equal to *a finite* function of sines and cosines of similar angles, which is impossible; therefore [879'] the functions k, k', k'', &c., are identically nothing.

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* (610) The computation in [876*b*] shows that Y, Z, S, &c., are respectively of the orders α, α^2, α^3, &c., in the example there given.

† (611) This inversion might be made by La Grange's formulas, [629*c*], which by changing x into t, and t into x, to conform to the present notation, become

$$t = x + F(t), \qquad \text{and} \qquad \psi(t) = \psi(x) + F(x) \, . \, \psi'(x) + \&c., \qquad [879a]$$

and if we put $\psi(x) = x$, which makes $\psi'(x) = 1$, also for brevity, $F(x) = X$, this last expression will become

$$t = x + X + \frac{d \, . \, X^2}{1 \, . \, 2 \, . \, dx} + \frac{d^2 \, . \, X^3}{1 \, . \, 2 \, . \, 3 \, . \, dx^2} + \&c., \qquad [879b]$$

Now if the arc t be raised only to the first power, under the signs of *sine* and *cosine*, as is the case in the theory of the celestial motions, this arc will not be produced by the successive differentials of y;* substituting therefore the preceding value of y, in the function $\frac{d^i y}{d t^i} + P + \alpha Q$, the function $k + k' t + \&c.$, into which it is transformed, will not contain the
[879''] arc t out of the signs of *sin.* and *cos.*, except as it is already contained in that form in y; therefore by changing in the expression of y, the arc t, without the periodical signs into $t - \theta$, θ being any constant quantity, the
[879'''] function $k + k' t + \&c.$, will become $k + k' . (t - \theta) + \&c.$; and since this last function becomes identically nothing, in consequence of the identical equations $k = 0$, $k' = 0$, &c., it follows that the expression

[880] $$y = X + (t - \theta) . Y + (t - \theta)^2 . Z + \&c.$$

will also satisfy the differential equation [876]

[881] $$0 = \frac{d^i y}{d t^i} + P + \alpha Q.$$

Although this second value of y seems to contain $i+1$ arbitrary quantities, namely θ and the i terms c, c', c'', &c.; yet it cannot actually contain more than i such quantities, which are really independent of each other. It therefore necessarily follows, that an appropriate change† in the constant

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Now if we divide [879] by k', we shall get $t = -\frac{k}{k'} - \frac{k''}{k'} . t^2 - \&c.$ Comparing this with t, [879a], we get $x = -\frac{k}{k'}$, and $F(t) = -\frac{k''}{k'} . t^2 - \&c.$; hence

$$X = F(x) = -\frac{k''}{k'} . x^2 - \frac{k'''}{k'} . x^3 - \&c.$$

Substituting this value of X in [879b], we shall get the required value of t, expressed in terms of k', k'', &c. When α is infinitely small, this value of t would, as in [879'], be a single finite function of sines and cosines of angles proportional to t, which would be impossible because there are an infinite number of values of t, corresponding to the same sine or cosine.

* (612) The successive differentials of any term like $b . \overset{\text{sin.}}{\text{cos.}} (m t + \varepsilon)$, taken relative to t, will not produce t out of the signs *sin.* and *cos.*, which would not be the case if the exponent of t should differ from unity, as $b . \overset{\text{sin.}}{\text{cos.}} (m t^2 + \varepsilon)$, the differential of which, divided by $d t$, would contain t without the signs of *sin.* and *cos.*

† (613) This consists in supposing c, c', c'', &c., to be functions of θ, as is shown hereafter.

quantities c, c', c'', &c., *will make the arbitrary term* θ *disappear from the second expression of* y [880], and in this manner it will be made to coincide with the first [877]. This consideration furnishes a method of making the arcs of a circle disappear from the quantities without the periodical signs. [881']

We shall put the second expression of y under the following form :*

$$y = X + (t - \theta) \,.\, R. \qquad [882]$$

Since we suppose that θ disappears from y, we shall have $\left(\frac{dy}{d\theta}\right) = 0$;† [882']
consequently

$$R = \left(\frac{dX}{d\theta}\right) + (t - \theta) \,.\, \left(\frac{dR}{d\theta}\right). \qquad [883]$$

Taking successively the differentials of this equation, we shall have

$$2 \,.\, \left(\frac{dR}{d\theta}\right) = \left(\frac{ddX}{d\theta^2}\right) + (t - \theta) \,.\, \left(\frac{ddR}{d\theta^2}\right) ;$$

$$3 \,.\, \left(\frac{ddR}{d\theta^2}\right) = \left(\frac{d^3X}{d\theta^3}\right) + (t - \theta) \,.\, \left(\frac{d^3R}{d\theta^3}\right) ; \qquad [884]$$

&c. ;

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* (614) The expression [877], by changing as above t into $t - \theta$, becomes as in [880], and if we put $R = Y + (t - \theta) \,.\, Z + \&c.$, it will become as in [882].

† (615) This follows from the value of y, [877], which being wholly independent of θ must evidently give $\left(\frac{dy}{d\theta}\right) = 0$. Substituting this in the differential of [882], relative to θ, we get $\left(\frac{dy}{d\theta}\right) = \left(\frac{dX}{d\theta}\right) - R + (t - \theta) \,.\, \left(\frac{dR}{d\theta}\right) = 0$, hence $R = \left(\frac{dX}{d\theta}\right) + (t - \theta) \,.\, \left(\frac{dR}{d\theta}\right)$, as in [883]. Taking the differential of this relative to θ, we get

$$\left(\frac{dR}{d\theta}\right) = \left(\frac{ddX}{d\theta^2}\right) - \left(\frac{dR}{d\theta}\right) + (t - \theta) \,.\, \left(\frac{d^2R}{d\theta^2}\right),$$

and by transposing $\left(\frac{dR}{d\theta}\right)$, we obtain the first of the equations [884]. The differential of this last equation being found relative to θ, and divided by $d\theta$, gives

$$2 \,.\, \left(\frac{ddR}{d\theta^2}\right) = \left(\frac{d^3X}{d\theta^3}\right) - \left(\frac{d^2R}{d\theta^2}\right) + (t - \theta) \,.\, \left(\frac{d^3R}{d\theta^3}\right),$$

and by transposing $-\left(\frac{d^2R}{d\theta^2}\right)$, we get the second of the equations [884], and so on.

hence it is easy to conclude, by eliminating R and its differentials from the preceding expression of y,*

[885] $$y = X + (t-\theta)\cdot\left(\frac{dX}{d\theta}\right) + \frac{(t-\theta)^2}{1\,.\,2}\cdot\left(\frac{ddX}{d\theta^2}\right) + \frac{(t-\theta)^3}{1\,.\,2\,.\,3}\cdot\left(\frac{d^3X}{d\theta^3}\right) + \&c.$$

X is a function of t [877'], and of the constant quantities c, c', c'', &c.; and *as these quantities are functions of* θ, X will be a function of t and θ, which we may represent by

[885'] $$X = \varphi\,(t, \theta).$$

The preceding expression of y is, by the formula (*i*) § 21 [617], the
development of the function $\varphi\,(t, \theta + t - \theta)$, according to the powers of
[885''] $t - \theta$;† therefore $y = \varphi\,(t, t)$; hence it follows that we shall have y, by
changing θ into t in the function X [617]. The problem is by this means
[885'''] reduced to the finding of X in a function of t and θ, and it will therefore
require the determination of c, c', c'', &c., in functions of θ.

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* (616) Substituting R [883] in [882], it becomes

$$y = X + (t-\theta)\cdot\left(\frac{dX}{d\theta}\right) + (t-\theta)^2\cdot\left(\frac{dR}{d\theta}\right).$$

Substituting in this the value of $\left(\frac{dR}{d\theta}\right)$, deduced from the first of the equations [884], we get, $y = X + (t-\theta)\cdot\left(\frac{dX}{d\theta}\right) + \frac{(t-\theta)^2}{2}\cdot\left(\frac{ddX}{d\theta^2}\right) + \frac{(t-\theta)^3}{2}\cdot\left(\frac{ddR}{d\theta^2}\right)$. Substituting in this the value of $\left(\frac{ddR}{d\theta^2}\right)$ deduced from the second of the equations [884], we shall find another value of y; and, by proceeding in this manner, we shall finally obtain [885].

† (617) Putting, in [617], $t = \theta$, $\alpha = t - \theta$, we shall get

$$\varphi\,.\,(\theta + t - \theta) = \varphi\,(\theta) + (t-\theta)\,.\,\frac{d\,.\,\varphi\,(\theta)}{d\theta} + \frac{(t-\theta)^2}{2}\,.\,\frac{d^2\,.\,\varphi\,(\theta)}{d\theta^2} + \&c.,$$

and as t is considered constant, in the differentials of the second member, we may introduce the term t under the function φ, and write $\varphi\,(t, \theta)$ for $\varphi\,(\theta)$, and $\varphi\,(t, \theta + t - \theta)$ for $\varphi\,(\theta + t - \theta)$, that is, we may write $X = \varphi\,(t, \theta)$, [885'], for $\varphi\,(\theta)$. By this means the preceding expression becomes $\varphi\,(t, \theta + t - \theta) = X + (t-\theta)\,.\left(\frac{dX}{d\theta}\right) + \&c.$, the second member of which is the same as in [885], therefore it is equal to its first member y; hence $y = \varphi\,(t, \theta + t - \theta)$, and as $\theta + t - \theta = t$, this becomes simply $y = \varphi\,(t, t)$. Hence it appears that the value of y may be obtained, by changing θ into t, in $X = \varphi\,(t, \theta)$; [885'].

For this purpose we shall resume the equation [880],

$$y = X + (t-\theta)\,.\,Y + (t-\theta)^2\,.\,Z + (t-\theta)^3\,.\,S + \&c. \qquad [886]$$

Since the constant quantity θ is supposed to disappear [881′] from this expression, we shall have the following identical equation :*

$$0 = \left(\frac{dX}{d\theta}\right) - Y + (t-\theta)\,.\left\{\left(\frac{dY}{d\theta}\right) - 2Z\right\} + (t-\theta)^2.\left\{\left(\frac{dZ}{d\theta}\right) - 3S\right\} + \&c. \quad (a) \qquad [887]$$

Applying to this equation, the same reasoning as in the case of

$$0 = k + k' t + k'' t^2 + \&c.,$$

[879, 879′], we shall easily perceive that the coefficients of the successive powers of $(t-\theta)$, ought to vanish. The functions X, Y, Z, &c., contain θ only as it is included in c, c', &c. [877′, 885‴]; so that to form the partial differentials $\left(\frac{dX}{d\theta}\right)$, $\left(\frac{dY}{d\theta}\right)$, $\left(\frac{dZ}{d\theta}\right)$, &c., it is only necessary to vary c, c', &c., in these functions; hence we get [887′]

$$\left(\frac{dX}{d\theta}\right) = \left(\frac{dX}{dc}\right).\frac{dc}{d\theta} + \left(\frac{dX}{dc'}\right).\frac{dc'}{d\theta} + \left(\frac{dX}{dc''}\right).\frac{dc''}{d\theta} + \&c.;$$

$$\left(\frac{dY}{d\theta}\right) = \left(\frac{dY}{dc}\right).\frac{dc}{d\theta} + \left(\frac{dY}{dc'}\right).\frac{dc'}{d\theta} + \left(\frac{dY}{dc''}\right).\frac{dc''}{d\theta} + \&c.; \qquad [888]$$

&c.

Now it may happen that the arc t is multiplied by some of the arbitrary quantities c, c', c'', &c., in the periodical functions X, Y, Z, &c.; the differential of these functions relative to θ, or in other words, the differentials relative to these arbitrary quantities, will develop this arc, and make it come forth, from under the signs of the periodical functions ;† the differentials [888′]

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* (618) This equation is found by computing $\left(\frac{dy}{d\theta}\right)$, from the equation [880], and putting, as in [882′], $\left(\frac{dy}{d\theta}\right) = 0$.

† (619) Suppose, for example, $X = c\,.\,\sin.\,at + b\,.\,\sin.\,ct$; a, b, being independent of c, c', &c., we should have $\left(\frac{dX}{d\theta}\right) = \left(\frac{dc}{d\theta}\right).\sin.\,at + b\,.\left(\frac{dc}{d\theta}\right).\,t\,.\cos.\,ct$, in which the last term contains t without the sign of $\cos.\,ct$, being produced in the manner above mentioned.

$\left(\frac{dX}{d\theta}\right)$, $\left(\frac{dY}{d\theta}\right)$, $\left(\frac{dZ}{d\theta}\right)$, &c., will then be of the form,

[889]

$$\begin{aligned}\left(\frac{dX}{d\theta}\right) &= X' + t\,.\,X'' ;\\ \left(\frac{dY}{d\theta}\right) &= Y' + t\,.\,Y'' ;\\ \left(\frac{dZ}{d\theta}\right) &= Z' + t\,.\,Z'' ;\\ &\&c. ;\end{aligned}$$

X', X'', Y', Y'', Z', Z'', &c., being periodical functions of t [877′, 888],
[889′] containing also the arbitrary quantities c, c', c'', &c., and their first differentials divided by $d\theta$, which differentials appear under a linear form in these functions;* therefore we shall have†

[890]

$$\begin{aligned}\left(\frac{dX}{d\theta}\right) &= X' + \theta\,.\,X'' + (t-\theta)\,.\,X'' ;\\ \left(\frac{dY}{d\theta}\right) &= Y' + \theta\,.\,Y'' + (t-\theta)\,.\,Y'' ;\\ \left(\frac{dZ}{d\theta}\right) &= Z' + \theta\,.\,Z'' + (t-\theta)\,.\,Z'' ;\\ &\&c.\end{aligned}$$

Substituting these values in the equation (a) [887], we shall have

[891]

$$\begin{aligned}0 = {} & X' + \theta\,.\,X'' - Y\\ & + (t-\theta)\,.\{Y' + \theta\,.\,Y'' + X'' - 2\,Z\}\\ & + (t-\theta)^2\,.\{Z' + \theta\,.\,Z'' + Y'' - 3\,S\} + \&c. ;\end{aligned}$$

putting the coefficients of the powers of $t-\theta$ severally equal to nothing,‡ we shall have

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* (620) That the differentials dc, dc', dc'', &c., appear under a linear form is evident from the equations [888].

† (621) These values of $\left(\frac{dX}{d\theta}\right)$, $\left(\frac{dY}{d\theta}\right)$, &c., [890], are evidently identical with those in [889], writing $\theta + (t-\theta)$ for t; so that when they are substituted in the equation [887], they may be arranged according to the powers of $t-\theta$, as in [891].

‡ (622) For the same reason that k, k', k'', &c., [879], were severally put equal to nothing, [879′].

$$0 = X' + \theta . X'' - Y;$$
$$0 = Y' + \theta . Y'' + X'' - 2\,Z; \qquad [892]$$
$$0 = Z' + \theta . Z'' + Y'' - 3\,S;$$
&c.

If we take the differential of the first of these equations $i-1$ times successively, relative to t, we shall obtain the same number of equations, between the quantities c, c', c'', &c., and their first differentials divided by [892′]
$d\theta$; taking the integrals of these equations relative to θ, we shall have these constant quantities in functions of θ. By merely inspecting the first of the preceding equations, and comparing separately the coefficients of the sines and cosines it contains, we may almost always obtain the differential equations in* c, c', c'', &c. For it is evident that the values of c, c', &c., being [892″]

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* (623) To show the use of these equations by an example, we shall suppose the differential equation to be, $0 = \frac{d^2 y}{d\,t^2} + (a+\alpha)^2 . y$. The value of y deduced from this by a first approximation in [876c], neglecting α^2, is $y = b.\sin.(at+\varphi) + \alpha bt.\cos.(at+\varphi)$, the last term of which contains t without the sign of cosine $(a\,t+\varphi)$, the arbitrary constant terms being b, φ, corresponding to c, c', in the above rule. By comparing the expression of y with [877], we have $X = b . \sin. (a\,t+\varphi)$, $Y = \alpha\, b . \cos. (a\,t+\varphi)$, Z, S, &c., [892a]
being nothing. The differential of this value of X, taken relatively to θ, considering b, φ, only as variable, gives $\left(\frac{dX}{d\theta}\right) = \left(\frac{d\,b}{d\,\theta}\right) . \sin. (a\,t+\varphi) + b . \left(\frac{d\varphi}{d\,\theta}\right) . \cos. (a\,t+\varphi)$.
Comparing this with the first of the equations [889], we get

$$X' = \left(\frac{d\,b}{d\,\theta}\right) . \sin. (a\,t+\varphi) + b . \left(\frac{d\varphi}{d\,\theta}\right) . \cos. (a\,t+\varphi),$$

and $X'' = 0$. Substituting these in the first of the equations [892], namely, $0 = X' + \theta X'' - Y$, it becomes

$$0 = \left(\frac{d\,b}{d\,\theta}\right) . \sin. (a\,t+\varphi) + \left\{ b . \left(\frac{d\varphi}{d\,\theta}\right) - \alpha\, b \right\} . \cos. (a\,t+\varphi),$$

and as this ought to be identically nothing, we must have $\left(\frac{d\,b}{d\,\theta}\right) = 0$, $b . \left(\frac{d\varphi}{d\,\theta}\right) - \alpha b = 0$.
The first gives b constant. The second divided by b becomes $\left(\frac{d\varphi}{d\,\theta}\right) - \alpha = 0$, or $d\,\varphi = \alpha\, d\,\theta$, whose integral is $\varphi = \alpha\,\theta + \varphi'$; θ' being a constant quantity to complete the integral. Substituting this in X, [892a], it becomes $X = b . \sin. (a\,t + \alpha\,\theta + \varphi')$, and this gives y by changing θ into t, as in [885″]. Hence $y = b . \sin. (a\,t + \alpha\,t + \varphi')$, which is of the same form as the complete integral [876a, &c.], found by the method [870]. This example is here used merely as a convenient way of illustrating the formulas [892].

independent of t, the differential equations which determine them ought also to be independent of t. The simplicity of this manner of considering the subject, is one of the principal advantages of the method. In general these equations can be integrated only by successive approximations, which may introduce the arc θ, without the periodical signs, in the values of c, c', &c., [892‴] even when this arc does not really appear in that form in the complete integral; but in this case it may be made to disappear by the method we have just explained.

It may happen that the first of the preceding equations, and its $i-1$ differentials in t, do not give the requisite number i of distinct equations, [892ⁱᵛ] between the quantities c, c', c'', &c., and their differentials. In this case we must refer to the second equation, and to those following it.

When we shall have determined, in this manner, the values of c, c', c'', [892ᵛ] &c., in functions of θ, we must substitute them in X [885]; then changing θ into t, we shall obtain the value of y [885″], free from arcs of a circle, without the periodical signs, when that is possible. If this value yet contain such arcs, it will be a proof that they exist in the rigorous integral.

44. We shall now consider a number n of differential equations,*

[893]
$$0=\frac{d^i y}{dt^i}+P+\alpha Q\,; \qquad 0=\frac{d^i y'}{dt^i}+P'+\alpha Q'\,; \qquad \&c.\,;$$

P, Q, P', Q', &c., being functions of y, y', &c., and of their differentials as [893′] far as the order $i-1$ inclusively, also of sines and cosines of angles, increasing in proportion to t, whose differential is considered as constant. Suppose the approximate integrals of these equations to be

[894]
$$\begin{aligned} y &= X+t\,.\,Y+t^2\,.\,Z+t^3\,.\,S+\&c.\,; \\ y' &= X_{\prime}+t\,.\,Y_{\prime}+t^2\,.\,Z_{\prime}+t^3\,.\,S_{\prime}+\&c.\,; \\ &\&c.\,; \end{aligned}$$

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* (624) This method is merely a generalization of that in the preceding article, and the demonstrations will be easily found by comparing the similar parts of the two articles. Thus the equations [876] are similar to those in [893]; [877] is similar to [894]; [892] is the same as [895], and this last is of the same form as [896], the letters being accented with an additional mark, &c.

X, Y, Z, &c., $X_{,}$, $Y_{,}$, $Z_{,}$, &c., being periodical functions of t, containing the $i\,n$ arbitrary constant quantities c, c', c'', &c. We shall have, as in the preceding article [892],

$$0 = X' + \theta \,.\, X'' - Y;$$
$$0 = Y' + \theta \,.\, Y'' + X'' - 2\,Z; \qquad [895]$$
$$0 = Z' + \theta \,.\, Z'' + Y'' - 3\,S;$$
&c.

The value of y' will likewise give equations of this form,

$$0 = X_{,}' + \theta \,.\, X_{,}'' - Y_{,};$$
$$0 = Y_{,}' + \theta \,.\, Y_{,}'' + X_{,}'' - 2\,Z_{,}; \qquad [896]$$
&c.

The values of y'', y''', &c., produce similar expressions. We must determine from these different equations, by selecting the most simple and approximate forms, the values of c, c', c'', &c , in functions of θ. Substituting these values [896'] in X, $X_{,}$, &c., and then changing θ into t, we shall have the values of y, y', &c., free from arcs of a circle without the periodical signs, when it is possible to be done.

45. We shall resume the method explained in § 40; from which it will be found, that if instead of supposing the parameters c, c', c'', &c., to be constant, we make them variable, so that we may have*

$$d\,c = -\alpha\,d\,t \,.\, \{F\,Q + F'\,Q' + \text{\&c.}\};$$
$$d\,c' = -\alpha\,d\,t \,.\, \{H\,Q + H'\,Q' + \text{\&c.}\}; \qquad [897]$$
&c.;

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[*] (625) In all the preceding articles of this chapter it is supposed that the arbitrary terms c, c', &c., have been found in the form of the equations [851], namely, $c = V$, $c' = V'$, &c., The object of this article is to find c, c', &c., without being under the necessity of forming the equations $c = V$, $c' = V'$, &c., as is observed in [906']. Now whether α be nothing or finite, if we put for $d\,V$, $d\,V'$, &c., the values assumed in [854], we shall obtain the equations [857], &c., and if in these we substitute the values [897], we shall have $d\,c = d\,V$, $d\,c' = d\,V'$, &c., whose integrals [898] take place whether α be finite or nothing; in the former case c, c', &c., will be *variable*, in the latter *constant*.

we shall always have the in integrals of the order $i-1$

[898] $$c=V\,; \qquad c'=V'\,; \qquad c''=V''\,; \qquad \&c.\,;$$

as when α is nothing. Hence it follows, that not only the finite integrals, but also all the equations, in which are found no other differentials except those of an order inferior to i, preserve the same form, whether α be nothing [898′] or finite; since these equations can result only from the comparison of the preceding integrals [898] of the order $i-1$. We may therefore in both cases, take the differentials of the finite equations, $i-1$ times in succession, [898″] without varying c, c', c'', &c.; and as we are at liberty to vary them all at the same time, there will result some equations of condition between [898‴] the parameters c, c', &c., and their differentials.

In the two cases of α nothing and α finite, the values of y, y', and of their differentials as far as the order $i-1$ inclusively, are the same functions of t, and of the parameters c, c', c'', &c.; therefore let Y be any function [898iv] whatever of the variable quantities y, y', y'', &c., and of their differentials *of an order below* $i-1$, and put T for the function of t, which Y becomes, when we substitute in it the values of these variable quantities, and their differentials in functions of t. We may take the differential of $Y=T$, supposing the parameters c, c', c'', &c., to be constant;* we may even take [898v] the partial differential of Y, relative to one or more of the variable quantities y, y', &c.; provided we vary in T only those quantities which vary with them. In all these differentials, the parameters c, c', c'', &c., may be [898vi] considered as constant; since by substituting for y, y', &c., and their differentials, the corresponding values in t, we shall obtain equations which are identically nothing, in the two cases of α nothing and of α finite.

When the differential equations are of the order $i-1$, we must no longer suppose the parameters c, c', c'', &c., to be constant in taking the differentials. To find the differentials of such expressions, we shall consider the equation

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* (626) By hypothesis Y does not contain any differentials of the order $i-1$; its differential will not therefore contain any one of a higher order than $i-1$, and in finding differentials of this order, we may, as is observed in [898″], consider c, c', c'', &c., as constant. The same remark will apply to *partial* differentials of Y, relative to one, or more, of the quantities y, y', y'', &c.

$\varphi = 0$, φ being a differential function of the order* $i-1$, containing the [898^{vii}] parameters c, c', c'', &c. Let $\delta\varphi$ be the differential of this function, supposing c, c', &c., and the differentials $d^{i-1}y$, $d^{i-1}\,.\,y'$, &c., to be constant. Put S for the coefficient of $\frac{d^i y}{dt^{i-1}}$, in the complete differential of φ; S' for the [898^{viii}] coefficient of $\frac{d^i y'}{dt^{i-1}}$, in the same differential; and so on. The complete differential of $\varphi = 0$, will become

$$0 = \delta\varphi + \left(\frac{d\varphi}{dc}\right).dc + \left(\frac{d\varphi}{dc'}\right).dc' + \&\text{c.}$$
$$+ S.\frac{d^i y}{dt^{i-1}} + S'.\frac{d^i y'}{dt^{i-1}} + \&\text{c.} \qquad [899]$$

Substituting, for $\frac{d^i y}{dt^{i-1}}$, its value [893], $-dt.\{P + \alpha Q\}$; for $\frac{d^i y'}{dt^{i-1}}$, [899'] its value [893], $-dt.\{P' + \alpha Q'\}$, &c., we shall have

$$0 = \delta\varphi + \left(\frac{d\varphi}{dc}\right).dc + \left(\frac{d\varphi}{dc'}\right).dc' + \&\text{c.} \qquad (t)$$
$$-dt.\{SP + S'P' + \&\text{c.}\} - \alpha\,dt.\{SQ + S'Q' + \&\text{c.}\} \qquad [900]$$

When α is supposed to be nothing, the parameters c, c', c'', &c. [897], will be constant; and the preceding equation will become

$$0 = \delta\varphi - dt.\{SP + S'P' + \&\text{c.}\} \qquad [901]$$

If we substitute in this equation, for c, c', c'', their values V, V', V'', &c., [898], we shall have a differential equation of the order $i-1$ without [901'] arbitrary constant quantities, which is impossible, except the equation be identically nothing. The function

$$\delta\varphi - dt.\{SP + S'P' + \&\text{c.}\} \qquad [902]$$

therefore becomes identically nothing, by means of the equations $c = V$, $c = V'$, &c.; and as these equations exist, when the parameters c, c', c'', &c., [902'] are variable [898'], it is evident that in this case the preceding function will yet remain identically nothing; the equation (t) [900] will therefore become

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* (627) After showing how to find the differential of a quantity of this order, and of this nature, as in the result given in [903']; the author applies the method, in [904, 905], to the investigation of i successive integrals of the finite quantity ψ, and by this means determines c, c', &c., as in [906'], without being under the necessity of reducing them to the form [898].

[903]
$$0=\left(\frac{d\varphi}{dc}\right).dc+\left(\frac{d\varphi}{dc'}\right).dc'+\&c.$$
$$-\alpha dt.\{SQ+S'Q'+\&c.\}.$$

Hence we see that to find the differential of the equation $\varphi=0$, *it is only requisite to vary in* φ, *the parameters* c, c', &c., *and the differentials* $d^{i-1}y$,
[903′] $d^{i-1}y'$, &c., *and to substitute, after taking the differentials,* $-\alpha Q$, $-\alpha Q'$, &c., *for* $\frac{d^iy}{dt^i}$, $\frac{d^iy'}{dt^i}$, &c.

Let $\psi=0$ be a *finite* equation between y, y', &c., and the variable quantity t; if we denote by $\delta\psi$, $\delta^2\psi$, $\delta^3\psi$, &c., the successive differentials
[903″] of ψ,* supposing c, c', &c., to be constant, we shall have, by what has been said, in case c, c', &c., are variable, the following equations:

[904]
$$\psi=0, \qquad \delta\psi=0, \qquad \delta^2\psi=0\ldots.\delta^{i-1}\psi=0.$$

By changing therefore successively, in the equation (x) [903, 903′], the function φ into ψ, $\delta\psi$, $\delta^2\psi$, &c., we shall have†

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* (628) These differentials being divided by dt, dt^2, &c., respectively, to conform to the last of the equations [905].

† (629) None of the equations [904], except the last, $\delta^{i-1}\psi=0$, contain the terms $d^{i-1}y$, $d^{i-1}y'$, &c., on which S, S', &c., [898$^{\text{viii}}$], depend; therefore in all these equations except the last, we may suppose S, S', &c., to be nothing, and this would be in conformity to the method given in [898$^{\text{v}}$], and we shall then get, by writing successively ψ, $\delta\psi$, &c., for φ, the whole system of equations [905] except the last, which being derived from $\delta^{i-1}\psi=0$, will contain S, S', &c., and will become

[905a]
$$0=\left(\frac{d.\delta^{i-1}\psi}{dc}\right).dc+\left(\frac{d.\delta^{i-1}\psi}{dc'}\right).dc'+\&c.-\alpha dt.\left\{SQ+S'Q'+\&c.\right\}$$

Now a little attention will show that S, [898$^{\text{viii}}$], which is the coefficient of $d^{i-1}y$ in $d^{i-1}\psi$, is also the coefficient of $d^{i-2}y$ in $d^{i-2}\psi$, or the coefficient of $d^{i-3}y$ in $d^{i-3}\psi$, and so on, till we get to the coefficient of dy in $d\psi$, which gives $S=\left(\frac{d\psi}{dy}\right)$. In like manner, $S'=\left(\frac{d\psi}{dy'}\right)$, &c. Substituting these values of S, S', &c., in [905a], we obtain the last of the equations [905].

$$0=\left(\frac{d\psi}{dc}\right).\,dc+\left(\frac{d\psi}{dc'}\right).\,dc'+\&c.\,;$$

$$0=\left(\frac{d.\delta\psi}{dc}\right).\,dc+\left(\frac{d.\delta\psi}{dc'}\right).\,dc'+\&c.\,;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad [905]$$

$$0=\left(\frac{d.\delta^{i-1}\psi}{dc}\right).\,dc+\left(\frac{d.\delta^{i-1}\psi}{dc'}\right).\,dc'+\&c.$$

$$-\alpha\,dt.\left\{Q.\left(\frac{d\psi}{dy}\right)+Q'.\left(\frac{d\psi}{dy'}\right)+\&c.\right\}$$

Thus the equations $\psi=0$, $\psi'=0$, &c., being supposed to be the n finite integrals of the differential equations

$$0=\frac{d^iy}{dt^i}+P\,; \qquad 0=\frac{d^iy'}{dt^i}+P'\,; \qquad \&c.\,; \qquad [906]$$

we shall have $i\,n$ equations,* by means of which we may determine the parameters c, c', c'', &c., without being under the necessity of forming, for that purpose, the equations $c=V$, $c'=V'$, &c.; but when the integrals appear in that form, the determination of c, c', c'', &c., will be more simple.† [906′]

45′. This manner of varying the parameters is of great importance in analysis, and in its applications. To point out a new use of it, we shall consider the differential equation

$$0=\frac{d^iy}{dt^i}+P\,; \qquad [907]$$

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* (630) Each of the n quantities ψ, ψ', ψ'', &c., will produce i equations of the form [905], making in all $i\,n$ equations.

† (631) When the arbitrary terms c, c', &c., are given under the form $c=V$, $c'=V'$, &c., as in the equations [851], which were found by supposing α to be nothing, it is easy to deduce from them the values of F, F', &c., [863′′′], and thence the values of V, V', &c., [859], when α is finite. But if the equations do not appear under the form of [851], we may use the equations [905, &c.], to determine the values of dc, dc', &c., and thence c, c', &c., when α is finite. As an example of this method, we shall apply it to the equation [865], $0=\frac{d^2y}{dt^2}+a^2y+\alpha\,Q$, already computed. The integral of this, when α is

P being a function of t, y, and its differentials as far as the order $i-1$, and of quantities q, q', &c., which are functions of t. Suppose we have the finite integral of this differential equation, when q, q', &c.,* are constant; and let us represent it by $\varphi=0$, φ being supposed to contain the i arbitrary [907′] constant quantities c, c', &c., We shall denote by $\delta\varphi$, $\delta^2\varphi$, $\delta^3\varphi$, &c., the successive differentials of φ, considering q, q', &c., c, c', c'', &c., as constant. If we suppose all these quantities to vary, the differential of φ will be

[908] $$\delta\varphi+\left(\frac{d\varphi}{dc}\right).dc+\left(\frac{d\varphi}{dc'}\right).dc'+\&c.+\left(\frac{d\varphi}{dq}\right).dq+\left(\frac{d\varphi}{dq'}\right).dq'+\&c.\,;$$

[906a] nothing is by [866] $y=\frac{c}{a}.\sin.at+\frac{c'}{a}.\cos.at$; and if we consider α as finite, this will still be the value of y, supposing c, c', to be variable, and to be determined, by the equations [905]. Now from [906a] we get $y-\frac{c}{a}.\sin.at-\frac{c'}{a}.\cos.at=0$, which is equivalent to $\psi=0$, [904], hence

$$\psi=y-\frac{c}{a}.\sin.at-\frac{c'}{a}.\cos.at, \qquad \delta\psi=\frac{d\psi}{dt}=\frac{dy}{dt}-c.\cos.at+c'.\sin.at,$$

Now the first and last equations of [905], are

$$0=\left(\frac{d\psi}{dc}\right).dc+\left(\frac{d\psi}{dc'}\right).dc'\,; \qquad 0=\left(\frac{d.\delta\psi}{dc}\right).dc+\left(\frac{d.\delta\psi}{dc'}\right).dc'-\alpha dt.Q.\left(\frac{d\psi}{dy}\right);$$

and by substituting the values of ψ, $\delta\psi$, they become

$$0=-\frac{dc}{a}.\sin.at-\frac{dc'}{a}.\cos.at\,; \qquad 0=-dc.\cos.at+dc'.\sin.at-\alpha dt.Q.$$

Multiplying them by $a.\sin.at$, $\cos.at$, respectively, and taking the sum of these products, we get, $-dc-\alpha dt.Q.\cos.at=0.$ Again multiplying them by $-a.\cos.at$, $\sin.at$, respectively, and taking the sum of these products we get $dc'-\alpha dt.Q.\sin.at=0$; hence $dc=-\alpha dt.Q.\cos.at$, $dc'=\alpha dt.Q.\sin.at$, whose integrals are $c=C-\alpha\int Q.dt.\cos.at$, $c'=C'+\alpha\int Q.dt.\sin.at$; C, C', being constant quantities. These being substituted in y, [906a], it becomes as in [870], changing c, c', into C, C', respectively.

* (632) The object of this article is to show how to find the values of y, when q, q', &c., vary with extreme slowness, supposing y to be known for the case of c, c', &c., constant, as is observed in [912′].

therefore by putting

$$0=\left(\frac{d\varphi}{dc}\right).dc+\left(\frac{d\varphi}{dc'}\right).dc'+\&\text{c.}+\left(\frac{d\varphi}{dq}\right).dq+\left(\frac{d\varphi}{dq'}\right).dq'+\&\text{c.}; \qquad [909]$$

$\delta\varphi$ will yet be the first differential of φ, supposing c, c', &c., q, q', &c., to be variable. If we also put

$$0=\left(\frac{d.\delta\varphi}{dc}\right).dc+\left(\frac{d.\delta\varphi}{dc'}\right).dc'+\&\text{c.}+\left(\frac{d.\delta\varphi}{dq}\right).dq+\left(\frac{d.\delta\varphi}{dq'}\right).dq'+\&\text{c.};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots \qquad [910]$$

$$0=\left(\frac{d.\delta^{i-1}\varphi}{dc}\right).dc+\left(\frac{d.\delta^{i-1}\varphi}{dc'}\right).dc'+\&\text{c.}+\left(\frac{d.\delta^{i-1}\varphi}{dq}\right).dq+\left(\frac{d.\delta^{i-1}\varphi}{dq'}\right).dq'+\&\text{c.};$$

$\delta^2\varphi$, $\delta^3\varphi\dots\delta^i\varphi$, will yet be the second, third, &c. differentials of φ, as far as the order i, when c, c', &c., q, q', &c., are supposed variable.

Now in the case of c, c', &c., q, q', &c., being constant, the differential equation

$$0=\frac{d^iy}{dt^i}+P, \qquad [911]$$

is the result of the elimination of the parameters c, c', &c., by means of the equations

$$\varphi=0; \qquad \delta\varphi=0; \qquad \delta^2\varphi=0; \qquad \dots\delta^i\varphi=0; \qquad [912]$$

and as these last equations take place when q, q', &c. are supposed variable, the equation $\varphi=0$ satisfies also in this case the proposed differential equations, provided the parameters c, c', &c., are determined by means of the i differential equations [909, 910]; and as their integration gives i arbitrary constant quantities, the function φ will contain those quantities, and $\varphi=0$ will be the complete integral of the proposed equation. [912′] [912″]

This manner of varying the parameters may be employed with advantage, when the quantities q, q', &c., change with extreme slowness, because this generally renders the integration of the equations which determine the variable parameters c, c', &c., much easier by approximation. [912‴]

CHAPTER VI.

SECOND APPROXIMATION OF THE CELESTIAL MOTIONS, OR THEORY OF THEIR PERTURBATIONS.

46. We shall now apply the preceding methods, to the perturbations of the celestial motions, with a view to obtain the most simple expressions of their periodical and secular equations. We shall resume for this purpose the differential equations of § 9 [416, 417, 418], from which the relative motion of m about M can be determined. If we put

[913] $$R=\frac{m'.(xx'+yy'+zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}+\frac{m''.(xx''+yy''+zz'')}{(x''^2+y''^2+z''^2)^{\frac{3}{2}}}+\&c.-\frac{\lambda}{m};$$

in which by the above mentioned article [412]*

[914] $$\lambda=\frac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}+\frac{mm''}{\{(x''-x)^2+(y''-y)^2+(z''-z)^2\}^{\frac{1}{2}}}$$
$$+\frac{m'm''}{\{(x''-x')^2+(y''-y')^2+(z''-z')^2\}^{\frac{1}{2}}}+\&c.$$

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* (634) The above value of R, [913], gives

$$\left(\frac{dR}{dx}\right)=\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&c.-\frac{1}{m}.\left(\frac{d\lambda}{dx}\right)=-\frac{mx}{r^3}+\Sigma.\frac{mx}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dx}\right),$$

because $\Sigma.\frac{mx}{r^3}=\frac{mx}{r^3}+\frac{m'x'}{r'^3}+\frac{m''x''}{r''^3}+\&c.$ Now if in the term $-\frac{mx}{r^3}$, we substitute for m its value, [914′], $\mu-M$, we shall have $\left(\frac{dR}{dx}\right)=\frac{x.(M-\mu)}{r^3}+\Sigma.\frac{mx}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dx}\right)$, hence $\frac{Mx}{r^3}+\Sigma.\frac{mx}{r^3}-\frac{1}{m}.\left(\frac{d\lambda}{dx}\right)=\frac{\mu x}{r^3}+\left(\frac{dR}{dx}\right)$; adding $\frac{ddx}{dt^2}$ to each side, and using [416], we shall get the first equation [915]. The two remaining equations in y, z, are deduced in like manner from [417, 418], which in fact are the same as [416], merely changing the axis of x into that of y or z.

supposing also [530$^{\text{iv}}$, 411],

$$M+m=\mu\,;\qquad r=\sqrt{x^2+y^2+z^2}\,;\qquad r'=\sqrt{x'^2+y'^2+z'^2}\,;\qquad \&c.\,;$$ [914']

we shall have

$$\left.\begin{aligned}0&=\frac{d\,d\,x}{d\,t^2}+\frac{\mu\,x}{r^3}+\left(\frac{d\,R}{d\,x}\right)\\0&=\frac{d\,d\,y}{d\,t^2}+\frac{\mu\,y}{r^3}+\left(\frac{d\,R}{d\,y}\right)\\0&=\frac{d\,d\,z}{d\,t^2}+\frac{\mu\,z}{r^3}+\left(\frac{d\,R}{d\,z}\right)\end{aligned}\right\}\,;\qquad (P)$$ [915]

The sum of these three equations, multiplied respectively by $2\,d\,x$, $2\,d\,y$, $2\,d\,z$, gives, by integration,*

$$0=\frac{d\,x^2+d\,y^2+d\,z^2}{d\,t^2}-\frac{2\,\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}\,R\,;\qquad (Q)$$ [916]

the differential $\mathrm{d}R$ referring only to the co-ordinates x, y, z, of the body m, [916'] and a being an arbitrary constant quantity, which, when R is nothing, becomes, by § 18, 19, [596'], the semi-transverse axis of the ellipsis† described by m about M. [916'']

Use of the symbol d

The equations (P) [915] multiplied respectively by x, y, z, and added to the integral (Q) [916], will give‡

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* (635) In finding this integral, we must substitute in the term $2\,\mu\,.\,\frac{x\,d\,x+y\,d\,y+z\,d\,z}{r^3}$, the value of the numerator, $x\,d\,x+y\,d\,y+z\,d\,z=r\,d\,r$, [571*b*], which reduces it to $2\,\mu\,.\,\frac{r\,dr}{r^3}=2\,\mu\,.\,\frac{d\,r}{r^2}$, whose integral is $-\frac{2\mu}{r}$. Again, as the symbol d, [916'], only affects the co-ordinates x, y, z, of the body m, we shall have as in [13*b*, 14*a*],

$$\mathrm{d}\,R=\left(\frac{d\,R}{d\,x}\right).\,d\,x+\left(\frac{d\,R}{d\,y}\right).\,d\,y+\left(\frac{d\,R}{d\,z}\right).\,d\,z\,,$$ [916*a*]

the integral of the double of this second member, which occurs in [916], will therefore be represented by $2\,.\int\mathrm{d}\,R$. Lastly, $\frac{\mu}{a}$ is the constant quantity to complete the integral.

† (636) When $R=0$, the equation [916] becomes identical with the last of the equations [572], and in [596'], it is shown that in this case the transverse diameter is $2\,a$.

‡ (637) After making this addition we must put $\frac{\mu}{r^3}\,.\,(x^2+y^2+z^2)=\frac{\mu}{r^3}\,.r^2=\frac{\mu}{r}$, [914']; also

$x\,d\,d\,x+y\,d\,d\,y+z\,d\,d\,z+d\,x^2+d\,y^2+d\,z^2=d\,.(x\,d\,x+y\,d\,y+z\,d\,z)=d\,.(r\,d\,r)=\tfrac{1}{2}\,.\,d^2.\,r^2$, [914', 549'].

[917] $$0=\tfrac{1}{2}\cdot\frac{d^2\,.\,r^2}{d\,t^2}-\frac{\mu}{r}+\frac{\mu}{a}+2\int \mathrm{d}\,R+x\,.\left(\frac{d\,R}{d\,x}\right)+y\,.\left(\frac{d\,R}{d\,y}\right)+z\,.\left(\frac{d\,R}{d\,z}\right).\qquad (R)$$

Now we may conceive that the disturbing masses m', m'', &c., are multiplied by a coefficient α; and then the value of r will be a function of the time t [917′] and α. If we develop this function according to the powers of α, and after the development put $\alpha=1$, it will be arranged according to the powers and products of the disturbing forces. We shall denote by the characteristic δ placed before any quantity, its differential taken with respect to α and [917″] divided by $d\,\alpha$. After having determined $\delta\,r$, in a series arranged according to the powers of α; we may obtain the radius r, by multiplying this series by $d\,\alpha$, and taking the integral relative to α, then adding to the integral a function of t independent of α, which function will evidently be equal to the [917‴] value of r when the disturbing forces are nothing, and the body describes a conic section. The determination of r is therefore reduced to the finding and integrating the differential equation upon which $\delta\,r$ depends.

Use of the symbol δ

For this purpose, we shall resume the differential equation (R) [917], and for greater simplicity, we shall put

[918] $$x\,.\left(\frac{d\,R}{d\,x}\right)+y\,.\left(\frac{d\,R}{d\,y}\right)+z\,.\left(\frac{d\,R}{d\,z}\right)=r\,.\,R'\,;$$

taking the differential of [917] relative to α, we shall have*

[919] $$0=\frac{d^2\,.\,r\,\delta\,r}{d\,t^2}+\frac{\mu\,.\,r\,\delta\,r}{r^3}+2\int\delta\,.\,\mathrm{d}\,R+\delta\,.\,r\,R'\,;\qquad (S)$$

Put $d\,v$ for the infinitely small angle included between the two radii vectores r and $r+d\,r$; the element of the curve described by m about M, will be [919′] $\sqrt{d\,r^2+r^2\,.\,d\,v^2}$ [583]; hence we shall have

[919″] $$d\,x^2+d\,y^2+d\,z^2=d\,r^2+r^2\,.\,d\,v^2\,;$$

and the equation (Q) [916] will become

[920] $$0=\frac{r^2\,.\,d\,v^2+d\,r^2}{d\,t^2}-\frac{2\,\mu}{r}+\frac{\mu}{a}+2\int\mathrm{d}\,R.$$

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* (638) The differential of $\tfrac{1}{2}\,r^2$ relative to α, or, in other words, relative to the symbol δ, is $r\,\delta\,r$, consequently $\tfrac{1}{2}\,.\,\delta\,.\,d^2\,.\,r^2=d^2\,.\,r\,\delta\,r$. Also, $-\delta\,.\,\frac{\mu}{r}=\frac{\mu\,\delta\,r}{r^2}=\frac{\mu\,r\,\delta\,r}{r^3}$. Substituting these and [918] in the differential of [917], taken relative to δ, it becomes as in [919].

Eliminating $\frac{\mu}{a}$ from this equation, by means of the equation, (R) [917], we shall obtain*

$$\frac{r^2.dv^2}{dt^2}=\frac{r.ddr}{dt^2}+\frac{\mu}{r}+r.R'; \qquad [921]$$

taking the differential of this relative to α, we shall find†

$$\frac{2r^2.dv.d.\delta v}{dt^2}=\frac{r.dd.\delta r-\delta r.ddr}{dt^2}-\frac{3\mu.r\delta r}{r^3}+r.\delta R'-R'.\delta r. \qquad [922]$$

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* (639) Substituting in [917] the value rR', [918], and then subtracting from it the equation [920], it becomes, $0=\frac{1}{2}.\frac{d^2.r^2}{dt^2}+\frac{\mu}{r}+rR'-\frac{(r^2.dv^2+dr^2)}{dt^2}$, and since by development $\frac{1}{2}.d^2.r^2=rddr+dr^2$, it gives by reduction [921].

† (640) The differential of [921], relative to δ, is

$$\frac{2r^2dv.d\delta v+2r\delta r.dv^2}{dt^2}=\frac{\delta r.ddr+r.d^2\delta r}{dt^2}-\frac{\mu\delta r}{r^2}+R'.\delta r+r.\delta R';$$

and [921] multiplied by $-\frac{2\delta r}{r}$ is

$$\frac{-2r\delta r.dv^2}{dt^2}=\frac{-2\delta r.ddr}{dt^2}-\frac{2\mu\delta r}{r^2}-2R'.\delta r,$$

adding these two expressions together we get

$$\frac{2r^2dv.d\delta v}{dt^2}=\frac{r.d^2\delta r-\delta r.ddr}{dt^2}-\frac{3\mu\delta r}{r^2}+r.\delta R'-R'.\delta r,$$

which, by writing $-\frac{3\mu r\delta r}{r^3}$ for $-\frac{3\mu\delta r}{r^2}$, becomes as in [922]. This, added to three times the equation [919], becomes

$$\frac{2r^2dv.d\delta v}{dt^2}=\frac{3.d^2.r\delta r+r.d^2\delta r-\delta r.ddr}{dt^2}+6.\int\delta.\mathrm{d}R+3.\delta.rR'+r.\delta R'-R'.\delta r.$$

Substituting for $3.d^2.r\delta r$, its developed value, $3.(d^2r.\delta r+2dr.d\delta r+rd^2\delta r)$, also $3\delta.rR'=3.(R'.\delta r+r.\delta R')$, it becomes

$$\frac{2r^2dv.d\delta v}{dt^2}=\frac{2d^2r.\delta r+4r.d^2\delta r+6dr.d\delta r}{dt^2}+6\int\delta.\mathrm{d}R+4r.\delta R'+2R'.\delta r,$$

of which the terms free from R, R', in the second member are equal to $\frac{d.(2dr.\delta r+4r.d\delta r)}{dt^2}$, as is easily proved by development. Substituting this and dividing by $\frac{2r^2dv}{dt^2}$, we shall obtain [923].

If we substitute in this equation, for $\frac{\mu . r \delta r}{r^3}$, its value, deduced from the equation (S) [919], we shall have

[923] $$d . \delta v = \frac{d . (d r . \delta r + 2 r . d \delta r) + d t^2 . \{3 \int \delta . \mathrm{d} R + 2 r . \delta R' + R' . \delta r\}}{r^2 . d v}; \qquad (T)$$

We may, by means of the equations (S) [919], and (T) [923], obtain as accurately as may be necessary, the values of δr and δv; but we must observe, that $d v$ being the angle included between the radii r and $r + d r$, the integral v of these angles is not in the same plane.* To determine the value of the angle described about M by the projection of the radius vector r upon a fixed plane, we shall denote this last angle by $v_{,}$, and shall put s for
[923'] the tangent of the latitude of m above the plane; $r . (1 + s s)^{-\frac{1}{2}}$ [680], will
[923''] be the expression of the projection of the radius vector; and the square of the element of the curve described by m will be†

[924] $$\frac{r^2 . d v_{,}^2}{1 + s s} + d r^2 + \frac{r^2 . d s^2}{(1 + s s)^2};$$

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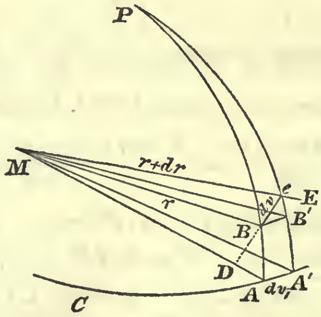
* (641) The mutual attraction of the bodies m and M, would make the orbit of the body m wholly in the *same* plane, as is observed in [533'']. The other attracting bodies m', m'', &c., not being situated in that plane, their disturbing forces will tend to change the orbit of m, so that two consecutive infinitely small parts of the orbit, will not be accurately in the same plane.



† (642) Let M be the place of the body M, supposed to be at rest; $MB = r$, $ME = r + d r$, the radii vectores of the body m, including the angle $BME = d v$; $CAA'M$ the fixed plane; $PBAA'B'e$ a spherical surface described about the centre M, with the radius MB, and meeting ME in e. Draw the quadrantal arcs PBA, $PeB'A'$, perpendicular to the arc AA', and the arc BB' parallel to AA'. Then putting the latitude of the point B equal to l, we shall have the angle $AMB = A'MB' = l$, $\quad B'Me = d l$, $\quad AMA' = d v_{,}$, $\quad AA' = r d v_{,}$, $BB' = AA' . \text{cos. lat.} = r d v_{,} . \cos. l$, and since $\text{tang.}\, l = s$, gives $\cos. l = \frac{1}{\sqrt{1 + s s}}$, we
[924a] shall get $BB' = \frac{r d v_{,}}{\sqrt{1 + s s}}$; $e E = d r$, and the arc $B'e = r d l = \frac{r d s}{1 + s s}$, [51] Int. Now the lines BB', $B'e$, eE, being perpendicular to each other, the sum of their

but the square of this element is $r^2 . d v^2 + d r^2$ [919′]; we shall therefore [924′] have, by putting these two expressions equal to each other,

$$d v_{,} = \frac{d v . \sqrt{(1+ss)^2 - \frac{d s^2}{d v^2}}}{\sqrt{1+ss}}. \qquad [925]$$

Hence $d v_{,}$ may be determined from $d v$, when s is known.

If we take the plane of the orbit of m at a given epoch for the fixed plane, s and $\frac{d s}{d v}$ will evidently be of the same order as the disturbing forces; by [925′] neglecting therefore the squares and products of these forces, we shall have $v = v_{,}$. In the theory of the planets and comets, we may neglect these [925″] squares and products except in a few terms of that order, which are rendered sensible by particular circumstances, and which may be easily determined by means of the equations (S) and (T) [919, 923]. These last equations assume a more simple form, when only the first power of the disturbing force is noticed. For we may then suppose δr and δv to be the parts of r and v arising from these forces;* δR and $\delta . r R'$ will be what R and $r R'$ become, [925‴]

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squares is equal to the square of the distance of the points B, E, and, by substituting the preceding values of $B B'$, $B' e$, $e E$, we shall get for $B E^2$, the same expression as in [924]. Putting this equal to $r^2 d v^2 + d r^2$, [924′], and rejecting $d r^2$ from both sides of the equation, we get $\frac{r^2 d v_{,}^2}{1+ss} + \frac{r^2 d s^2}{(1+ss)^2} = r^2 d v^2$; multiplying this by $\frac{1+ss}{r^2}$, transposing the second term and extracting the square root, we get

$$d v_{,} = \sqrt{d v^2 . (1+ss) - \frac{d s^2}{1+ss}}, \text{ which is easily reduced to the form [925];} \qquad [924b]$$

and by neglecting terms of the order of the square of s, or $\frac{d s}{d v}$, it becomes $d v_{,} = d v$, hence $v_{,} = v$, as in [925″].

* (643) The radius r being developed according to the powers of α, as in [917′], if we neglect the second and higher powers of α, it will become of the form $r = r' + \alpha r''$. Hence $\delta r = \left(\frac{d r}{d \alpha}\right) = r''$, therefore $r = r' + \alpha \delta r$. Now when the terms depending on the disturbing forces, or upon α, vanish, we shall have $r = r'$. The difference of these two values of r, namely, $\alpha \delta r$ or δr, will represent the part of r depending on the disturbing forces, as in [925‴]; in like manner δv will represent the part of v depending on

when, for the co-ordinates of the body, we substitute their values relative to the elliptical motion: we may therefore denote them by these last quantities, subjected to that condition. The equation (S) [919], in this manner, will become

[926] $$0=\frac{d^2.r\,\delta r}{d\,t^2}+\frac{\mu.r\,\delta r}{r^3}+2\int \mathrm{d}R+r\,R'.$$

The fixed plane of x and y being supposed that of the orbit of m, at a given
[926′] epoch, z will be of the same order as the disturbing forces; and since we neglect the square of these forces, we may neglect the quantity $z.\left(\frac{d\,R}{d\,z}\right)$.*
[926″] Moreover, as the radius r differs from its projection but by quantities of the order z^2;† the angle which this radius makes with the axis of x, will differ from that made by its projection, by quantities of the same order [924*b*]; this angle may therefore be supposed equal to v, and we shall have, by neglecting quantities of the same order,

[927] $$x=r.\cos.v; \qquad\qquad y=r.\sin.v;$$

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the disturbing forces. The quantity R, [913], being of the order m', or α, its development according to the powers of α, will be of the form $R=\alpha R''+\alpha^2.R'''+\&c.$, in which R'' evidently represents the value of R, found by using the values of r, v, &c., corresponding to the elliptical motion. If we retain only the first power of α, it will become $R=\alpha R''$. Its differential relative to δ gives $\delta R=R''$, as in [925‴]. In like manner $\delta.r R'$ is equal to $r R'$, corresponding to the elliptical motion.

* (644) The values of R, λ, [913, 914], give

$$z.\left(\frac{d\,R}{d\,z}\right)=\frac{m'\,z\,z'}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}+\frac{m''\,z\,z''}{(x''^2+y''^2+z''^2)^{\frac{3}{2}}}+\&c.-\frac{z}{m}.\left(\frac{d\lambda}{d\,z}\right),$$

and $$-\frac{z}{m}.\left(\frac{d\lambda}{d\,z}\right)=-z.\left\{\frac{m'.(z'-z)}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{\frac{3}{2}}}+\&c.\right\},$$

all the terms of which contain powers and products $m'z$, $m''z$, &c., of the second degree of the disturbing forces, [926′], therefore the whole expression, in the present hypothesis, will be of the order of the square of the disturbing forces.

† (645) The projection MD, fig. page 508, of the radius vector, $r=MB$, on the plane of AMA', is by [680], equal to $r.(1-\frac{1}{2}s^2+\&c.)$, which differs from r by quantities of the order s^2 or z^2, r being supposed nearly equal to unity.

hence we deduce*

$$x\cdot\left(\frac{dR}{dx}\right)+y\cdot\left(\frac{dR}{dy}\right)=r\cdot\left(\frac{dR}{dr}\right);$$ [928]

consequently

$$rR'=r\cdot\left(\frac{dR}{dr}\right)=x\cdot\left(\frac{dR}{dx}\right)+y\cdot\left(\frac{dR}{dy}\right).$$ [928′]

It is easy to prove, by taking the differential, that if we neglect the square of the disturbing forces, the preceding differential equation will give, by means of the two first of the equations (*P*) [915],†

$$r\cdot\delta r=\frac{x\cdot\int y\,dt\cdot\left\{2\int\mathrm{d}R+r\cdot\left(\frac{dR}{dr}\right)\right\}-y\cdot\int x\,dt\cdot\left\{2\int\mathrm{d}R+r\cdot\left(\frac{dR}{dr}\right)\right\}}{\left(\frac{x\,dy-y\,dx}{dt}\right)}.$$ [929]

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* (645*a*) The values of x, y, [927], are the same as those in [371]. These give $\left(\frac{dx}{dr}\right)=\cos.v$; $\left(\frac{dy}{dr}\right)=\sin.v$; hence $r.\left(\frac{dx}{dr}\right)=r.\cos.v=x$; $r.\left(\frac{dy}{dr}\right)=r.\sin.v=y$. Again, in the last of the equations [371], we have $r^2=x^2+y^2$, hence r may be considered as a function of x, y, consequently $\left(\frac{dR}{dr}\right)=\left(\frac{dR}{dx}\right).\left(\frac{dx}{dr}\right)+\left(\frac{dR}{dy}\right).\left(\frac{dy}{dr}\right)$. Multiplying this by r and substituting for $r.\left(\frac{dx}{dr}\right)$, $r.\left(\frac{dy}{dr}\right)$ their preceding values x, y, it becomes as in [928]. The first member of which, neglecting $z.\left(\frac{dR}{dz}\right)$, [926′], is the same as that of [918], consequently their second members are equal, hence

$$rR'=r.\left(\frac{dR}{dr}\right),\ [928'].$$

† (646) We may easily prove that the value of $r\,\delta r$, [929], satisfies the equation [926]. For R being of the order of the disturbing forces, [913], we may, by neglecting the squares and products of such forces, substitute the elliptical elements in the second member of the formula [929], and then we may put, as in the first equation [572], $\frac{x\,dy-y\,dx}{dt}$ constant and equal to c. Putting also for brevity, $Q=2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right)$, as in [934], the [929*a*] expression [929] becomes

$$r\,\delta r=\frac{x.\int y\,dt.Q-y.\int x\,dt.Q}{c},$$ [929*b*]

whose first differential gives $\frac{d.(r\delta r)}{dt}=\frac{dx.\int y\,dt.Q-dy.\int x\,dt.Q}{c\,dt}$; and second

For the co-ordinates in the second member of this equation, we may use their
[929′] elliptical values, which makes $\frac{x\,dy - y\,dx}{dt}$ constant [929*a*], and equal to

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differential gives $\frac{d^2.(r\delta r)}{dt^2} = \frac{ddx.\int y\,dt.\,Q - ddy.\int x\,dt.\,Q - Q\,dt.(x\,dy - y\,dx)}{c\,dt^2}$, which by substituting $x\,dy - y\,dx = c\,dt$, becomes

[929c]
$$\frac{d^2.(r\delta r)}{dt^2} = \frac{ddx.\int y\,dt.\,Q - ddy.\int x\,dt.\,Q}{c\,dt^2} - Q.$$

Again, the assumed value of $r\,\delta r$ [929*b*], gives

$$\frac{\mu.r\delta r}{r^3} = \frac{\mu x}{r^3}.\frac{\int y\,dt.\,Q}{c} - \frac{\mu y}{r^3}.\frac{\int x\,dt.\,Q}{c};$$

but by [915] we have $\frac{\mu x}{r^3} = -\frac{ddx}{dt^2} - \left(\frac{dR}{dx}\right)$; $-\frac{\mu y}{r^3} = \frac{ddy}{dt^2} + \left(\frac{dR}{dy}\right)$, which being substituted in $\frac{\mu.r\delta r}{r^3}$, neglecting the terms containing both quantities Q and $\mathrm{d}R$, because they are both of the order of the disturbing forces, and their product is of the order of the square of those forces, we shall have $\frac{\mu.r\delta r}{r^3} = \frac{-ddx.\int y\,dt.\,Q + ddy.\int x\,dt.\,Q}{c\,dt^2}$. This added to the value of $\frac{d^2.(r\delta r)}{dt^2}$, [929*c*], gives $\frac{d^2.(r\delta r)}{dt^2} + \frac{\mu.r\delta r}{r^3} = -Q$; transposing Q and resubstituting its value, [929*a*], $2\int \mathrm{d}R + r.\left(\frac{dR}{dr}\right)$, or by [928′], $2\int \mathrm{d}R + rR'$, it becomes $\frac{d^2.r\delta r}{dt^2} + \frac{\mu.r\delta r}{r^3} + 2\int \mathrm{d}R + rR' = 0$, as in [926]. Hence the expression [929] satisfies the proposed differential [926], and as the former contains two signs of integration including two arbitrary constant quantities, it is the complete integral. Besides this demonstration, it has been thought proper to give the following direct investigation of the value of $r\,\delta r$, as it answers the purpose of illustrating the use of the formulas [864′].

Put $r\,\delta r = Y$, and Q as in [929*a*], then the equation [926] becomes

$$0 = \frac{d^2 Y}{dt^2} + \frac{\mu Y}{r^3} + Q,$$

which may be integrated by the method of § 40, &c. Comparing it with the first of the equations [864], writing Y instead of y in [864] to distinguish it from y of this article. For if we suppose $R = 0$, and $Q = 0$, the equation will become $0 = \frac{d^2 Y}{dt^2} + \frac{\mu Y}{r^3}$; multiplying this by x, and substituting for $\frac{\mu x}{r^3}$ its value $-\frac{ddx}{dt^2}$, deduced from the first of the equations [915], it becomes $\frac{x\,d^2Y - Y\,ddx}{dt^2} = 0$, whose integral, using c [863$^{\text{vi}}$], is

$$c = \frac{x\,dY - Y\,dx}{dt} = -Y.\frac{dx}{dt} + x.\frac{dY}{dt}.$$

$\sqrt{\mu a.(1-e^2)}$, by § 19 [596c], ae being the excentricity of the orbit of m [377'']. If, in this expression of $r\delta r$, we substitute, for x and y, their values [927], $r.\cos.v$, and $r.\sin.v$; and for $\frac{x\,dy-y\,dx}{dt}$, the quantity [929'] $\sqrt{\mu a.(1-e^2)}$; observing also that by § 20 [605'], $\mu=n^2a^3$, we shall have*

If we multiply by y instead of x, and substitute the value of $\frac{\mu y}{r^3}$, deduced from the second of the equations [915], we shall get $0=\frac{y\,d^2Y-Y\,d^2y}{dt^2}$, whose integral gives

$$c'=\frac{y\,dY-Y\,dy}{dt}=-Y.\frac{dy}{dt}+y.\frac{dY}{dt}.$$

Comparing these values of c, c', with the equations $c=V$, $c'=V'$, [863$^{\text{vi}}$], in which y, y', &c., are changed into Y, Y', &c., we shall find $F=x$, $H=y$, these being the coefficients of $\frac{dY}{dt}$ in c, c', respectively, and the terms of the equations [864'], namely, $c-\alpha\int dt.FQ$, $c'-\alpha\int dt.HQ$, become in the present case, where Q is put for αQ, $c-\int x\,dt.Q$, $c'-\int y\,dt.Q$. These being substituted instead of c, c', in the two preceding integrals give

$$c-\int x\,dt.Q=-Y.\frac{dx}{dt}+x.\frac{dY}{dt}, \qquad c'-\int y\,dt.Q=-Y.\frac{dy}{dt}+y.\frac{dY}{dt},$$

which are the two first complete integrals of [926], and we may even neglect c, and c', supposing the constant quantities to be included under the terms $\int x\,dt.Q$, $\int y\,dt.Q$, by which means we shall have

$$-\int x\,dt.Q=-Y.\frac{dx}{dt}+x.\frac{dY}{dt}, \qquad -\int y\,dt.Q=-Y.\frac{dy}{dt}+y.\frac{dY}{dt}.$$

Eliminating $\left(\frac{dY}{dt}\right)$ from these, we shall obtain Y. This is done by multiplying the first by y, the second by $-x$, and adding the products, from which we get $-y.\int x\,dt.Q+x.\int y\,dt.Q=Y.\left(\frac{xdy-ydx}{dt}\right)$. Dividing by $\frac{xdy-ydx}{dt}$, and resubstituting $Y=r\delta r$, we obtain $r\delta r=\frac{x.\int y\,dt.Q-y.\int x\,dt.Q}{\left(\frac{xdy-ydx}{dt}\right)}$, and by [929d] resubstituting the value of Q, [929a] it becomes as in [929].

* (647) Neglecting the squares and products of the disturbing forces, as in [929a], we we may use in the second member of the equation [929], the elliptical values of the elements,

[930]
$$\delta r=\frac{\left\{\begin{array}{l} a\,.\cos.v\,.\int n\,dt\,.\,r\,.\sin.v\,.\left\{2\int \mathrm{d}R+r\,.\left(\frac{dR}{dr}\right)\right\} \\ -a\,.\sin.v\,.\int n\,dt\,.\,r\,.\cos.v\,.\left\{2\int \mathrm{d}R+r\,.\left(\frac{dR}{dr}\right)\right\}\end{array}\right\}}{\mu\,.\sqrt{1-e^2}}. \qquad (X)$$

The equation (T) [923], being integrated, neglecting the squares of the disturbing forces, gives*

[931]
$$\delta v=\frac{\frac{2r\,.\,d\,.\,\delta r+dr\,.\,\delta r}{a^2\,.\,n\,dt}+\frac{3a}{\mu}\,.\iint n\,dt\,.\,\mathrm{d}R+\frac{2a}{\mu}\,.\int n\,dt\,.\,r\,.\left(\frac{dR}{dr}\right)}{\sqrt{1-e^2}}. \qquad (Y)$$

[931′] This expression will give easily the perturbations of the motion of m in longitude, after those of the radius vector have been ascertained.

[930a] and put $\frac{x\,dy-y\,dx}{dt}=c=\sqrt{\mu a.(1-e^2)}$, [596c]. Substituting this value of $\frac{x\,dy-y\,dx}{dt}$ and those of x, y, [927], in $r\,\delta r$, [929d], it becomes

$$r\,\delta r=\frac{r\,.\cos.v\,.\int r\,.\sin.v\,.dt\,.\,Q-r\,.\sin.v\,.\int r\,.\cos.v\,.dt\,.\,Q}{\sqrt{\mu}\,.\sqrt{a}\,.\sqrt{1-e^2}};$$

dividing this by r, and multiplying the second member by $\frac{n a^{\frac{3}{2}}}{\sqrt{\mu}}=1$, [605′], we shall obtain, δr, [930].

* (648) By [372] we have $r^2\,.\,dv=x\,dy-y\,dx=dt\,.\sqrt{\mu a\,.(1-e^2)}$, [930a], and since $\sqrt{\mu}=n\,a^{\frac{3}{2}}$, [605′], it becomes $r^2\,.\,dv=a^2\,n\,dt\,.\sqrt{1-e^2}$. Substituting this in [923], neglecting the term $R'\,.\,\delta r$, which is of the order of the square of the disturbing forces, because both R' and δr, are of the same order as these forces, it becomes $d\,.\,\delta v=\frac{d\,.(dr\,.\,\delta r+2r\,.\,d\delta r)+dt^2\,.\{3\int\delta\,\mathrm{d}R+2r\,.\,\delta R'\}}{a^2.n\,dt\,.\sqrt{1-e^2}}$, in which the denominator is constant. Taking the integral of this, we obtain

$$\delta v=\frac{\frac{dr\,.\,\delta r+2r\,.\,d\delta r}{a^2\,n\,dt}+\frac{3}{a^2 n}\,.\iint dt\,.\,\delta\,.\,\mathrm{d}R+\frac{2}{a^2 n}\,.\int r\,.\,\delta R'\,.\,dt}{\sqrt{1-e^2}};$$

Substituting in the two last terms, for $\frac{1}{a^2 n}$, its value $\frac{an}{\mu}$, [605′]; introducing the constant quantity n, under the signs of integration; putting, as in [925‴], $\mathrm{d}R$ for $\delta\,\mathrm{d}R$, R' for $\delta R'$, and then making $r R'=r\,.\left(\frac{dR}{dr}\right)$, [928′], it becomes as in [931].

It now remains to determine the perturbations of the motion in latitude. For this purpose we shall resume the third of the equations (P) [915]. Integrating it, as we have done the equation (S) [919], and putting $z = r\delta s$, [931''] we shall have*

$$\delta s = \frac{a.\cos.v.\int n dt.r.\sin.v.\left(\frac{dR}{dz}\right) - a.\sin.v.\int n dt.r.\cos.v.\left(\frac{dR}{dz}\right)}{\mu.\sqrt{1-e^2}}; \qquad (Z) \qquad [932]$$

δs is the latitude of m above the plane of its primitive orbit. If we wish to refer the motion of m to a plane which is a little inclined to that orbit, we may put s for the latitude, when it is supposed not to quit the plane of the [932'] primitive orbit, and then $s+\delta s$ will be very nearly the latitude of m above the proposed plane.

47. The formulas (X), (Y), (Z), [930, 931, 932], have the advantage of presenting the perturbations under a finite form. This is very useful in the theory of comets, in which those perturbations cannot be found, except [932''] by the quadrature of curves. But the smallness of the excentricities, and the inclinations of the orbits of the planets to each other, enables us to develop their perturbations, in converging series, of sines and cosines of angles, increasing in proportion to the time, and we can then arrange them in tables which will answer for an indefinite time. Instead of the preceding expressions [930, 932] of δr and δs, it will, for this purpose, be more convenient to use the differential equations, by which these variable [932'''] quantities are determined. If we arrange these equations according to the powers and products of the excentricities and inclinations of the orbits, we

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* (649) When the fixed plane is supposed to be that of the orbit at a given epoch, s will become δs, and the elevation of the point B, in the figure, page 508, above the fixed plane AMA' will be nearly $r\delta s$, substituting this for z in the last of the equations [915], it becomes $0 = \frac{d^2.(r\delta s)}{dt^2} + \frac{\mu.(r\delta s)}{r^3} + \left(\frac{dR}{dz}\right)$, which is of the same form as the equation [926], changing δr into δs and $2\int dR + rR'$, or $2\int dR + r.\left(\frac{dR}{dr}\right)$ into $\left(\frac{dR}{dz}\right)$, and by making these changes in the value of δr, [930], deduced from [926], we shall obtain the value of δs, [932].

may always reduce the computation of the values of δr and δs, to the integration of equations of this form,

[933] $$0 = \frac{d^2 y}{d t^2} + n^2 y + Q\,;$$

the integral of which has been given in § 42 [870]. This very simple form may be given to the preceding differential equations in the following direct manner.

Let us resume the equation (R) [917] of the preceding article, putting for brevity

[934] $$Q = 2 \int \mathrm{d} R + r \,.\left(\frac{d R}{d r}\right);$$

by this means it becomes*

[935] $$0 = \tfrac{1}{2} \,.\, \frac{d^2 \,.\, r^2}{d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + Q. \qquad (R')$$

In the case of the elliptical motion, where $Q = 0$, r^2 is by § 22, a function†
[935′] of $e \,.\, \cos.\,(n t + \varepsilon - \varpi)$, $a e$ being the excentricity of the orbit, and $n t + \varepsilon - \varpi$ the mean anomaly of the planet. Put

[935″] $$e \,.\, \cos.\,(n t + \varepsilon - \varpi) = u\,;$$
$$r^2 = \varphi\,(u)\,;$$

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* 650) Substituting in [917] the value $r R'$, [918], and then its value $r \,.\left(\frac{d R}{d r}\right)$, [928′], neglecting terms of the order of the squares of the disturbing forces, we shall get [935], using the abridged symbol Q, [934].

† (651) If in the value of $\frac{r}{a}$, [659], we change $n t$ into $n t + \varepsilon - \varpi$, as in [669], and put for brevity $n t + \varepsilon - \varpi = z'$, we shall have r in a series of the following form

$$r = A + B \,.\, \cos.\, z' + C \,.\, \cos.\, 2\, z' + D \,.\, \cos.\, 3\, z' + F \,.\, \cos.\, 4\, z' + \&c.$$

Now from [6, 7, &c.] Int. we find $\cos.\, 2\, z' = 2 \cos.\, z'^2 - 1$, $\cos.\, 3\, z' = 4 \cos.\, z'^3 - 3 \cos.\, z'$, and so on for $\cos.\, 3\, z' \,.\, \cos.\, 4\, z'$, &c., all of which will be expressed in terms of $\cos.\, z'$, and its powers. The general form of such expressions may be obtained by taking half the sum of the formulas [15, 16] Int., from which we get

[935a] $$\cos.\, n\, z' = \tfrac{1}{2} \,.\, \{\cos.\, z' + \sqrt{-1} \,.\, \sin.\, z'\}^n + \tfrac{1}{2} \,.\, \{\cos.\, z' - \sqrt{-1} \,.\, \sin.\, z'\}^n.$$

Developing the second member, and putting $\sin.\, z'^2 = 1 - \cos.\, z'^2$, we shall find $\cos.\, n z'$, in terms of $\cos.\, z'$, and its powers. Therefore the preceding value of r will become a function of $\cos.\, z'$, or of $\frac{u}{e}$, or simply a function of u, as in [935″].

and we shall have*

$$0 = \left(\frac{d^2 u}{d t^2}\right) + n^2 u. \qquad [936]$$

In case the motion is troubled, we may also put $r^2 = \varphi(u)$, but u will not then be equal to $e \,.\, \cos.\, (n t + \varepsilon - \varpi)$; but will be given by the preceding [936′] differential equation [936], increased by a term depending on these disturbing forces. To ascertain this term, we shall observe that if we put

$$u = \psi(r^2)\,; \qquad [936'']$$

we shall have†

$$\frac{d d u}{d t^2} + n^2 \,.\, u = \frac{d^2 \,.\, r^2}{d t^2} \,.\, \psi'(r^2) + \frac{4 r^2 \,.\, d r^2}{d t^2} \,.\, \psi''(r^2) + n^2 \,.\, \psi(r^2)\,; \qquad [937]$$

$\psi'(r^2)$ being the differential of $\psi(r^2)$ divided by $d \,.\, r^2$, and $\psi''(r^2)$ the [937′] differential of $\psi'(r^2)$ divided by $d \,.\, r^2$. The equation (R') [935] gives $\frac{d^2 \,.\, r^2}{d t^2}$ equal to a function of r, increased by a quantity depending on the disturbing force.‡ If we multiply this equation by $2 r d r$, and then

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* (652) Having $u = e \,.\, \cos.\, (n t + e - \varpi)$, its second differential gives

$$\frac{d d u}{d t^2} = -n^2 e \,.\, \cos.\, (n t + \varepsilon - \varpi) = -n^2 u,$$

transposing $-n^2 u$, we get [936].

† (652a) Substituting in the first member of [937] the value of u, [936″], it becomes as in the second member.

‡ (653) The equation [935] gives $\frac{d^2 \,.\, r^2}{d t^2} = 2 \,.\, \left(\frac{\mu}{r} - \frac{\mu}{a}\right) - 2\,Q = f(r) - 2\,Q$, putting for brevity, $2 \,.\, \left(\frac{\mu}{r} - \frac{\mu}{a}\right)$ equal to a function of r denoted by $f(r)$. Multiplying this by $2\, d \,.\, r^2$ or $4\, r\, d r$, it becomes $\frac{2 d \,.\, r^2 \,.\, d^2 \,.\, r^2}{d t^2} = 2 \,.\, f(r) \,.\, d \,.\, r^2 - 8\, Q \,.\, r\, d r$, whose integral, by putting $\int 2 f(r) \,.\, d \,.\, r^2 = F(r)$ is

$$\frac{(d \,.\, r^2)^2}{d t^2} = F(r) - \int 8\, Q \,.\, r\, d r \quad \text{or} \quad \frac{4 r^2 \,.\, d r^2}{d t^2} = F(r) - 8 \int Q \,.\, r\, d r.$$

The parts of these expressions depending on Q are as in [938]. Substituting the complete values in [937], it becomes

$$\frac{d d u}{d t^2} + n^2 u = \{f(r) \,.\, \psi'(r^2) + F(r) \,.\, \psi''(r^2) + n^2 \,.\, \psi(r^2)\} - 2\,Q \,.\, \psi'(r^2) - 8 \,.\, \psi''(r^2) \,.\, \int Q \,.\, r d r.$$

integrate it, we shall have $\frac{r^2 \, . \, d r^2}{d t^2}$ equal to a function of r, increased by a quantity depending on the disturbing force. Substituting these values of
[937''] $\frac{d^2 \, . \, r^2}{d t^2}$ and $\frac{r^2 \, . \, d r^2}{d t^2}$ in the preceding expression of $\frac{d^2 u}{d t^2} + n^2 u$; we shall find that the function of r, independent of the disturbing force, will disappear of itself, since it is identically nothing, when the force is nothing; we shall therefore have the value of $\frac{d d u}{d t^2} + n^2 u$, by substituting in it, instead of
[937'''] $\frac{d^2 \, . \, r^2}{d t^2}$, and $\frac{r^2 \, . \, d r^2}{d t^2}$, the parts of these expressions, which depend on the disturbing force. But by noticing only these parts, the equation (R') and its integral will give

[938]
$$\frac{d^2 \, . \, r^2}{d t^2} = -2Q;$$
$$\frac{4 r^2 \, . \, d r^2}{d t^2} = -8 \int Q \, . \, r \, d r;$$

therefore

[939]
$$\frac{d d u}{d t^2} + n^2 u = -2 Q \, . \, \psi'(r^2) - 8 \psi''(r^2) \, . \int Q \, . \, r \, d r.$$

[939'] Now from the equation $u = \psi(r^2)$ [936''], we deduce $d u = 2 r \, d r \, . \, \psi'(r^2)$; and $r^2 = \varphi(u)$ [935''], gives $2 r \, d r = d u \, . \, \varphi'(u)$, consequently,*

[940]
$$\psi'(r^2) = \frac{1}{\varphi'(u)}.$$

Taking the differential of this equation, and substituting $\varphi'(u)$ for $\frac{2 r \, d r}{d u}$, we shall obtain†

Now when $Q = 0$, the term $f(r) \, . \, \psi'(r^2) + F(r) \, . \, \psi''(r^2) + n^2 \, . \, \psi(r^2)$ of the second member must, by [936], be equal to nothing, and as this is simply a function of r and constant quantities, it must be identically nothing, and the preceding expression will become as in [939].

* (654) Substituting in $d u = 2 r \, d r \, . \, \psi'(r^2)$, the value of $2 r \, d r = d u \, . \, \varphi'(u)$, and dividing by $d u \, . \, \varphi'(u)$, we get [940].

† (655) The differential of [940] is $2 r \, d r \, . \, \psi''(r^2) = \frac{-d u \, . \, \varphi''(u)}{\varphi'(u)^2}$, substituting $2 r \, d r = d u \, . \, \varphi'(u)$, and dividing by $d u \, . \, \varphi'(u)$, we obtain [941].

$$\psi''(r^2) = -\frac{\varphi''(u)}{\varphi'(u)^3};$$ [941]

$\varphi''(u)$ being equal to $\frac{d.\varphi'(u)}{du}$, in the same manner as $\varphi'(u)$ is equal to [941′] $\frac{d.\varphi(u)}{du}$. This being premised; if we put

$$u = e.\cos.(nt + \varepsilon - \varpi) + \delta u,$$ [942]

the differential equation in u will become*

$$0 = \frac{d^2.\delta u}{dt^2} + n^2.\delta u - \frac{4.\varphi''(u)}{\varphi'(u)^3}.\int Q.du.\varphi'(u) + \frac{2Q}{\varphi'(u)};$$ [943]

and if we neglect the square of the disturbing force, u may be supposed equal [943′] to $e.\cos.(nt + \varepsilon - \varpi)$ in the terms depending on Q.

The value of $\frac{r}{a}$ found in § 22 [669], gives, by carrying the approximation to quantities of the order e^3 inclusively†

$$r = a.\{1 + e^2 - u.(1 - \tfrac{3}{2}e^2) - u^2 - \tfrac{3}{2}u^3\};$$ [944]

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* (656) From u [942] we get $\frac{ddu}{dt^2} = -n^2 e.\cos.(nt + \varepsilon - \varpi) + \frac{d^2.\delta u}{dt^2}$, hence $\frac{ddu}{dt^2} + n^2 u = \frac{dd.\delta u}{dt^2} + n^2.\delta u$, which being substituted in [939], as also the values of $\psi'(r^2)$, $\psi''(r^2)$ [940, 941], and $2rdr = du.\varphi'(u)$, [939′], we shall obtain by reduction the expression [943].

† (657) In finding $\varphi(u)$ or r^2, and its differentials to be used in [943], we may, as in [943′], put $u = e.\cos.(nt + \varepsilon - \varpi)$, which gives $\cos.(nt + \varepsilon - \varpi) = \frac{u}{e}$, also by [944a] [6, 7] Int. $\cos.(2nt + 2\varepsilon - 2\varpi) = 2.\cos.^2(nt + \varepsilon - \varpi) - 1 = \frac{2u^2}{e^2} - 1$, and

$$\cos.3.(nt + \varepsilon - \varpi) = 4.\cos.^3(nt + \varepsilon - \varpi) - 3.\cos.(nt + \varepsilon - \varpi) = \frac{4u^3}{e^3} - \frac{3u}{e}.$$

These values being substituted in r, [659], altered as in [669], namely

$$r = a.\left\{\begin{array}{l} 1 + \frac{1}{2}e^2 - e.\cos.(nt + \varepsilon - \varpi) - \frac{e^2}{2}.\cos.2.(nt + \varepsilon - \varpi) \\ -\frac{3e^3}{8}.[\cos.3.(nt + \varepsilon - \varpi) - \cos.(nt + \varepsilon - \varpi)] \end{array}\right\},$$

it becomes $r = a.\left\{1 + \frac{1}{2}e^2 - u - \frac{e^2}{2}.\left(\frac{2u^2}{e^2} - 1\right) - \frac{3e^3}{8}.\left(\frac{4u^3}{e^3} - \frac{4u}{e}\right)\right\}$, which, by reduction, is as in [944], terms of the order e^4 or u^4, being always neglected; squaring this we get r^2, which, by [935″], is equal to $\varphi(u)$.

hence we deduce

[945] $$r^2 = a^2 . \{1 + 2e^2 - 2u . (1 - \tfrac{1}{2}e^2) - u^2 - u^3\} = \varphi(u).$$

If we substitute this value of $\varphi(u)$, in the differential equation in δu [943], and resubstitute $Q = 2\int \mathrm{d}R + r . \left(\frac{dR}{dr}\right)$ [934], and $u = e . \cos.(nt + \varepsilon - \varpi)$; we shall have, by neglecting quantities of the order e^3,*

$$0 = \frac{d^2 . \delta u}{dt^2} + n^2 \delta u \qquad (X')$$

[946] $$- \frac{1}{a^2} . \{1 + \tfrac{1}{4}e^2 - e . \cos.(nt + \varepsilon - \varpi) - \tfrac{1}{4}e^2 . \cos.2(nt + \varepsilon - \varpi)\} . \left\{2\int \mathrm{d}R + r . \left(\frac{dR}{dr}\right)\right\}$$

$$- \frac{2e}{a^2} . \int n\,dt . \left[\sin.(nt + \varepsilon - \varpi) . [1 + e . \cos.(nt + \varepsilon - \varpi)] . \left\{2\int \mathrm{d}R + r . \left(\frac{dR}{dr}\right)\right\}\right].$$

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* (657a) Taking the first and second differentials of $\varphi(u)$, [945], we get

[946a] $$\varphi'(u) = a^2 . \{-2 . (1 - \tfrac{1}{2}e^2) - 2u - 3u^2\}; \qquad \varphi''(u) = a^2 . \{-2 - 6u\}.$$

This value of $\varphi'(u)$ gives

$$\frac{2}{\varphi'(u)} = \frac{-1}{a^2 . \{1 - \tfrac{1}{2}e^2 + u + \tfrac{3}{2}u^2\}} = - \frac{1}{a^2} . \left\{1 + \tfrac{1}{2}e^2 - u - \tfrac{1}{2}u^2\right\},$$

neglecting terms of the order e^3 or u^3. If in this we substitute for u, $\tfrac{1}{2}u^2$ their values [944a],

$$-u = -e . \cos.(nt + \varepsilon - \varpi), \qquad -\tfrac{1}{2}u^2 = -\tfrac{1}{4}e^2 - \tfrac{1}{4}e^2 . \cos.(2nt + 2\varepsilon - 2\varpi),$$

it becomes

$$\frac{2}{\varphi'(u)} = - \frac{1}{a^2} \left\{1 + \tfrac{1}{4}e^2 - e . \cos.(nt + \varepsilon - \varpi) - \tfrac{1}{4}e^2 . \cos.(2nt + 2\varepsilon - 2\varpi)\right\}.$$

In like manner we obtain $- \frac{4\varphi''(u)}{\varphi'(u)^3} = \frac{-4a^2 . (-2 - 6u)}{\{a^2 . [-2 . (1 - \frac{1}{2}e^2) - 2u \,\&c.]\}^3}$, or, by neglecting terms of the order e^2, u^2,

$$- \frac{4\varphi''(u)}{(\varphi' u)^3} = \frac{-4a^2 . (-2 - 6u)}{a^6 . \{-2 - 2u\}^3} = \frac{8 + 24u}{a^4 . (-2 - 2u)^3} = \frac{-(8 + 24u)}{a^4 . (8 + 24u)} = - \frac{1}{a^4}.$$

In the term $\int Q\,du . \varphi'(u)$, we may put $du = -e . n\,dt . \sin.(nt + \varepsilon - \varpi)$, [944a], $\varphi'(u) = a^2 . (-2 - 2u) = a^2 . \{-2 - 2e . \cos.(nt + \varepsilon - \varpi)\}$, and it becomes

$$\int Q\,du . \varphi' . (u) = \int e . n\,dt . \sin.(nt + \varepsilon - \varpi) . a^2 . \{+2 + 2e . \cos.(nt + \varepsilon - \varpi)\} . Q$$
$$= 2a^2 e . \int n\,dt . \sin.(nt + \varepsilon - \varpi) . \{1 + e . \cos.(nt + \varepsilon - \varpi)\} . Q.$$

Now by substituting, in [943], the preceding values of $- \frac{4\varphi''(u)}{\varphi'(u)^3}$; $\frac{2}{\varphi'(u)}$; $\int Q\,du . \varphi'(u)$; and Q [934] we shall get [946], which is exact in terms of the second degree in e, because the term $\frac{-4\varphi''(u)}{\varphi'(u)^3}$, which was computed to the first degree, is multiplied by $du . \varphi'(u)$, of the first degree, so that their product must be exact in the second degree.

When we have found δu, by means of this differential equation, we shall have δr by taking the differential of r relative to the characteristic δ, which gives*

$$\delta r = -a\delta u.\{1+\tfrac{3}{4}e^2+2e.\cos.(nt+\varepsilon-\varpi)+\tfrac{9}{4}e^2.\cos.(2nt+2\varepsilon-2\varpi)\}. \qquad [947]$$

This value of δr will give δv, by means of the formula (Y) [931] of the preceding article.

It remains now to determine δs. If we compare the formulas (X), (Z), [930, 932] of the preceding article, we shall find that δr is changed into δs, by writing $\left(\frac{dR}{dz}\right)$ for $2\int \mathrm{d}R+r.\left(\frac{dR}{dr}\right)$ in [930]; hence it follows, [947′] that to obtain δs, it will be sufficient to make this change in the differential equation in δu [946], and then to substitute the value of δu, given by this equation, in [947]. This value of δu we shall denote by $\delta u'$, and we shall have, [946, 947],†

$$\begin{aligned} 0 = {} & \frac{d^2.\delta u'}{dt^2}+n^2.\delta u' \\ & -\frac{1}{a^2}.\{1+\tfrac{1}{4}e^2-e.\cos.(nt+\varepsilon-\varpi)-\tfrac{1}{4}e^2.\cos.(2nt+2\varepsilon-2\varpi)\}.\left(\frac{dR}{dz}\right) \\ & -\frac{2e}{a^2}.\int ndt.\left\{\sin.(nt+\varepsilon-\varpi).\{1+e.\cos.(nt+\varepsilon-\varpi)\}.\left(\frac{dR}{dz}\right)\right\}; \qquad (Z') \\ \delta s = {} & -a\delta u'.\{1+\tfrac{3}{4}e^2+2e.\cos.(nt+\varepsilon-\varpi)+\tfrac{9}{4}e^2.\cos.(2nt+2\varepsilon-2\varpi)\}. \end{aligned} \qquad [948]$$

The system of equations (X'), (Y), (Z'), [946, 931, 948], will give in a very simple manner the motion of m, taking notice only of the first power of the disturbing force. The consideration of the terms of this order, is very [948′] nearly sufficient in the theory of the planets; we shall therefore proceed to deduce, from these equations, such formulas as may be convenient, for the determination of the planetary motions.

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* (658) The differential of the value of r, [944], taken relative to the characteristic δ, is $\delta r = -a\delta u.\{1-\tfrac{3}{2}e^2+2u+\tfrac{9}{2}u^2\}$, but by [944*a*], $2u=2e.\cos.(nt+\varepsilon-\varpi)$; $\tfrac{9}{2}u^2=\tfrac{9}{4}e^2+\tfrac{9}{4}e^2.\cos.(2nt+2\varepsilon-2\varpi)$, hence δr becomes as in [947].

† (659) That is, we must change in [946], δu into $\delta u'$, and $2\int \mathrm{d}R+r.\left(\frac{dR}{dr}\right)$, into $\left(\frac{dR}{dz}\right)$, this gives the first of the equations [948]. The second is obtained by changing δu into $\delta u'$ in the equation [947].

48. It is necessary, for this purpose, to develop the function R in a series. If we notice only the action of m' upon m, we shall have by § 46 [913]

[949] $$R = \frac{m' . (x x' + y y' + z z')}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}}}.$$

This function is wholly independent of the position of the plane of x and y;

[949'] for the radical $\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$, denoting the distance of m' from m,* is independent of it; therefore the function

$$x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 - 2 x x' - 2 y y' - 2 z z',$$

is also independent of it; but the squares of the radii vectores $x^2 + y^2 + z^2$, $x'^2 + y'^2 + z'^2$, do not depend on the position of that plane, hence the quantity $x x' + y y' + z z'$, and therefore the function R, must be independent of it. Suppose now in this function†

[950] $$x = r . \cos. v; \qquad y = r . \sin. v;$$
$$x' = r' . \cos. v'; \qquad y' = r' . \sin. v';$$

we shall have

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* (660) Developing the terms denoting the square of the distance of m from m', namely, $(x'-x)^2 + (y'-y)^2 + (z'-z)^2$, [109*a*], it becomes

$$x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 - 2 . (x x' + y y' + z z'),$$

which ought to be independent of the situation of the plane of x, y, and as $x^2 + y^2 + z^2 = r^2$, $x'^2 + y'^2 + z'^2 = r'^2$, [914'], they are also independent as observed above. The difference $2 . (x x' + y y' + z z')$, and $x x' + y y' + z z'$, must also be independent, hence every one of these terms of R, and therefore R itself must be independent.

[950*a*] † (661) The values of x, x', y, y', [950], are like those in [927]. They give

$$x^2 + y^2 = r^2; \qquad x'^2 + y'^2 = r'^2;$$

$$x x' + y y' = r r' . (\cos. v . \cos. v' + \sin. v . \sin. v') = r r' . \cos. (v' - v),$$

[24] Int. These being substituted in [949] give [951], by developing $(x'-x)^2 + (y'-y)^2$ as in last note.

$$R = \frac{m' . \{ r r' . \cos. (v' - v) + z z' \}}{(r'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\{ r^2 - 2 r r' . \cos. (v' - v) + r'^2 + (z' - z)^2 \}^{\frac{1}{2}}} . \qquad [951]$$

The orbits of the planets being nearly circular, and but little inclined to each other, we may select the plane of x, y, so that z, z', may be very small. In this case r and r' will differ but very little from the semi-transverse axes a and a' of the elliptical orbits; we shall therefore suppose [951']

$$r = a . (1 + u_{,}) ; \qquad r' = a' . (1 + u'_{,}) ; \qquad [952]$$

$u_{,}$ and $u'_{,}$ being very small quantities. The angles v and v' differ very little from the mean longitudes $n t + \varepsilon$, $n' t + \varepsilon'$; we shall suppose [952']

$$v = n t + \varepsilon + v_{,} ; \qquad v' = n' t + \varepsilon' + v'_{,} ; \qquad [953]$$

$v_{,}$ and $v'_{,}$ being very small angles. Then by reducing R to a series arranged according to the powers and products of $u_{,}$, $v_{,}$, z, $u'_{,}$, $v'_{,}$, and z', this series will be very converging. We shall put* [953']

$$\frac{a}{a'^2} . \cos. (n' t - n t + \varepsilon' - \varepsilon) - \{ a^2 - 2 a a' . \cos. (n' t - n t + \varepsilon' - \varepsilon) + a'^2 \}^{-\frac{1}{2}}$$

$$= \tfrac{1}{2} A^{(0)} + A^{(1)} . \cos. (n' t - n t + \varepsilon' - \varepsilon) + A^{(2)} . \cos. 2 (n' t - n t + \varepsilon' - \varepsilon) \qquad [954]$$

$$+ A^{(3)} . \cos. 3 (n' t - n t + \varepsilon' - \varepsilon) + \&c.$$

$$= \tfrac{1}{2} \Sigma . A^{(i)} . \cos. i . (n' t - n t + \varepsilon' - \varepsilon).$$

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* (662) After substituting the values [952] and [953] in R [951], and developing it according to the powers and products of $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, $z'^2 . (z' - z)^2$. The part which is wholly independent of the quantities $u_{,}$, $u'_{,}$, &c., is evidently equal to the first member of [954] multiplied by m'; and the first term of the factor of $(z' - z)^2 . \frac{m'}{2}$ is equal to the first member of [956]. The form of the series in the second members of [954, 956] is evident from the usual rules of development. Now if for brevity we put

$$T = n' t - n t + \varepsilon' - \varepsilon, \qquad \text{and} \qquad W = f t + \varpi, \qquad [954a]$$

the second member of [954] will become

$$\tfrac{1}{2} A^{(0)} + A^{(1)} . \cos. T + A^{(2)} . \cos. 2 T + A^{(3)} . \cos. 3 T + \&c. ;$$

and as $\cos. T = \cos. (-T)$, $\cos. 2 T = \cos. (-2 T)$, &c., this may be written

$$\tfrac{1}{2} A^{(0)} + \tfrac{1}{2} A^{(1)} . \{ \cos. T + \cos. (-T) \} + \tfrac{1}{2} A^{(2)} . \{ \cos. 2 T + \cos. (-2 T) \} + \&c.,$$

and by putting $A^{(1)} = A^{(-1)}$, $A^{(2)} = A^{(-2)}$, &c., [954''], it changes into

$$\left. \begin{array}{l} \tfrac{1}{2} A^{(0)} + \tfrac{1}{2} A^{(1)} . \cos. T + \tfrac{1}{2} A^{(2)} . \cos. 2 T + \&c. \\ \quad + \tfrac{1}{2} A^{(-1)} . \cos. (-T) + \tfrac{1}{2} A^{(-2)} . \cos. (-2 T) + \&c. \end{array} \right\} ,$$

which is evidently expressed by the general formula $\tfrac{1}{2} \Sigma . A^{(i)} . \cos. i T$, taking i from $-\infty$ to $+\infty$, including $i = 0$.

[954'] We may give to this series the form, $\frac{1}{2}.\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$, in
Use of the symbol Σ which the characteristic of finite integrals Σ, refers to the number i, and includes
all whole numbers from $i=-\infty$, to $i=\infty$; the value $i=0$, being also
[954''] comprised in this infinite number of terms; and we must also put $A^{(-i)}=A^{(i)}$.
This form has not only the advantage of expressing, in a very abridged manner, the preceding series, but it gives also the product of this series by the sine or cosine of any angle $ft+\varpi$; since it is easy to perceive that this product is equal to[*]

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[*] (663) Using T, W, [954a], we shall have, for the two terms of $\frac{1}{2}\Sigma.A^{(i)}.\cos.iT$, depending on any integer i, $\frac{1}{2}A^{(i)}.\cos.iT+\frac{1}{2}A^{(-i)}.\cos.(-iT)$, or as it may be written $A^{(i)}.\cos.iT$, this being multiplied by $\sin.W$, and reduced, as in [18] Int., becomes $\frac{1}{2}A^{(i)}.\sin.(iT+W)+\frac{1}{2}A^{(i)}.\sin.(-iT+W)$, or

$$\tfrac{1}{2}A^{(i)}.\sin.(iT+W)+\tfrac{1}{2}A^{(-i)}.\sin.(-iT+W),$$

which are evidently the two terms depending on the same integer i in $\frac{1}{2}\Sigma.A^{(i)}.\cos.(iT+W)$. Hence

[954b] $$\sin.W.\tfrac{1}{2}\Sigma.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.A^{(i)}.\sin.(iT+W),$$

which is the upper formula of [955]. The lower formula is obtained by writing in the preceding $\frac{1}{2}\pi+W$ for W, $\frac{1}{2}\pi$ being a right angle, for this changes $\sin.W$ into $\cos.W$, and $\sin.(iT+W)$ into $\cos.(iT+W)$, hence we get

[954c] $$\cos.W.\tfrac{1}{2}\Sigma.A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.A^{(i)}.\cos.(iT+W).$$

From these two equations the following may be deduced,

[955a] $$\sin.W.\tfrac{1}{2}\Sigma.iA^{(i)}.\sin.iT=-\tfrac{1}{2}\Sigma.iA^{(i)}.\cos.(iT+W),$$
[955b] $$\cos.W.\tfrac{1}{2}\Sigma.iA^{(i)}.\sin.iT=\tfrac{1}{2}\Sigma.iA^{(i)}.\sin.(iT+W),$$
[955c] $$\sin.W.\tfrac{1}{2}\Sigma.iA^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.iA^{(i)}.\sin.(iT+W),$$
[955d] $$\cos.W.\tfrac{1}{2}\Sigma.iA^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.iA^{(i)}.\cos.(iT+W),$$

For the equations [954b, c], being identical, exist with all values of T; we may therefore take their differentials relative to T, and divide by $-dT$ and we shall obtain [955a, b]; and if in these we change iT into $iT+\frac{1}{2}\pi$, we shall get [955c, d].

In like manner we may take the differentials of the equations [955a—d], relative to T, and dividing by $\pm dT$, we shall obtain the four following equations

[955e] $$\sin.W.\tfrac{1}{2}\Sigma.i^2A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.i^2A^{(i)}.\sin.(iT+W),$$
[955f] $$\cos.W.\tfrac{1}{2}\Sigma.i^2A^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.i^2A^{(i)}.\cos.(iT+W),$$
[955g] $$\sin.W.\tfrac{1}{2}\Sigma.i^2A^{(i)}.\sin.iT=-\tfrac{1}{2}\Sigma.i^2A^{(i)}.\cos.(iT+W),$$
[955h] $$\cos.W.\tfrac{1}{2}\Sigma.i^2A^{(i)}.\sin.iT=\tfrac{1}{2}\Sigma.i^2A^{(i)}.\sin.(iT+W),$$

and so on for others.

$$\tfrac{1}{2}\,\Sigma\,.\,A^{(i)}\,.\,\begin{matrix}\text{sin.}\\ \text{cos.}\end{matrix}\,\{i\,.\,(n't-nt+\varepsilon'-\varepsilon)+ft+\varpi\}. \qquad [955]$$

From this property we may obtain very convenient expressions of the perturbations of the motion of the planets. We shall also suppose

$$\begin{aligned}&\{a^2-2\,a\,a'\,.\,\text{cos.}\,(n't-nt+\varepsilon'-\varepsilon)+a'^2\}^{-\frac{3}{2}}\\ &\quad=\tfrac{1}{2}\,.\,\Sigma\,.\,B^{(i)}\,.\,\text{cos.}\,i\,.\,(n't-nt+\varepsilon'-\varepsilon)\,;\end{aligned} \qquad [956]$$

$B^{(-i)}$ being equal to $B^{(i)}$. This being premised, we shall find, from the theorems of § 21,* [956′]

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* (664) If we develop the terms relative to z, z', in the value of R, [951], it becomes

$$\begin{aligned}R=&\frac{m'r\,.\,\text{cos.}\,(v'-v)}{r'^2}-\frac{m'}{\{r^2-2rr'\,.\,\text{cos.}\,(v'-v)+r'^2\}^{\frac{1}{2}}}+\frac{m'zz'}{r'^3}\\ &-\frac{3\,m'\,.\,rz'^2}{2\,r'^4}\,.\,\text{cos.}\,(v'-v)+\frac{m'\,.\,(z'-z)^2}{2\,.\,\{r^2-2rr'\,.\,\text{cos.}\,(v'-v)+r'^2\}^{\frac{3}{2}}}+\&\text{c.},\end{aligned} \qquad [955i]$$

each of the terms of which may be further developed by substituting for r, r', v, v', their values [952, 953]. It has been supposed sufficiently accurate to put in the three last terms a, a', $nt+\varepsilon$, $n't+\varepsilon'$, for r, r', v, v', and using for brevity T, [954a], they will become $\frac{m'zz'}{a'^3}-\frac{3\,m'a\,z'^2}{2\,a'^4}\,.\,\text{cos.}\,T+\frac{m'\,.\,(z'-z)^2}{2\,.\,(a^2-2\,a\,a'\,.\,\text{cos.}\,T+a'^2)^{\frac{3}{2}}}$. If we substitute, for the [956a] denominator of the last term, its value deduced from [956], the three preceding terms will become like the three last of the expression [957]. The two remaining terms of R, [955i] being taken for u, [607, &c.], we shall have

$$u=\frac{m'r\,.\,\text{cos.}\,(v'-v)}{r'^2}-\frac{m'}{\{r'^2-2rr'\,.\,\text{cos.}\,(v'-v)+r^2\}^{\frac{1}{2}}}, \qquad [956b]$$

and if we use the values [952, 953], putting $\alpha=a\,u_{,}$, $\alpha'=a\,u'_{,}$, $\alpha''=v'_{,}-v_{,}$, we [956c] may, as in [607—612], develop u according to the powers of α, α', α'', observing that the two terms $v'_{,}-v_{,}$ are connected together, because they occur only in this form in u; by this means we shall get

$$\begin{aligned}u=\,&U+\alpha\,.\left(\frac{du}{d\alpha}\right)+\alpha'\,.\left(\frac{du}{d\alpha'}\right)+\alpha''\,.\left(\frac{du}{d\alpha''}\right)+\tfrac{1}{2}\,.\,\alpha\alpha\,.\left(\frac{ddu}{d\alpha^2}\right)+\alpha\alpha'\,.\left(\frac{ddu}{d\alpha\,.\,d\alpha'}\right)\\ &+\tfrac{1}{2}\,.\,\alpha'^2\,.\left(\frac{ddu}{d\alpha'^2}\right)+\alpha\alpha''\,.\left(\frac{ddu}{d\alpha\,.\,d\alpha''}\right)+\alpha'\alpha''\,.\left(\frac{ddu}{d\alpha'\,.\,d\alpha''}\right)+\tfrac{1}{2}\,.\,\alpha''^2\,.\left(\frac{ddu}{d\alpha''^2}\right)+\&\text{c.}\end{aligned} \qquad [957a]$$

in which $U=\frac{m'a}{a'^2}\,.\,\text{cos.}\,T-\frac{m'}{\{a^2-2\,a\,a'\,.\,\text{cos.}\,T+a'^2\}^{\frac{1}{2}}}$, or by [954′], [957b] $=\frac{m'}{2}\,.\,\Sigma A^{(i)}\,.\,\text{cos.}\,i\,T$, this being the value of u, when α, α', α'', are nothing. In the terms $\left(\frac{du}{d\alpha}\right)$, $\left(\frac{du}{d\alpha'}\right)$, &c., we must also put α, α', α'', equal to nothing, or in other words $r=a$, [957c] $r'=a'$, and $v'-v=n't-nt+\varepsilon'-\varepsilon=T$, but as α or $a\,u_{,}$ is found in u, only as it

$$
\begin{aligned}
R = {} & \frac{m'}{2} . \Sigma . A^{(i)} . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m'}{2} . u_{,} . \Sigma . a . \left(\frac{d A^{(i)}}{d a}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m'}{2} . u'_{,} . \Sigma . a' . \left(\frac{d A^{(i)}}{d a'}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& - \frac{m'}{2} . (v'_{,} - v_{,}) . \Sigma . i A^{(i)} . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m'}{4} . u_{,}^{2} . \Sigma . a^{2} . \left(\frac{d d A^{(i)}}{d a^{2}}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m'}{2} . u_{,} u'_{,} . \Sigma . a a' . \left(\frac{d d A^{(i)}}{d a \, d a'}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
\text{[957]} \qquad & + \frac{m'}{4} . u'^{2}_{,} . \Sigma . a'^{2} . \left(\frac{d d A^{(i)}}{d a'^{2}}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& - \frac{m'}{2} . (v'_{,} - v_{,}) . u_{,} . \Sigma . i a . \left(\frac{d A^{(i)}}{d a}\right) . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\
& - \frac{m'}{2} . (v'_{,} - v_{,}) . u'_{,} . \Sigma . i a' . \left(\frac{d A^{(i)}}{d a'}\right) . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\
& - \frac{m'}{4} . (v'_{,} - v_{,})^{2} . \Sigma . i^{2} . A^{(i)} . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m' . z z'}{a'^{3}} - \frac{3 m' . a z'^{2}}{2 a'^{4}} . \cos . (n't - nt + \varepsilon' - \varepsilon) \\
& + \frac{m' . (z' - z)^{2}}{4} . \Sigma . B^{(i)} . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\
& + \&c.
\end{aligned}
$$

If we substitute in this expression of R, for $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, z, z', their values relative to the elliptical motion, which values are functions of the sines and

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enters into the values of r, [952], we shall have $\left(\frac{du}{d\alpha}\right) = \left(\frac{du}{dr}\right) . \left(\frac{dr}{d\alpha}\right)$, and $r = a + a u_{,} = a + \alpha$, [952], gives $\left(\frac{dr}{d\alpha}\right) = 1$, hence $\left(\frac{du}{d\alpha}\right) = \left(\frac{du}{dr}\right)$. Again, if we compare the functions u, [956*b*], and U, [957*b*], we shall find that u is composed of r, r', $v' - v$, in exactly the same manner as U is composed of a, a', T, and it is evident from a little consideration that $\left(\frac{dU}{da}\right)$ will be exactly equal to the value of $\left(\frac{du}{dr}\right)$, in which a, a', T, are written respectively for r, r', $v' - v$, as in [957*c*]; therefore in the above value of u, [957*a*], we must put $\left(\frac{du}{d\alpha}\right) = \left(\frac{du}{dr}\right) = \left(\frac{dU}{da}\right)$, and in like manner cosines of the angles $nt+\varepsilon$, $n't+\varepsilon'$, and of their multiples;* R will be expressed by an infinite series of cosines of the form $m'k.\cos.(i'n't-int+A)$, i and i' being whole numbers. [957′]

It is evident that the action of the bodies m'', m''', &c., on m, will produce in R terms like those which result from the action of m', and we may obtain these terms, by changing, in the preceding expression of R, all the quantities which refer to m', into the corresponding quantities relative to m'', m''', &c. [957″]

We shall now consider any term of R, represented generally by

$$m'k\,.\cos.\,(i'n't-int+A). \qquad [957'''] $$

If the orbits were circular and in the same plane, we should have† $i'=i$;

$$\left(\frac{du}{d\alpha'}\right)=\left(\frac{du}{dr'}\right)=\left(\frac{dU}{da'}\right);\quad \left(\frac{du}{d\alpha''}\right)=\left(\frac{du}{d.(v'-v)}\right)=\left(\frac{dU}{dT}\right);\quad \left(\frac{ddu}{d\alpha^2}\right)=\left(\frac{ddu}{dr^2}\right)=\left(\frac{ddU}{da^2}\right);$$

$$\left(\frac{ddu}{d\alpha d\alpha'}\right)=\left(\frac{ddu}{drdr'}\right)=\left(\frac{ddU}{dada'}\right);\quad \left(\frac{ddu}{d\alpha'^2}\right)=\left(\frac{ddU}{da'^2}\right),\quad \left(\frac{ddu}{d\alpha d\alpha''}\right)=\left(\frac{ddU}{dadT}\right); \qquad [957d]$$

$$\left(\frac{ddu}{d\alpha'd\alpha''}\right)=\left(\frac{ddU}{da'dT}\right);\qquad \left(\frac{ddu}{d\alpha''^2}\right)=\left(\frac{ddU}{dT^2}\right);$$

which being substituted in u, [957*a*] we get

$$u=U+\alpha.\left(\frac{dU}{da}\right)+\alpha'.\left(\frac{dU}{da'}\right)+\alpha''.\left(\frac{dU}{dT}\right)+\tfrac{1}{2}\alpha^2.\left(\frac{ddU}{da^2}\right)+\alpha\alpha'.\left(\frac{ddU}{dada'}\right)$$
$$+\tfrac{1}{2}\alpha'^2.\left(\frac{ddU}{da'^2}\right)+\alpha\alpha''.\left(\frac{ddU}{dadT}\right)+\alpha'\alpha''.\left(\frac{ddU}{da'dT}\right)+\tfrac{1}{2}\alpha''^2.\left(\frac{ddU}{dT^2}\right). \qquad [957e]$$

Substituting in this the value of $U=\frac{m'}{2}.\Sigma A^{(i)}.\cos.\,i\,T$, [957*b*], and those of α,α',α'', [956*c*], it becomes, term for term, like those in the ten first lines of [957]. These, with the terms in z, z', above found, [956*a*], constitute the complete value of R, [957].

* (665) This is evident from the equations [669], or from [659] and [668].

† (666) The orbits being circular, we should have $u=0$, $u'=0$, [952], also $r=a$, $r'=a'$. Moreover, the motions being uniform, and in the same plane, we shall have $v_{,}=0$, $v_{,}'=0$, [953]; there being no reductions like those in [675, 676′]; hence $v'-v=n't-nt+\varepsilon'-\varepsilon=T$. Substituting these and $z=0$, $z'=0$, in [951] we shall get $R=\frac{m'a.\cos.T}{a'^2}-\frac{m'}{\{a^2-2aa'.\cos.T+a'^2\}^{\frac{1}{2}}}$, which, by [954], is [957*f*] $=\frac{m'}{2}.\Sigma A^{(i)}.\cos.\,i\,(n't-nt+\varepsilon'-\varepsilon)$, and in this last expression the coefficient of $n't$ [957*g*] and nt is the same quantity i.

therefore i' cannot exceed or fall short of i, except by means of the sines or cosines of the expressions of $u_{,}$, $v_{,}$, z, $u'_{,}$, $v'_{,}$, z', which, being combined
[957iv] with the sines and cosines of the angle $n't-nt+\varepsilon'-\varepsilon$, and its multiples, will produce sines and cosines of angles, in which i' differs from i.

If we consider the excentricities and inclinations of the orbits as very small
[957v] quantities of the first order, it follows, from the formulas of § 22,* that in the expressions of $u_{,}$, $v_{,}$, z, or rs, s being the tangent of the latitude of m, [1027a],
[957vi] the coefficient of the sine or cosine of an angle like $f.(nt+\varepsilon)$ is expressed by a series, whose first term is of the order f; the second term of the order
[957vii] $f+2$; the third term of the order $f+4$; and so on. It is the same with the coefficient of the sine or cosine of the angle $f'.(n't+\varepsilon')$ in the expressions of $u'_{,}$, $v'_{,}$, z'. Hence it follows, that if i' and i are supposed positive, and i' greater than i; the coefficient k, in the term

[957viii] $$m'k.\cos.(i'n't-int+A)$$

is of the order $i'-i$, and in the series which expresses it, the first term is of the order $i'-i$, the second term of the order $i'-i+2$, and so on;
[957ix] so that this series is very converging. If i be greater than i', the terms of the series would be successively of the orders $i-i'$, $i-i'+2$, &c.

Let ϖ be the longitude of the perihelion of the orbit of m, θ the longitude
[957x] of its node. In like manner let ϖ' be the longitude of the perihelion of m', and θ' that of its node; these longitudes being counted on a plane but little inclined to that of the orbits. It is evident, by the formulas of § 22, that in the expressions of $u_{,}$, $v_{,}$, [669, 952, 953], and z, [679, &c.], the angle $nt+\varepsilon$ is always accompanied by $-\varpi$ or $-\theta$; and in the expressions of $u'_{,}$,
[957xi] $v'_{,}$, z', the angle $n't+\varepsilon'$ is always accompanied by $-\varpi'$, or $-\theta'$; hence it follows that the term $m'k.\cos.(i'n't-int+A)$, [957'''] is of the following form:

[958] $$m'k.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta-g'''\theta'),$$

[958'] g, g', g'', g''', being whole numbers positive or negative, so that we shall have*

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* (667) The formula [659], altered as in [669] gives $u_{,}$, $u'_{,}$; also [668] altered as in [669], gives $v_{,}$, $v'_{,}$; the reduction to the fixed plane is made as in [675, 676', &c.] Now by examining all these, it will evidently appear, that the order of the coefficient of the sine or cosine of any angle, as $f(nt+\varepsilon)$, is as in [957vi, &c.]

$$0 = i' - i - g - g' - g'' - g''' ; \qquad [959]$$

which also follows from the consideration that the values of R and its different
terms are independent of the position of the right line from which the
longitude is computed. Moreover, in the formulas of § 22, the coefficient of
the sine and cosine of the angle ϖ [669, &c.], has always for a factor the [959']
excentricity e of the orbit of m; the coefficient of the sine and cosine of the
angle 2ϖ, has for a factor the square of the excentricity e^2, and so on.†
Likewise the coefficient of the sine and cosine of the angle θ, [676', &c.],
has for a factor $\text{tang.}\,\frac{1}{2}\varphi$, φ being the inclination of the orbit of m upon the [959'']
fixed plane. The coefficient of the sine and cosine of the angle 2θ, has for
a factor $\text{tang.}^2\,\frac{1}{2}\varphi$, and so on; hence it follows, that the coefficient k has for
a factor $e^{g} \,.\, e'^{g'} \,.\, (\text{tang.}\,\frac{1}{2}\varphi)^{g''} \,.\, (\text{tang.}\,\frac{1}{2}\varphi')^{g'''}$; the numbers g, g', g'', g''', being
taken positively in the exponents of this factor. If all these numbers are [959''']

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* (668) It is shown in [957g], that when $u_{,}, u'_{,}, v_{,}, v'_{,}, z', z$, are nothing, which takes place when $g = 0$, $g' = 0$, $g'' = 0$, $g''' = 0$, the term $i' - i$ must be equal to 0. Now if g' becomes 1, the term i' must be increased by unity, because the change must arise from multiplying the sine or cosine of an angle like $i\,(n't - nt + \varepsilon' - \varepsilon)$, by one depending on $n't + \varepsilon' - \varpi'$, which must increase the coefficient of $n't$, as much as it does that of $-\varpi'$, and must still leave the expression [959] equal to nothing. By following this method we shall easily perceive that the equation [959] always takes place for all values of i', i, g, &c. The same conclusion may also be drawn from the other consideration mentioned relative to the position of the line from which the longitudes are computed. For the function R is not affected by the position of the plane of $x\,y$, [949'], it cannot therefore be affected by the position of the axis of x, from which the angles ε, ε', ϖ, ϖ', θ, θ', are computed. If we now suppose the origin to be altered so as to augment these quantities by the angle b, the angle $i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - g''\theta - g'''\theta'$, of the expression [958] will be varied by $i'b - ib - gb - g'b - g''b - g'''b$, which expression ought to be equal to nothing, in order that the part of R denoted by [958] should remain unaltered. Putting it therefore equal to nothing, and dividing by b, we obtain

$$0 = i' - i - g - g' - g'' - g''', \qquad \text{as in [959].}$$

† (669) This appears from the formulas [659, 668], altered as in [669]. The remarks relative to $(\text{tang.}\,\frac{1}{2}\varphi')^{g''}$, appear from the formulas [675, 676'].

really positive, this factor will be of the order $i'-i$, by means of the equation [959],

[960] $$0=i'-i-g-g'-g''-g''';$$

but if one of them as g is negative and equal to $-g$, this factor will be of the order $i'-i+2g$.* Retaining therefore, among the terms of R, only those depending on the angle $i'n't-int$, which are of the order $i'-i$, [960'] and rejecting those which depend on the same angle, but which are of the orders $i'-i+2$, $i'-i+4$, &c., the expression of R will be composed of terms of the form

[961] $$H\,.\,e^{g}\,.\,e'^{g'}\,.\,(\text{tang.}\,\tfrac{1}{2}\,\varphi)^{g''}\,.\,(\text{tang.}\,\tfrac{1}{2}\,\varphi')^{g'''} \\ .\cos.\,(i'n't-int+i'\varepsilon'-i\varepsilon-g\,.\,\varpi-g'\,.\,\varpi'-g''\,.\,\theta-g'''\,.\,\theta');$$

H being a coefficient independent of the excentricities and inclinations of [961'] the orbits; the numbers g, g', g'', g''', being all positive, and such that their sum is equal to $i'-i$.

If we substitute in R the value of r [952],

[961''] $$r=a\,.\,(1+u_{,}),$$

we shall have†

[962] $$r\,.\left(\frac{dR}{dr}\right)=a\,.\left(\frac{dR}{da}\right).$$

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* (670) Suppose the negative value of g to be $-G$, G being a positive number; the factor k will contain the terms $e^{G}\,.\,e'^{g'}\,.\,(\text{tang.}\,\frac{1}{2}\,\varphi)^{g''}\,.\,(\text{tang.}\,\frac{1}{2}\,\varphi')^{g'''}$, and it will be of the order $G+g'+g''+g'''$. In this case the equation [959] will become $0=i'-i+G-g'-g''-g'''$, which gives $G+g'+g''+g'''=i'-i+2\,G$. consequently that term will be of the order $i'-i+2\,G$, exceeding $i'-i$ by the positive quantity $2\,G$.

† (671) After substituting $r=a\,.\,(1+u_{,})$, [952], in R we must consider $u_{,}$ as not containing a explicitly, and then the partial differential of $r=a\,.\,(1+u_{,})$, relative to a, will give $\left(\frac{dr}{da}\right)=(1+u_{,})$, whence $\left(\frac{dr}{da}\right)=\frac{r}{a}$. Now, by considering R as a function of a and then as a function of r, we shall have $\left(\frac{dR}{da}\right)=\left(\frac{dR}{dr}\right)\cdot\left(\frac{dr}{da}\right)=\left(\frac{dR}{dr}\right)\cdot\frac{r}{a}$, whence we easily get [962]. In like manner, since by [953], $v=nt+\varepsilon+v_{,}$, we may first suppose R to be a function of v and then of $nt+\varepsilon+v_{,}$, and we shall have

If in the same function we substitute, for $u_{,}$, $v_{,}$, z, their values given by the formulas of § 22, we shall have

$$\left(\frac{dR}{dv}\right)=\left(\frac{dR}{d\varepsilon}\right), \qquad [963]$$

[963'] provided we suppose $\varepsilon-\varpi$ and $\varepsilon-\theta$ constant, in the differential of R, taken with respect to ε; for then $u_{,}$, $v_{,}$, z, will be constant in that differential; and as we have $v=nt+\varepsilon+v_{,}$, it is evident that the preceding equation takes place. We may therefore easily obtain the values of $r.\left(\frac{dR}{dr}\right)$, and $\left(\frac{dR}{dv}\right)$, which enter in the differential equations of the preceding articles, [963''] when we shall have the value of R, developed in a series of cosines of angles increasing in proportion to the time t. The differential $\text{d}R$ will likewise be very easy to determine; taking care to vary in R, only the angle nt, supposing $n't$ to be constant; since $\text{d}R$ is the differential of R, taken on [963'''] the supposition that the co-ordinates of m', which are functions of $n't$, are constant [916'].

49. The difficulty of developing R in a series, is therefore reduced to that of forming the quantities $A^{(i)}$, $B^{(i)}$, [954, 956], and their differentials, relative to a or a'. For this purpose we shall consider the function

$$(a^2-2aa'.\cos.\theta+a'^2)^{-s},$$

and develop it according to the cosines of the angle θ, and its multiples; if we put

$$\frac{a}{a'}=\alpha; \qquad [963^{iv}]$$

it will become

$$(a^2-2aa'.\cos.\theta+a'^2)^{-s}=a'^{-2s}.\{1-2\alpha.\cos.\theta+\alpha^2\}^{-s}. \qquad [963^{v}]$$

$\left(\frac{dR}{d\varepsilon}\right)=\left(\frac{dR}{dv}\right).\left(\frac{dv}{d\varepsilon}\right)$; and if we take the differential of $v=nt+\varepsilon+v_{,}$, [953], relative to ε, without varying $v_{,}$, we shall have $\left(\frac{dv}{d\varepsilon}\right)=1$, hence $\left(\frac{dR}{d\varepsilon}\right)=\left(\frac{dR}{dv}\right)$. Now by comparing the second equation [669] with [675, &c.], it appears that in $v_{,}$ the term ε always occurs in the form $\varepsilon-\varpi$ or $\varepsilon-\theta$, we must therefore suppose $\varepsilon-\varpi$, $\varepsilon-\theta$ to be constant, in finding [963] in the method here used.

Suppose

[964] $$(1-2\alpha.\cos.\theta+\alpha^2)^{-s}=\tfrac{1}{2}.b_s^{(0)}+b_s^{(1)}.\cos.\theta+b_s^{(2)}.\cos.2\theta+b_s^{(3)}.\cos.3\theta+\&c.\,;$$

$b_s^{(0)}$, $b_s^{(1)}$, $b_s^{(2)}$, &c., being functions of α and s. If we take the differentials of the logarithms of both sides of this equation, with respect to the quantity θ, we shall have

[965] $$\frac{-2s.\alpha.\sin.\theta}{1-2\alpha.\cos.\theta+\alpha^2}=\frac{-b_s^{(1)}.\sin.\theta-2.b_s^{(2)}.\sin.2\theta-\&c.}{\tfrac{1}{2}.b_s^{(0)}+b_s^{(1)}.\cos.\theta+b_s^{(2)}.\cos.2\theta+\&c.}$$

Multiplying by the denominators, to clear from fractions, and comparing the similar cosines, we shall have in general*

[966] $$b_s^{(i)}=\frac{(i-1).(1+\alpha^2).b_s^{(i-1)}-(i+s-2).\alpha.b_s^{(i-2)}}{(i-s).\alpha}\,; \qquad (a)$$

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* (672) Putting for brevity $\mathcal{N}=b_s^{(1)}.\sin.\theta+2b_s^{(2)}.\sin.2\theta+3b_s^{(3)}.\sin.3\theta+\&c.$ $D=\tfrac{1}{2}b_s^{(0)}+b_s^{(1)}.\cos.\theta+b_s^{(2)}.\cos.2\theta+b_s^{(3)}.\cos.3\theta+\&c.$, the equation [965] will become $\dfrac{-2s.\alpha.\sin.\theta}{1-2\alpha.\cos.\theta+\alpha^2}=\dfrac{-\mathcal{N}}{D}$. Multiplying this by $D.(1-2\alpha.\cos.\theta+\alpha^2)$, and reducing we get $\mathcal{N}.(1+\alpha^2)-2\mathcal{N}.\alpha.\cos.\theta-2Ds.\alpha.\sin.\theta=0$. Resubstituting the values of $\mathcal{N}$, D, performing the multiplications, and putting as in [18, 19] Int.

[966a] $$2.\cos.\theta.\sin.m\theta=\sin.(m+1).\theta+\sin.(m-1).\theta,$$
$$2.\sin.\theta.\cos.m\theta=\sin.(m+1).\theta-\sin.(m-1).\theta,$$

we shall get the following expression of the different terms of the preceding equation. The first line is the value of $\mathcal{N}.(1+\alpha^2)$, the second and third lines are $-2\mathcal{N}\alpha.\cos.\theta$, the fourth and fifth lines are $-2Ds\alpha.\sin.\theta$.

$$(1+\alpha^2).b_s^{(1)}.\sin.\theta+(1+\alpha^2).2b_s^{(2)}.\sin.2\theta+(1+\alpha^2).3b_s^{(3)}.\sin.3\theta+(1+\alpha^2).4b_s^{(4)}.\sin.4\theta+\&c.$$

$$\left\{\begin{array}{llll} & -\alpha.b_s^{(1)}.\sin.2\theta- & 2\alpha.b_s^{(2)}.\sin.3\theta- & 3\alpha.b_s^{(3)}.\sin.4\theta-\&c. \\ -2\alpha.b_s^{(2)}.\sin.\theta- & 3\alpha.b_s^{(3)}.\sin.2\theta- & 4\alpha.b_s^{(4)}.\sin.3\theta- & 5\alpha.b_s^{(5)}.\sin.4\theta-\&c.\end{array}\right\}$$

$$\left\{\begin{array}{llll} -s\alpha.b_s^{(0)}.\sin.\theta- & s\alpha.b_s^{(1)}.\sin.2\theta- & s\alpha.b_s^{(2)}.\sin.3\theta- & s\alpha.b_s^{(3)}.\sin.4\theta-\&c. \\ +s\alpha.b_s^{(2)}.\sin.\theta+ & s\alpha.b_s^{(3)}.\sin.2\theta+ & s\alpha.b_s^{(4)}.\sin.3\theta+ & s\alpha.b_s^{(5)}.\sin.4\theta+\&c.\end{array}\right\}$$

The sum of these three expressions being equal to nothing, the coefficient of each cosine must be equal to nothing. Now the coefficient of $\sin.(i-1).\theta$, in the preceding sum, i being any positive integer greater than unity, is

$$(1+\alpha^2).(i-1).b_s^{(i-1)}-(i-2).\alpha.b_s^{(i-2)}-i\alpha.b_s^{(i)}-s\alpha.b_s^{(i-2)}+s\alpha.b_s^{(i)}\,;$$

for this is evidently the case if i be 2, 3, 4, or 5, and the law of continuation is manifest. Putting this coefficient equal to nothing we get [966].

we shall therefore have $b_s^{(2)}$, $b_s^{(3)}$, &c., when we know $b_s^{(0)}$, $b_s^{(1)}$.

If we change s into $s+1$, in the preceding expression of

$$(1-2\alpha\,.\,\cos.\theta+\alpha^2)^{-s},$$

[964], we shall have

$$(1-2\alpha\,.\,\cos.\theta+\alpha^2)^{-s-1}=\tfrac{1}{2}\,.\,b_{s+1}^{(0)}+b_{s+1}^{(1)}\,.\,\cos.\theta+b_{s+1}^{(2)}\,.\,\cos.2\theta+b_{s+1}^{(3)}\,.\,\cos.3\theta+\&c. \qquad [967]$$

Multiplying both sides of this equation by $1-2\alpha\,.\,\cos.\theta+\alpha^2$, and substituting, for $(1-2\alpha\,.\,\cos.\theta+\alpha^2)^{-s}$, its value [964], we shall have,

$$\begin{aligned}&\tfrac{1}{2}\,.\,b_s^{(0)}+b_s^{(1)}\,.\,\cos.\theta+b_s^{(2)}\,.\,\cos.2\theta+\&c.\\&=(1-2\alpha\,.\,\cos.\theta+\alpha^2)\,.\,\{\tfrac{1}{2}\,.\,b_{s+1}^{(0)}+b_{s+1}^{(1)}\,.\,\cos.\theta+b_{s+1}^{(2)}\,.\,\cos.2\theta+b_{s+1}^{(3)}\,.\,\cos.3\theta+\&c.\};\end{aligned} \qquad [968]$$

hence we deduce, by comparing the similar cosines,*

$$b_s^{(i)}=(1+\alpha^2)\,.\,b_{s+1}^{(i)}-\alpha\,.\,b_{s+1}^{(i-1)}-\alpha\,.\,b_{s+1}^{(i+1)}. \qquad [969]$$

The formula (*a*) [966] gives,†

$$b_{s+1}^{i+1}=\frac{i\,.\,(1+\alpha^2)\,.\,b_{s+1}^{(i)}-(i+s)\,.\,\alpha\,.\,b_{s+1}^{(i-1)}}{(i-s)\,.\,\alpha}; \qquad [970]$$

the preceding expression of $b_s^{(i)}$ [969], will therefore become

$$b_s^{(i)}=\frac{2s\,.\,\alpha\,.\,b_{s+1}^{(i-1)}-s\,.\,(1+\alpha^2)\,.\,b_{s+1}^{(i)}}{(i-s)}. \qquad [971]$$

By changing i into $i+1$ in this equation, we shall have

$$b_s^{(i+1)}=\frac{2s\,.\,\alpha\,.\,b_{s+1}^{(i)}-s\,.\,(1+\alpha^2)\,.\,b_{s+1}^{(i+1)}}{i-s+1}; \qquad [972]$$

and if we substitute for $b_{s+1}^{(i+1)}$, its preceding value [970], we shall find

$$b_s^{(i+1)}=\frac{s\,.\,(i+s)\,.\,\alpha\,.\,(1+\alpha^2)\,.\,b_{s+1}^{(i-1)}+s\,.\,\{2\,.\,(i-s)\,.\,\alpha^2-i\,.\,(1+\alpha^2)^2\}\,.\,b_{s+1}^{(i)}}{(i-s)\,.\,(i-s+1)\,.\,\alpha}. \qquad [973]$$

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* (673) Putting as in [20] Int., $2\cos.\theta\,.\,\cos.m\theta=\cos.(m+1)\,.\,\theta+\cos.(m-1)\,.\,\theta$, [969*a*] and then making the coefficients of the term $\cos.i\theta$, equal to each other, in both members of the equation [968], we shall obtain the expression [969].

† (674) By writing $i+1$ for i, and $s+1$ for s, in [966], we get [970]. Substituting this value of $b_{s+1}^{(i+1)}$ in [969], we shall obtain, by reduction [971].

These two expressions of $b_s^{(i)}$ [971], $b_s^{(i+1)}$ [973], give*

[974] $$b_{s+1}^{(i)}=\frac{\frac{(i+s)}{s}\cdot(1+\alpha^2)\cdot b_s^{(i)}-2\cdot\frac{(i-s+1)}{s}\cdot\alpha\cdot b_s^{(i+1)}}{(1-\alpha^2)^2};\qquad(b)$$

substituting for $b_s^{(i+1)}$, its value, deduced from the equation (a) [966], we shall get†

[975] $$b_{s+1}^{(i)}=\frac{\frac{(s-i)}{s}\cdot(1+\alpha^2)\cdot b_s^{(i)}+\frac{2\cdot(i+s-1)}{s}\cdot\alpha\cdot b_s^{(i-1)}}{(1-\alpha^2)^2};\qquad(c)$$

an expression which might have been deduced from the preceding [974], by changing i into $-i$, and observing that

[975′] $$b^{(i)}=b^{(-i)}.$$

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* (675) Multiplying [971] by $\frac{(i+s)}{s}\cdot(1+\alpha^2)$, and [973] by $-\frac{2\cdot(i-s+1)\cdot\alpha}{s}$, and adding the products we shall get

[974a] $$\begin{aligned}&\left(\frac{i+s}{s}\right)\cdot(1+\alpha^2)\cdot b_s^{(i)}-2\cdot\frac{(i-s+1)}{s}\cdot\alpha\cdot b_s^{(i+1)}\\&\quad=\left\{-\frac{(i+s)}{i-s}\cdot(1+\alpha^2)^2-\frac{2}{(i-s)}\cdot[2\cdot(i-s)\cdot\alpha^2-i\cdot(1+\alpha^2)^2]\right\}\cdot b_{s+1}^{(i)}\\&\quad+\left\{2\,\alpha\cdot\frac{(i+s)}{i-s}\cdot(1+\alpha^2)-2\,\alpha\cdot\frac{(i+s)}{i-s}\cdot(1+\alpha^2)\right\}\cdot b_{s+1}^{(i-1)},\end{aligned}$$

in which the terms of the coefficient of $b_{s+1}^{(i-1)}$ mutually destroy each other; and if we connect together the terms of the coefficient of $b_{s+1}^{(i)}$ multiplied by $(1+\alpha^2)^2$, the second member of [974a], will become

$$\left\{\frac{(i-s)}{i-s}\cdot(1+\alpha^2)^2-4\,\alpha^2\right\}\cdot b_{s+1}^{(i)},\qquad\text{or}\qquad\{(1+\alpha^2)^2-4\,\alpha^2\}\cdot b_{s+1}^{(i)},$$

or simply $(1-\alpha^2)^2\cdot b_{s+1}^{(i)}$. Dividing this by $(1-\alpha^2)^2$ we shall obtain [974].

† (676) Changing i into $i+1$ in [966] gives $b_s^{(i+1)}=\frac{i\cdot(1+\alpha^2)\cdot b_s^{(i)}-(i+s-1)\cdot\alpha\cdot b_s^{(i-1)}}{(i-s+1)\cdot\alpha}$. Multiplying this by $-2\cdot\frac{(i-s+1)}{s}\cdot\alpha$, it becomes

$$-2\cdot\frac{(i-s+1)}{s}\cdot\alpha\cdot b_s^{(i+1)}=-\frac{2\,i}{s}\cdot(1+\alpha^2)\cdot b_s^{(i)}+2\cdot\frac{(i+s-1)}{s}\cdot\alpha\cdot b_s^{(i-1)};$$

substituting this in [974], and reducing, we shall get [975].

Therefore we shall have, by means of this formula, the values of $b_{s+1}^{(0)}$, $b_{s+1}^{(1)}$, $b_{s+1}^{(2)}$, &c., when those of $b_s^{(0)}$, $b_s^{(1)}$, $b_s^{(2)}$, &c., shall be known.

Putting for brevity,

$$\lambda = 1 - 2\,\alpha \,.\, \cos.\,\theta + \alpha^2, \qquad [975'']$$

and then taking the differential of [964]

$$\lambda^{-s} = \tfrac{1}{2} \,.\, b_s^{(0)} + b_s^{(1)} \,.\, \cos.\,\theta + b_s^{(2)} \,.\, \cos.\,2\,\theta + \&c. \qquad [976]$$

relative to α, we shall get

$$-2\,s \,.\, (\alpha - \cos.\,\theta) \,.\, \lambda^{-s-1} = \tfrac{1}{2} \,.\, \frac{d\,b_s^{(0)}}{d\,\alpha} + \frac{d\,b_s^{(1)}}{d\,\alpha} \,.\, \cos.\,\theta + \frac{d\,b_s^{(2)}}{d\,\alpha} \,.\, \cos.\,2\,\theta + \&c.\,; \qquad [977]$$

but we have*

$$-\alpha + \cos.\,\theta = \frac{1 - \alpha^2 - \lambda}{2\,\alpha}\,; \qquad [978]$$

therefore we shall have

$$\frac{s \,.\, (1 - \alpha^2)}{\alpha} \,.\, \lambda^{-s-1} - \frac{s \,.\, \lambda^{-s}}{\alpha} = \tfrac{1}{2} \,.\, \frac{d\,b_s^{(0)}}{d\,\alpha} + \frac{d\,b_s^{(1)}}{d\,\alpha} \,.\, \cos.\,\theta + \&c.\,; \qquad [979]$$

whence we deduce generally†

$$\frac{d\,b_s^{(i)}}{d\,\alpha} = \frac{s \,.\, (1 - \alpha^2)}{\alpha} \,.\, b_{s+1}^{(i)} - \frac{s}{\alpha} \,.\, b_s^{(i)}. \qquad [980]$$

Substituting the value of $b_{s+1}^{(i)}$, given by the formula (*b*) [974], we shall find

$$\frac{d\,b_s^{(i)}}{d\,\alpha} = \left\{ \frac{i + (i + 2\,s) \,.\, \alpha^2}{\alpha \,.\, (1 - \alpha^2)} \right\} \,.\, b_s^{(i)} - \frac{2 \,.\, (i - s + 1)}{1 - \alpha^2} \,.\, b_s^{(i+1)}. \qquad [981]$$

If we take the differential of this equation, we shall have‡

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* (677) Adding $-1 + \alpha^2$, to both sides of the equation [975''], and dividing by $-2\,\alpha$, we shall get [978]. Substituting this in [977] we get [979].

† (678) Substituting in [979] the value λ^{-s} [976], also that of λ^{-s-1}, deduced from the same formula, namely, $\tfrac{1}{2}\,b_{s+1}^{(0)} + b_{s+1}^{(1)} \,.\, \cos.\,\theta + \&c.$; then putting the coefficients of $\cos.\,i\,\theta$ equal to each other, on both sides of that equation, we shall obtain [980].

‡ (679) In finding the differential of the coefficient of $b_s^{(i)}$, it will be convenient to put it under the form $\left\{ \frac{2 \,.\, (i+s) \,.\, \alpha}{1 - \alpha^2} + \frac{i}{\alpha} \right\}$, which is evidently equal to $\frac{i + (i + 2s) \,.\, \alpha^2}{\alpha \,.\, (1 - \alpha^2)}$, and the differential of the first of these expressions, relative to α, being divided by $d\,\alpha$, will give the coefficient of $b_s^{(i)}$, [982].

[982]
$$\frac{d^2 b_s^{(i)}}{d\alpha^2} = \left\{ \frac{i+(i+2s)\,.\,\alpha^2}{\alpha\,.\,(1-\alpha^2)} \right\} . \frac{d b_s^{(i)}}{d\alpha} + \left\{ \frac{2\,.\,(i+s)\,.\,(1+\alpha^2)}{(1-\alpha^2)^2} - \frac{i}{\alpha^2} \right\} . b_s^{(i)}$$
$$- \frac{2\,.\,(i-s+1)}{1-\alpha^2} . \frac{d b_s^{(i+1)}}{d\alpha} - \frac{4\,.\,(i-s+1)\,.\,\alpha}{(1-\alpha^2)^2} . b_s^{(i+1)}.$$

and again taking the differential, we shall get*

[983]
$$\frac{d^3 b_s^{(i)}}{d\alpha^3} = \left\{ \frac{i+(i+2s)\,.\,\alpha^2}{\alpha\,.\,(1-\alpha^2)} \right\} . \frac{d^2 b_s^{(i)}}{d\alpha^2} + 2\,.\left\{ \frac{2\,.\,(i+s)\,.\,(1+\alpha^2)}{(1-\alpha^2)^2} - \frac{i}{\alpha^2} \right\} . \frac{d b_s^{(i)}}{d\alpha}$$
$$+ \left\{ \frac{4\,.\,(i+s)\,.\,\alpha\,.\,(3+\alpha^2)}{(1-\alpha^2)^3} + \frac{2\,i}{\alpha^3} \right\} . b_s^{(i)} - \frac{2\,.\,(i-s+1)}{1-\alpha^2} . \frac{d\,d\,.\,b_s^{(i+1)}}{d\alpha^2}$$
$$- \frac{8\,.\,(i-s+1)\,.\,\alpha}{(1-\alpha^2)^2} . \frac{d\,b_s^{(i+1)}}{d\alpha} - \frac{4\,.\,(i-s+1)\,.\,(1+3\,\alpha^2)}{(1-\alpha^2)^3} . b_s^{(i+1)}.$$

Hence we perceive, that to determine the values of $b_s^{(i)}$, and its successive
[983′] differentials, it is sufficient to know those of $b_s^{(0)}$ and $b_s^{(1)}$. We shall find
these two quantities in the following manner.

If we put c for the number whose hyperbolic logarithm is unity, we may put the expression λ^{-s} [975″] under this form,†

[984]
$$\lambda^{-s} = \left(1 - \alpha\,.\,c^{\theta\,.\,\sqrt{-1}}\right)^{-s} . \left(1 - \alpha\,.\,c^{-\theta\,.\,\sqrt{-1}}\right)^{-s}.$$

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* (680) In finding this differential, it is only necessary to compute the coefficients of $b_s^{(i)}$, $b_s^{(i+1)}$, those of the other terms having been already computed in [982]. Thus if the coefficient of $b_s^{(i)}$, in [981], were put equal to C, the term $C b_s^{(i)}$ would produce in [982] the terms $C\,.\,\frac{d b_s^{(i)}}{d\alpha} + \left(\frac{dC}{d\alpha}\right) . b_s^{(i)}$; and in the expression [983], the terms

$$C\,.\left(\frac{d\,d\,b_s^{(i)}}{d\alpha^2}\right) + 2\,.\left(\frac{dC}{d\alpha}\right) . \frac{d b_s^{(i)}}{d\alpha} + \left(\frac{d\,d\,C}{d\alpha^2}\right) . b_s^{(i)}.$$

Now C and $\left(\frac{dC}{d\alpha}\right)$ having been found in [982], we have only to compute $\left(\frac{d\,d\,C}{d\alpha^2}\right)$ arising from this term, and the similar one from the coefficient of $b_s^{(i+1)}$ in [981].

† (681) Substituting in [975″], $2\cos.\theta = c^{\theta\,.\,\sqrt{-1}} + c^{-\theta\,.\,\sqrt{-1}}$, [12] Int., we shall get

[984a]
$$\lambda = 1 - \alpha\,.\left(c^{\theta\,.\,\sqrt{-1}} + c^{-\theta\,.\,\sqrt{-1}}\right) + \alpha^2 = \left(1 - \alpha\,.\,c^{\theta\,.\,\sqrt{-1}}\right) . \left(1 - \alpha\,.\,c^{-\theta\,.\,\sqrt{-1}}\right),$$

whose power $-s$, gives λ^{-s}, as in [984]. The two factors of this expression being developed, by the binomial theorem, become as in [985].

Developing the second member of this equation, according to the powers of $c^{\theta.\sqrt{-1}}$, and $c^{-\theta.\sqrt{-1}}$; it is evident that the two exponentials $c^{i\theta.\sqrt{-1}}$, and $c^{-i\theta.\sqrt{-1}}$, will have the same coefficient, which we shall denote by k. The sum of these two terms $k.c^{i\theta.\sqrt{-1}}$, and $k.c^{-i\theta.\sqrt{-1}}$, is $2k.\cos.i\theta$ [984'] [12 Int.]; this will be the value of $b_s^{(i)}.\cos.i\theta$ [964]; therefore we shall have $b_s^{(i)}=2k$. Now the expression of λ^{-s} is equal to the product of the two series

$$1+s\alpha.c^{\theta.\sqrt{-1}}+\frac{s.(s+1)}{1.2}.\alpha^2.c^{2\theta.\sqrt{-1}}+\&c.;$$
$$1+s\alpha.c^{-\theta.\sqrt{-1}}+\frac{s.(s+1)}{1.2}.\alpha^2.c^{-2\theta.\sqrt{-1}}+\&c.;$$ [985]

multiplying these two series together, we shall find, in the case of $i=0$,*

$$k=1+s^2.\alpha^2+\left(\frac{s.(s+1)}{1.2}\right)^2.\alpha^4+\&c.;$$ [986]

and in the case of $i=1$,

$$k=\alpha.\left\{s+s.\frac{s.(s+1)}{1.2}.\alpha^2+\frac{s.(s+1)}{1.2}.\frac{s.(s+1).(s+2)}{1.2.3}.\alpha^4+\&c.\right\};$$ [987]

therefore

$$b_s^{(0)}=2.\left\{1+s^2\alpha^2+\left(\frac{s.(s+1)}{1.2}\right)^2.\alpha^4+\left(\frac{s.(s+1).(s+2)}{1.2.3}\right)^2.\alpha^6+\&c.\right\};$$
$$b_s^{(1)}=2\alpha.\left\{s+s.\frac{s.(s+1)}{1.2}.\alpha^2+\frac{s.(s+1)}{1.2}.\frac{s.(s+1).(s+2)}{1.2.3}.\alpha^4+\&c.\right\}.$$ [988]

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* (681a) The two factors [985] being multiplied together, we shall find that the terms free from θ, or in other words, those which correspond to $i=0$, are evidently produced by multiplying each term of the lower series, [985], by that immediately above it, and adding these products together; this gives the expression [986]. In like manner, the coefficient k of the term $kc^{\theta.\sqrt{-1}}$ is found by multiplying each term of the *lower* factor [985], by the term immediately following it in the *upper* factor, and adding these products; the sum will be the value of k in [987]. This product contains therefore the terms

$$k.c^{\theta.\sqrt{-1}}+k.c^{-\theta.\sqrt{-1}}=2k.\cos.\theta, \qquad \text{[12] Int.}$$

Comparing these with [976], we get, $b_s^{(0)}$, $b_s^{(1)}$, as in [988].

To make these series converge, it is necessary that α should be less than unity. This condition can always be satisfied, by putting α equal to the ratio of the least to the greatest, of the distances a, a'; having therefore assumed

[988'] $\alpha = \frac{a}{a'}$ [963iv], we shall suppose a to be less than a'.

In the theory of the motion of the bodies m, m', m'', &c., we shall have

[988''] occasion to ascertain the values of $b_s^{(0)}$, $b_s^{(1)}$, when $s=\frac{1}{2}$, and $s=\frac{3}{2}$. In these two cases, these values converge but slowly, unless α be a small fraction. These series converge more rapidly when $s=-\frac{1}{2}$, and we shall then have [988],

[989]

$$\tfrac{1}{2} b_{-\frac{1}{2}}^{(0)} = 1 + \left(\frac{1}{2}\right)^2 . \alpha^2 + \left(\frac{1.1}{2.4}\right)^2 . \alpha^4 + \left(\frac{1.1.3}{2.4.6}\right)^2 . \alpha^6 + \left(\frac{1.1.3.5}{2.4.6.8}\right)^2 . \alpha^8 + \&c.$$

$$b_{-\frac{1}{2}}^{(1)} = -\alpha . \left\{ 1 - \frac{1.1}{2.4} . \alpha^2 - \frac{1}{4} . \frac{1.1.3}{2.4.6} . \alpha^4 - \frac{1.3}{4.6} . \frac{1.1.3.5}{2.4.6.8} . \alpha^6 - \frac{1.3.5}{4.6.8} . \frac{1.1.3.5.7}{2.4.6.8.10} . \alpha^8 - \&c. \right\}$$

[989'] In the theory of the planets and satellites, it will be sufficient to take the sum of the eleven or twelve first terms, and to neglect the rest; or more accurately, to take the sum of the remaining terms, as a geometrical progression, whose ratio is $1-\alpha^2$.* When we have ascertained, in this

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* (682) The expressions [989] may be put under the following form, in which C_1, C_2, C_3, &c.; D_1, D_2, &c., denote the terms of the series, immediately preceding those, in which these symbols respectively occur,

$$\tfrac{1}{2} b_{-\frac{1}{2}}^{(0)} = 1 + \left(\frac{1}{2}\right)^2 . \alpha^2 + \left(\frac{1}{4}\right)^2 . \alpha^2 . C_1 + \left(\frac{3}{6}\right)^2 . \alpha^2 . C_2 + \left(\frac{5}{8}\right)^2 . \alpha^2 . C_3 + \&c. \ldots\ldots + \left(\frac{2n-1}{2n+2}\right)^2 . \alpha^2 . C_n + \&c.$$

$$b_{-\frac{1}{2}}^{(1)} = -\alpha + \left(\frac{1.1}{2.4}\right) . \alpha^3 + \left(\frac{1}{4} . \frac{3}{6}\right) . \alpha^2 . D_1 + \left(\frac{3}{6} . \frac{5}{8}\right) . \alpha^2 . D_2 + \left(\frac{5}{8} . \frac{7}{10}\right) . \alpha^2 . D_3 + \&c. \ldots\ldots$$

$$+ \left(\frac{(2n-1).(2n+1)}{(2n+2).(2n+4)}\right) . \alpha^2 D_n + \&c.$$

Now, when n is very large, the coefficients of $\alpha^2 C_n$, $\alpha^2 D_n$, are very nearly equal to unity, and then the terms of the upper series following C_n become nearly

$$\alpha^2 . C_n + \alpha^2 . C_{n+1} + \alpha^2 . C_{n+2} + \&c. = \alpha^2 . C_n + \alpha^4 . C_n + \alpha^6 . C_n + \&c.$$

$$= C_n . \{\alpha^2 + \alpha^4 + \&c.\} = \frac{\alpha^2}{1-\alpha^2} . C_n,$$

and the similar terms of the lower series are nearly equal to $\frac{\alpha^2}{1-\alpha^2} . D_n$. Before seeing this publication I had used this method of finding the last terms of the series [989], and had computed the values of $b_{\frac{3}{2}}^{(0)}$, $b_{\frac{3}{2}}^{(1)}$, &c., corresponding to the orbits of the planet Mars and the Earth, by rapidly converging series, like those in [989].

manner, $b^{(0)}_{-\frac{1}{2}}$, and $b^{(1)}_{-\frac{1}{2}}$, we shall obtain $b^{(0)}_{\frac{1}{2}}$, by making $i=0$, and $s=-\tfrac{1}{2}$, [989'']
in the formula (*b*) [974], and we shall find

$$b^{(0)}_{\frac{1}{2}}=\frac{(1+\alpha^2)\,.\,b^{(0)}_{-\frac{1}{2}}+6\,\alpha\,.\,b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}\,. \qquad [990]$$

If in the formula (*c*) [975], we suppose $i=1$, and $s=-\tfrac{1}{2}$, we shall get

$$b^{(1)}_{\frac{1}{2}}=\frac{2\,\alpha\,.\,b^{(0)}_{-\frac{1}{2}}+3\,.\,(1+\alpha^2)\,.\,b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}\,. \qquad [991]$$

With these values of $b^{(0)}_{\frac{1}{2}}$, $b^{(1)}_{\frac{1}{2}}$, we may find, by the preceding formulas, the
values of $b^{(i)}_{\frac{1}{2}}$, and its partial differentials, whatever be the value of i; thence [991']
we may determine the values of $b^{(i)}_{\frac{3}{2}}$ and its differentials. The values of
$b^{(0)}_{\frac{3}{2}}$ and $b^{(1)}_{\frac{3}{2}}$ may be found very easily by the following formulas,*

$$b^{(0)}_{\frac{3}{2}}=\frac{b^{(0)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}\,; \qquad b^{(1)}_{\frac{3}{2}}=-3\,.\,\frac{b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}\,. \qquad [992]$$

Now to find the quantities $A^{(0)}$, $A^{(1)}$, &c., and their differentials, we shall

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* (683) Putting $i=0$, and $s=\tfrac{1}{2}$, in [974], we get

$$b^{(0)}_{\frac{3}{2}}=\frac{(1+\alpha^2)\,.\,b^{(0)}_{\frac{1}{2}}-2\,\alpha\,b^{(1)}_{\frac{1}{2}}}{(1-\alpha^2)^2}\,,$$

and by substituting the values of $b^{(0)}_{\frac{1}{2}}$, $b^{(1)}_{\frac{1}{2}}$, [990, 991], it becomes

$$b^{(0)}_{\frac{3}{2}}=\frac{(1+\alpha^2)\,.\,\left\{(1+\alpha^2)\,.\,b^{(0)}_{-\frac{1}{2}}+6\,\alpha\,.\,b^{(1)}_{-\frac{1}{2}}\right\}-2\,\alpha\,.\,\left\{2\,\alpha\,.\,b^{(0)}_{-\frac{1}{2}}+3\,.\,(1+\alpha^2)\,.\,b^{(1)}_{-\frac{1}{2}}\right\}}{(1-\alpha^2)^4}$$

$$=\frac{(1+\alpha^2)^2-4\,\alpha^2}{(1-\alpha^2)^4}\,.\,b^{(0)}_{-\frac{1}{2}}=\frac{b^{(0)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}\,.$$

Putting $i=1$ and $s=\tfrac{1}{2}$, in [975], we get $b^{(1)}_{\frac{3}{2}}=\dfrac{-(1+\alpha^2)\,.\,b^{(1)}_{\frac{1}{2}}+2\,\alpha\,.\,b^{(0)}_{\frac{1}{2}}}{(1-\alpha^2)^2}$, and by substituting $b^{(0)}_{\frac{1}{2}}$, $b^{(1)}_{\frac{1}{2}}$, [990, 991], it becomes

$$b^{(1)}_{\frac{3}{2}}=\frac{-(1+\alpha^2)\,.\,\left\{2\,\alpha\,.\,b^{(0)}_{-\frac{1}{2}}+3\,.\,(1+\alpha^2)\,.\,b^{(1)}_{-\frac{1}{2}}\right\}+2\,\alpha\,.\,\left\{(1+\alpha^2)\,.\,b^{(0)}_{-\frac{1}{2}}+6\,\alpha\,.\,b^{(1)}_{-\frac{1}{2}}\right\}}{(1-\alpha^2)^4}$$

$$=\frac{-3.(1+\alpha^2)^2+12\,\alpha^2}{(1-\alpha^2)^4}\,.\,b^{(1)}_{-\frac{1}{2}}=\frac{-3\,.\,(1-\alpha^2)^2}{(1-\alpha^2)^4}\,.\,b^{(1)}_{-\frac{1}{2}}=\frac{-3}{(1-\alpha^2)^2}\,.\,b^{(1)}_{-\frac{1}{2}}\,,$$

as in [992].

observe that by the preceding article [954] the series*

[993] $$\tfrac{1}{2}\,.\,A^{(0)}+A^{(1)}\,.\cos.\,\theta+A^{(2)}\,.\cos.\,2\,\theta+\&c.$$

arises from the development of the function

[994] $$\frac{a\,.\cos.\,\theta}{a'^2}-(a^2-2\,a\,a'\,.\cos.\,\theta+a'^2)^{-\frac{1}{2}},$$

in a series of cosines of the angle θ and its multiples. Putting $\frac{a}{a'}=\alpha$, this function becomes [964]

[995] $$-\frac{1}{2a'}\,.\,b_{\frac{1}{2}}^{(0)}+\left(\frac{a}{a'^2}-\frac{1}{a'}\,.\,b_{\frac{1}{2}}^{(1)}\right).\cos.\,\theta-\frac{1}{a'}\,.\,b_{\frac{1}{2}}^{(2)}\,.\cos.\,2\,\theta-\&c.\,;$$

which gives generally

[996] $$A^{(i)}=-\frac{1}{a'}\,.\,b_{\frac{1}{2}}^{(i)}\,;$$

when i is nothing, or greater than 1 independent of its sign. If $i=1$, we shall have

[997] $$A^{(1)}=\frac{a}{a'^2}-\frac{1}{a'}\,.\,b_{\frac{1}{2}}^{(1)}.$$

Hence we get†

[998] $$\left(\frac{d\,A^{(i)}}{d\,a}\right)=-\frac{1}{a'}\,.\,\frac{d\,b_{\frac{1}{2}}^{(i)}}{d\,\alpha}\,.\left(\frac{d\,\alpha}{d\,a}\right);$$

now we have $\left(\frac{d\,\alpha}{d\,a}\right)=\frac{1}{a'}$; therefore

[999] $$\left(\frac{d\,A^{(i)}}{d\,a}\right)=-\frac{1}{a'^2}\,.\,\frac{d\,b_{\frac{1}{2}}^{(i)}}{d\,\alpha}\,;$$

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* (684) Putting $\frac{a}{a'}=\alpha$, in [994], it becomes $\frac{a.\cos.\theta}{a'^2}-\frac{1}{a'}.(1-2\,\alpha.\cos.\theta+\alpha^2)^{-\frac{1}{2}}$, and this, by [964], is equal to

$$\frac{a\,.\cos.\,\theta}{a'^2}-\frac{1}{a'}\,.\left\{\tfrac{1}{2}b_{\frac{1}{2}}^{(0)}+b_{\frac{1}{2}}^{(1)}\,.\cos.\,\theta+b_{\frac{1}{2}}^{(2)}\,.\cos.\,2\,\theta+\&c.\right\},$$

as in [995]. Now as this is equal to the expression [993], we shall get, by comparing the terms depending on the same multiple of θ, the equations [996, 997].

† (685) Taking the differential of [996], relative to a, always considering $b_{\frac{1}{2}}^{(i)}$, as a function of α, and α as a function of a, we get [998], and as $\alpha=\frac{a}{a'}$, [963$^{\text{iv}}$], we shall have $\left(\frac{d\,\alpha}{d\,a}\right)=\frac{1}{a'}$, which, being substituted in [998], gives [999].

and in the case of $i=1$, we shall have*

$$\left(\frac{d\,A^{(1)}}{d\,a}\right)=\frac{1}{a'^2}\cdot\left\{1-\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}\right\}. \qquad [1000]$$

Lastly we have, even when $i=1$,

$$\left(\frac{d^2 A^{(i)}}{d\,a^2}\right)=-\frac{1}{a'^3}\cdot\frac{d^2 b_{\frac{1}{2}}^{(i)}}{d\,\alpha^2}\,;$$

$$\left(\frac{d^3 A^{(i)}}{d\,a^3}\right)=-\frac{1}{a'^4}\cdot\frac{d^3 b_{\frac{1}{2}}^{(i)}}{d\,\alpha^3}\,. \qquad [1001]$$

To obtain the differentials of $A^{(i)}$ relative to a', we shall observe that $A^{(i)}$ [1001′] being a homogeneous function in a and a', of the dimension -1, we shall have, by the nature of such functions,†

$$a\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)+a'\,.\left(\frac{d\,A^{(i)}}{d\,a'}\right)=-A^{(i)}\,; \qquad [1002]$$

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* (686) Taking the differential of [997], relative to a, dividing it by $d\,a$, and putting $\left(\frac{d\alpha}{da}\right)=\frac{1}{a'}$, as in the last note, we shall get [1000]. In like manner, the differential of [999], will give the first equation [1001], which is correct even when $i=1$, because the differential of [1000] agrees with [1001]. The differential of the first equation [1001] gives the second, and so on, always substituting $\left(\frac{d\alpha}{da}\right)=\frac{1}{a'}$.

† (687) A homogeneous function of a, a', &c., of the degree denoted by m, is a function **Homogeneous function.** in which the sum of the exponents of a, a', &c., noticing their signs, is equal to m, in every term of the function; as for example $a^3+3\,a^2\,a'+a\,a'\,a''+\frac{a'^5}{a^2}$, is a homogeneous function in a, a', a'', of the third degree. If we put $a=t\,y$, $a'=t\,y'$, $a''=t\,y''$, &c., in a homogeneous function $A^{(i)}$ of the degree m, it will become of the form $A^{(i)}=t^m V$, V being [1000a] a function of y, y', y'', &c., independent of t. Taking now the differential of this expression relative to t, and observing that $A^{(i)}$ contains t, only as it is found in a, a', &c., we shall get

$$\left(\frac{d\,A^{(i)}}{d\,a}\right)\cdot\left(\frac{d\,a}{d\,t}\right)+\left(\frac{d\,A^{(i)}}{d\,a'}\right)\cdot\left(\frac{d\,a'}{d\,t}\right)+\left(\frac{d\,A^{(i)}}{d\,a''}\right)\cdot\left(\frac{d\,a''}{d\,t}\right)+\&c.=m\,t^{m-1}\,.\,V. \qquad [1000b]$$

But $a=t\,y$, gives $\left(\frac{d\,a}{d\,t}\right)=y$; hence $t\,.\left(\frac{d\,a}{d\,t}\right)=t\,y=a$; in like manner

$$t\,.\left(\frac{d\,a'}{d\,t}\right)=t\,y'=a', \qquad t\,.\left(\frac{d\,a''}{d\,t}\right)=t\,y''=a'', \qquad \&c.$$

Substituting these in [1000b], multiplied by t, and in the second member of the equation putting for $m\,t^m\,.\,V$, its value $m\,A^{(i)}$, [1000a], we shall get hence we deduce*

$$a' \cdot \left(\frac{d A^{(i)}}{d a'}\right) = - A^{(i)} - a \cdot \left(\frac{d A^{(i)}}{d a}\right);$$

$$a' \cdot \left(\frac{d d A^{(i)}}{d a \, d a'}\right) = -2 \cdot \left(\frac{d A^{(i)}}{d a}\right) - a \cdot \left(\frac{d d A^{(i)}}{d a^2}\right);$$

[1003] $$a'^2 \cdot \left(\frac{d d A^{(i)}}{d a'^2}\right) = 2 A^{(i)} + 4 a \cdot \left(\frac{d A^{(i)}}{d a}\right) + a^2 \cdot \left(\frac{d d A^{(i)}}{d a^2}\right);$$

$$a'^2 \cdot \left(\frac{d^3 A^{(i)}}{d a'^2 d a}\right) = 6 \cdot \left(\frac{d A^{(i)}}{d a}\right) + 6 a \cdot \left(\frac{d d A^{(i)}}{d a^2}\right) + a^2 \cdot \left(\frac{d^3 A^{(i)}}{d a^3}\right);$$

$$a'^3 \cdot \left(\frac{d^3 A^{(i)}}{d a'^3}\right) = -6 A^{(i)} - 18 a \cdot \left(\frac{d A^{(i)}}{d a}\right) - 9 a^2 \cdot \left(\frac{d d A^{(i)}}{d a^2}\right) - a^3 \cdot \left(\frac{d^3 A^{(i)}}{d a^3}\right);$$

&c.

[1001a] $$a \cdot \left(\frac{d A^{(i)}}{d a}\right) + a' \cdot \left(\frac{d A^{(i)}}{d a'}\right) + \&c. = m A^{(i)}.$$

Now it is evident that the first member of [954] is of the order — 1, in a, a', and therefore its development in the second member, must be of the same order, consequently $A^{(i)}$ in its second member, must be a homogeneous function, in a, a', of the order -1. Putting therefore $m = -1$, in [1001a] we shall get the equation [1002].

† (688) The first of the equations [1003] is deduced from [1002], by transposing $a \cdot \left(\frac{d A^{(i)}}{d a}\right)$. The next equation is the differential of this relative to a, divided by $d a$. The differential of the first, relative to a', being multiplied by $\frac{a'}{d a'}$ gives

$$a' \cdot \left(\frac{d A^{(i)}}{d a'}\right) + a'^2 \cdot \left(\frac{d d A^{(i)}}{d a'^2}\right) = - a' \cdot \left(\frac{d A^{(i)}}{d a'}\right) - a a' \cdot \left(\frac{d d A^{(i)}}{d a \, d a'}\right),$$

and if in this we substitute the values of $a' \cdot \left(\frac{d A^{(i)}}{d a'}\right)$, and $a a' \cdot \left(\frac{d d A^{(i)}}{d a \, d a'}\right)$, deduced from the first and second equations, we shall get the third. The differential of this, relative to a, gives the fourth. The differential of the third, relative to a', multiplied by $\frac{a'}{d a'}$ gives

$$2 a'^2 \cdot \left(\frac{d d A^{(i)}}{d a'^2}\right) + a'^3 \cdot \left(\frac{d d d A^{(i)}}{d a'^3}\right) = 2 a' \cdot \left(\frac{d A^{(i)}}{d a'}\right) + 4 a a' \cdot \left(\frac{d d A^{(i)}}{d a \, d a'}\right) + a^2 a' \cdot \left(\frac{d d d A^{(i)}}{d a^2 d a'}\right),$$

and by substituting the values $a' \cdot \left(\frac{d A'}{d a'}\right)$, $a' \cdot \left(\frac{d d A'}{d a \, d a'}\right)$, $a'^2 \cdot \left(\frac{d d A^{(i)}}{d a'^2}\right)$, given by the three first of the equations [1003], also the differential of the second, relative to $d a$, which is

$$a' \cdot \left(\frac{d d A^{(i)}}{d a^2 d a'}\right) = -3 \cdot \left(\frac{d d A^{(i)}}{d a^2}\right) - a \cdot \left(\frac{d^3 A^{(i)}}{d a^3}\right),$$

we shall get the last of the equations [1003].

We shall have $B^{(i)}$ and its differentials, by observing that by the preceding article [956, 963v], the series*

$$\tfrac{1}{2}\,.\,B^{(0)}+B^{(1)}\,.\,\cos.\,\theta+B^{(2)}\,.\,\cos.\,2\,\theta+\&c. \qquad [1004]$$

is the development of the function $a'^{-3}.(1-2\,\alpha\,.\,\cos.\,\theta+\alpha^2)^{-\frac{3}{2}}$, according to the cosines of the angle θ and its multiples; now this function being developed [964], is equal to [1004']

$$a'^{-3}\,.\left\{\tfrac{1}{2}\,.\,b_{\frac{3}{2}}^{(0)}+b_{\frac{3}{2}}^{(1)}\,.\,\cos.\,\theta+b_{\frac{3}{2}}^{(2)}\,.\,\cos.\,2\,\theta+\&c.\right\}; \qquad [1005]$$

therefore we shall have in general

$$B^{(i)}=\frac{1}{a'^3}\,.\,b_{\frac{3}{2}}^{(i)}\,; \qquad [1006]$$

hence we find

$$\left(\frac{d\,B^{(i)}}{d\,a}\right)=\frac{1}{a'^4}\,.\,\frac{d\,b_{\frac{3}{2}}^{(i)}}{d\,\alpha}\,; \qquad \left(\frac{d\,d\,B^{(i)}}{d\,a^2}\right)=\frac{1}{a'^5}\,.\,\frac{d\,d\,b_{\frac{3}{2}}^{(i)}}{d\,\alpha^2}\,; \qquad \&c. \qquad [1007]$$

Moreover, $B^{(i)}$ being a homogeneous function in a and a' [956], of the order -3, we shall have† [1007']

$$a\,.\left(\frac{d\,B^{(i)}}{d\,a}\right)+a'\,.\left(\frac{d\,B^{(i)}}{d\,a'}\right)=-3\,B^{(i)}. \qquad [1008]$$

Hence we may easily deduce the partial differentials of $B^{(i)}$ taken relative to a', from those of the partial differentials relative to a.

In the theory of the perturbations of m', by the action of m, the values of $A^{(i)}$ and $B^{(i)}$ are the same as above,‡ excepting $A^{(1)}$, which in this theory

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* (689) This follows from [956], putting $\alpha=\frac{a}{a'}$ [963iv], and developing

$$(1-2\,\alpha\,.\,\cos.\,\theta+\alpha^2)^{-\frac{3}{2}},$$

as in [964]. Comparing the coefficient of $\cos.\,i\,\theta$, in the expressions [1004, 1005], we get $B^{(i)}$, as in [1006]. Taking the differentials of this, relative to a, we shall get [1007].

† (690) This is deduced from [1001a], changing $A^{(i)}$ into $B^{(i)}$, and putting $m=-3$; it being evident, from [956], that $B^{(i)}$ is of the degree -3 in a, a'.

‡ (691) Changing a into a', and a' into a, in the first member of [956], its value would remain unaltered; therefore, the second member, or the value of $B^{(i)}$, would also remain unchanged. In like manner, the general value of $A^{(i)}$, found in [954], would remain

[1008'] becomes $\frac{a'}{a^2}-\frac{1}{a'}\cdot b_{\frac{1}{2}}^{(1)}$. Therefore the same calculations of $A^{(i)}$, $B^{(i)}$, and their differentials, will answer in the theory of the motions of both bodies, m and m'.

Investigation of the perturbations of the planets, neglecting the squares and products of the excentricities and inclinations.

50. After this digression, upon the development of R in a series, we shall resume the differential equations (X'), (Y), (Z'), [946, 931, 948], in order to determine, by means of these equations, the values of δr, δv, and δs; [1008''] and we shall continue the approximation only to quantities of the same order as the excentricities and inclinations of the orbits.

If in the elliptical orbits, we suppose

[1009]
$$r=a\cdot(1+u_{,})\,; \qquad r'=a'\cdot(1+u_{,}')\,;$$
$$v=nt+\varepsilon+v_{,}\,; \qquad v'=n't+\varepsilon'+v_{,}'\,;$$

we shall have, by § 22,*

[1010]
$$u_{,}=-e\cdot\cos.(nt+\varepsilon-\varpi)\,; \qquad u_{,}'=-e'\cdot\cos.(n't+\varepsilon'-\varpi')\,;$$
$$v_{,}=2e\cdot\sin.(nt+\varepsilon-\varpi)\,; \qquad v_{,}'=2e'\cdot\sin.(n't+\varepsilon'-\varpi')\,;$$

$nt+\varepsilon$, $n't+\varepsilon'$, are the mean longitudes of m, m' [543]; a and a', [534'], the semi-transverse axes of the orbits; e and e' the ratios of the [1010'] excentricities to the semi-transverse axes; lastly, ϖ and ϖ' are the longitudes of their perihelia. It is a matter of indifference whether these longitudes be referred to the planes of the orbits, or to a plane but little inclined to them, [1010''] since we shall neglect quantities of the order of the square or product of the excentricities and inclinations. Substituting the preceding values in the expression of R, § 48 [957], we shall find†

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unchanged, except in the case of $i=1$; $A^{(1)}$ being affected by the first term of the first member, so that it would become $A^{(1)}=\frac{a'}{a^2}-\frac{1}{a'}\cdot b_{\frac{1}{2}}^{(1)}$, instead of the value [997], as is evident from [954, 964].

* (692) The equations [1009] are like [952, 953], from which, by means of [669], we deduce [1010].

† (693) The first term, or line of the value of R, [957], produces the first term, independent of e, e', in [1011]. The terms multiplied by $u_{,}$ and $v_{,}$, in the second and fourth lines of [957], produce respectively the two terms of the second line of [1011], multiplied

$$R=\frac{m'}{2}.\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-\frac{m'}{2}.\Sigma.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+2i.A^{(i)}\right\}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \qquad [1011]$$

$$-\frac{m'}{2}.\Sigma.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\};$$

the sign Σ of finite integrals, embracing all integral values of i, positive or negative, including the value $i=0$. Hence we deduce,* [1011′]

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by e. The terms multiplied by $u_{\prime}'$ and $v_{\prime}'$, in the third and fourth lines of [957], produce respectively, by means of [954c, 955a], the two terms of the third line of [1011], multiplied by e'. These two last terms first appear under the form

$$-\tfrac{m'}{2}.\Sigma.\left\{a'.\left(\frac{dA^{(i)}}{da'}\right)-2i.A^{(i)}\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\},$$

and by changing i into $i-1$, which may be done, because i embraces all numbers, from $-\infty$ to $+\infty$, including $i=0$, it becomes

$$-\tfrac{m'}{2}.\Sigma.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}.e'.\cos.\{(i-1).(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\},$$

which is evidently equal to

$$-\tfrac{m'}{2}.\Sigma.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\},$$

as above.

* (694) If we exclude the value $i=0$ from the terms under the characteristic Σ, in [1011], it will become,

$$R=\tfrac{m'}{2}.\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)+\tfrac{m'}{2}.A^{(0)}-\tfrac{m'}{2}.a.\left(\frac{dA^{(0)}}{da}\right).e.\cos.(nt+\varepsilon-\varpi)$$

$$-\tfrac{m'e'}{2}.\left\{a'.\left(\frac{dA^{(1)}}{da'}\right)+2.A^{(1)}\right\}.\cos.(nt+\varepsilon-\varpi')$$

$$-\tfrac{m'}{2}.\Sigma.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+2i.A^{(i)}\right\}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \qquad [1012a]$$

$$-\tfrac{m'}{2}.\Sigma.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}.e'.\cos.[i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'],$$

because $A^{(-1)}=A^{(1)}$, [954″]. Taking the differential of R, relative to nt, we get $\mathrm{d}R$, [1012b] and its integral being doubled gives

$$Q=2\int \mathrm{d}R+r.\left(\frac{dR}{dr}\right)$$

$$=2m'.g+\frac{m'}{2}.a.\left(\frac{dA^{(0)}}{da}\right)+\frac{m'}{2}.\Sigma.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.A^{(i)}\right\}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-\frac{m'}{2}.\left\{a^2.\left(\frac{ddA^{(0)}}{da^2}\right)+3a.\left(\frac{dA^{(0)}}{da}\right)\right\}.e.\cos.(nt+\varepsilon-\varpi)$$

[1012] $$-\frac{m'}{2}.\left\{aa'.\left(\frac{ddA^{(1)}}{da\,da'}\right)+2a.\left(\frac{dA^{(1)}}{da}\right)+2a'.\left(\frac{dA^{(1)}}{da'}\right)+4A^{(1)}\right\}.e'.\cos.(nt+\varepsilon-\varpi')$$

$$-\frac{m'}{2}.\Sigma.\left\{\begin{array}{l}a^2.\left(\frac{ddA^{(i)}}{da^2}\right)+(2i+1).a.\left(\frac{dA^{(i)}}{da}\right)\\+\frac{2.(i-1).n}{i.(n-n')-n}.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+2i.A^{(i)}\right\}\end{array}\right\}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$-\frac{m'}{2}.\Sigma.\left\{\begin{array}{l}aa'.\left(\frac{ddA^{(i-1)}}{da\,da'}\right)-2.(i-1)a.\left(\frac{dA^{(i-1)}}{da}\right)\\+\frac{2.(i-1).n}{i.(n-n')-n}.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}\end{array}\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}\;;$$

$$2\int \mathrm{d}R=2m'.g+\frac{m'}{2}.\frac{2n}{n-n'}.\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-\frac{m'}{2}.2a.\left(\frac{dA^{(0)}}{da}\right).e.\cos.(nt+\varepsilon-\varpi)$$

[1012c] $$-\frac{m'}{2}.2.\left\{a'.\left(\frac{dA^{(1)}}{da'}\right)+2A^{(1)}\right\}.e'.\cos.(nt+\varepsilon-\varpi')$$

$$-\frac{m'}{2}.\Sigma.\frac{2.(i-1).n}{i.(n-n')-n}.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+2i.A^{(i)}\right\}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$-\frac{m'}{2}.\Sigma.\frac{2.(i-1).n}{i.(n-n')-n}.\left\{a'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).A^{(i-1)}\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}.$$

Moreover, since $r.\left(\frac{dR}{dr}\right)=a.\left(\frac{dR}{da}\right)$, [962], if we take the differential of R [1012*a*], relative to a, and multiply it by a, we shall we shall get

$$a.\left(\frac{dR}{da}\right)=r.\left(\frac{dR}{dr}\right)=\frac{m'}{2}.\Sigma a.\left(\frac{dA^{(i)}}{da}\right).\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$+\frac{m'}{2}.a.\left(\frac{dA^{(0)}}{da}\right)-\frac{m'}{2}.\left\{a.\left(\frac{dA^{(0)}}{da}\right)+a^2.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}.e.\cos.(nt+\varepsilon-\varpi)$$

[1012d] $$-\frac{m'}{2}.\left\{aa'.\left(\frac{ddA^{(1)}}{da\,da'}\right)+2a.\left(\frac{dA^{(1)}}{da}\right)\right\}.e'.\cos.(nt+\varepsilon-\varpi')$$

$$-\frac{m'}{2}\Sigma\left\{a^2.\left(\frac{ddA^{(i)}}{da^2}\right)+a.\left(\frac{dA^{(i)}}{da}\right)+2i.a.\left(\frac{dA^{(i)}}{da}\right)\right\}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$-\frac{m'}{2}\Sigma\left\{aa'.\left(\frac{ddA^{(i-1)}}{da\,da'}\right)-2.(i-1).a.\left(\frac{dA^{(i-1)}}{da}\right)\right\}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}.$$

Adding together the expressions [1012*c*, *d*], and connecting the terms depending on the same angles, we shall get [1012], equal to the value of Q, [934].

The sign Σ includes in this and in the following formulas, all the integral values of i, positive or negative, excepting $i=0$; the term depending on $i=0$, having been brought from under that sign; $m'g$ is a constant [1012'] quantity, added to the integral $\int \mathrm{d}R$. Now put

$$C=\tfrac{1}{2}a^3.\left(\frac{ddA^{(0)}}{da^2}\right)+3a^2.\left(\frac{dA^{(0)}}{da}\right)+6ag;$$

$$D=\tfrac{1}{2}a^2a'.\left(\frac{ddA^{(1)}}{da\,da'}\right)+a^2.\left(\frac{dA^{(1)}}{da}\right)+aa'.\left(\frac{dA^{(1)}}{da'}\right)+2aA^{(1)};$$

$$\begin{aligned}C^{(i)}=&\tfrac{1}{2}a^3.\left(\frac{ddA^{(i)}}{da^2}\right)+\frac{(2i+1)}{2}.a^2.\left(\frac{dA^{(i)}}{da}\right)\\&+\frac{\{i.(n-n')-3n\}}{2.\{i.(n-n')-n\}}.\left\{a^2.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.aA^{(i)}\right\}\\&+\frac{(i-1).n}{i.(n-n')-n}.\left\{a^2.\left(\frac{dA^{(i)}}{da}\right)+2i.aA^{(i)}\right\};\end{aligned}\qquad[1013]$$

$$\begin{aligned}D^{(i)}=&\tfrac{1}{2}a^2a'.\left(\frac{d^2A^{(i-1)}}{da\,da'}\right)-(i-1).a^2.\left(\frac{dA^{(i-1)}}{da}\right)\\&+\frac{(i-1).n}{i.(n-n')-n}.\left\{aa'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).aA^{(i-1)}\right\}.\end{aligned}$$

Supposing the sum of the two masses $M+m$ to be equal to unity, and observing that by § 20 [605'], $\frac{M+m}{a^3}=n^2$; the equation (X') [946] will [1013'] become*

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* (696) Putting $M+m=1$ in [605'], we shall get $n^2a^3=1$, which will be [1013a] used hereafter. Also putting, for brevity, $Q=2\int\mathrm{d}R+r.\left(\frac{dR}{dr}\right)$, [1012], the equation [946], neglecting e^2, will become

$$0=\frac{d^2.\delta u}{dt^2}+n^2.\delta u-\frac{1}{a^2}.\left\{1-e.\cos.(nt+\varepsilon-\varpi)\right\}.Q-\frac{2e}{a^2}.\int Q.ndt.\sin.(nt+\varepsilon-\varpi).$$

Substituting for $\frac{1}{a^2}$ its value n^2a, [1013a], we shall get

$$0=\frac{d^2.\delta u}{dt^2}+n^2.\delta u-n^2aQ+n^2aQ.e.\cos.(nt+\varepsilon-\varpi)-2n^2ae.\int Q.ndt.\sin.(nt+\varepsilon-\varpi).\qquad[1014a]$$

In the two last terms, multiplied by e, we may for Q, [1012], substitute the part of it independent of e, e', namely,

$$Q=2m'g+\frac{m'}{2}.a.\left(\frac{dA^{(0)}}{da}\right)+\frac{m'}{2}.\Sigma.\left\{a.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.A^{(i)}\right\}.\cos.i.(n't-nt+\varepsilon'-\varepsilon),$$

[1014]

$$\begin{aligned}0 = \frac{d^2 . \delta u}{d t^2} + n^2 . \delta u - 2 n^2 . m' a g - \frac{n^2 m'}{2} . a^2 . \left(\frac{d \mathcal{A}^{(0)}}{d a}\right) \\ - \frac{n^2 m'}{2} . \Sigma . \left\{ a^2 . \left(\frac{d \mathcal{A}^{(i)}}{d a}\right) + \frac{2 n}{n - n'} . a \mathcal{A}^{(i)} \right\} . \cos. i . (n' t - n t + \varepsilon' - \varepsilon) \\ + n^2 m' . C . e . \cos. (n t + \varepsilon - \varpi) + n^2 m' . D . e' . \cos. (n t + \varepsilon - \varpi') \\ + n^2 m' . \Sigma . C^{(i)} . e . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi \} \\ + n^2 m' . \Sigma . D^{(i)} . e' . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi' \} ;\end{aligned}$$

and by integration[*]

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which gives, by [954c],

[1014b]

$$\begin{aligned}n^2 a Q . e . \cos. (n t + \varepsilon - \varpi) = \left\{ 2 m' g + \frac{m'}{2} . a . \left(\frac{d \mathcal{A}^{(0)}}{d a}\right) \right\} . n^2 a e . \cos. (n t + \varepsilon - \varpi) \\ + \frac{m'}{2} . n^2 a e . \Sigma . \left\{ a . \left(\frac{d \mathcal{A}^{(i)}}{d a}\right) + \frac{2 n}{n - n'} . \mathcal{A}^{(i)} \right\} . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi \} .\end{aligned}$$

Also, by [954b],

$$\begin{aligned}- Q . n d t . \sin. (n t + \varepsilon - \varpi) = - \left\{ 2 m' g + \frac{m'}{2} . a . \left(\frac{d \mathcal{A}^{(0)}}{d a}\right) \right\} . n d t . \sin. (n t + \varepsilon - \varpi) \\ - \frac{m'}{2} . \Sigma . \left\{ a . \left(\frac{d \mathcal{A}^{(i)}}{d a}\right) + \frac{2 n}{n - n'} . \mathcal{A}^{(i)} \right\} . n d t . \sin. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi \} .\end{aligned}$$

Taking the integral of this last expression, and multiplying it by $2 n^2 a e$, we shall find,

[1014c]

$$\begin{aligned}- 2 n^2 a e . \int Q . n d t . \sin. (n t + \varepsilon - \varpi) = 2 n^2 a e . \left\{ 2 m' g + \frac{m'}{2} . a . \left(\frac{d \mathcal{A}^{(0)}}{d a}\right) \right\} . \cos. (n t + \varepsilon - \varpi) \\ - \frac{m'}{2} . 2 n^2 a e . \Sigma . \frac{n}{i . (n - n') - n} . \left\{ a . \left(\frac{d \mathcal{A}^{(i)}}{d a}\right) + \frac{2 n}{n - n'} . \mathcal{A}^{(i)} \right\} . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi \} .\end{aligned}$$

no constant term being added, because it would produce, in [946], a term independent of Q, which would prevent δu from being nothing, when $Q = 0$, which is contrary to the principle assumed in § 46. Substituting [1014b, c] in [1014a], we shall get, by reduction,

$$\begin{aligned}0 = \frac{d^2 . \delta u}{d t^2} + n^2 . \delta u - n^2 a . Q + m' . n^2 e . \left\{ 6 a g + \tfrac{3}{2} a^2 . \left(\frac{d \mathcal{A}^{(0)}}{d a}\right) \right\} . \cos. (n t + \varepsilon - \varpi) \\ + m' . n^2 e . \Sigma . \left\{ \frac{\{ i . (n - n') - 3 n \}}{2 . \{ i . (n - n') - n \}} . \left[a^2 . \left(\frac{d \mathcal{A}^{(i)}}{d a}\right) + \frac{2 n}{n - n'} . a \mathcal{A}^{(i)} \right] \right\} \\ . \cos. \{ i . (n' t - n t + \varepsilon' - \varepsilon) + n t + \varepsilon - \varpi \} .\end{aligned}$$

If in this we substitute the value of Q, [1012], and connect together the terms depending on the same angles, using the abridged expressions [1013], it will become as in [1014], as is very evident, from the mere inspection of the terms.

[*] (697) The equation [1014] is of the same form as [865], whose solution is given in [870—871″]; putting $y = \delta u$, and changing a [865] into n [1014]; representing also by αQ, [865], all the terms of the expression [1014], except the two first $\frac{d^2 . \delta u}{d t^2} + n^2 . \delta u$;

$$\delta u = 2\,m'\,a\,g + \frac{m'}{2}\,.\,a^2\,.\left(\frac{d\mathcal{A}^{(0)}}{d\,a}\right)$$

$$-\frac{m'}{2}\,.\,n^2\,.\,\Sigma\,.\,\frac{\left\{a^2\,.\left(\frac{d\,\mathcal{A}^{(i)}}{d\,a}\right)+\frac{2\,n}{n-n'}\,.\,a\,\mathcal{A}^{(i)}\right\}}{i^2\,.\,(n-n')^2-n^2}\,.\,\cos.\,i\,.\,(n't-nt+\varepsilon'-\varepsilon)$$

$$+m'\,.\,f_{\prime}\,.\,e\,.\,\cos.\,(n\,t+\varepsilon-\varpi)+m'\,.\,f'_{\prime}\,.\,e'\,.\,\cos.\,(n\,t+\varepsilon-\varpi')$$

$$-\frac{m'}{2}\,.\,C\,.\,n\,t.\,e\,.\sin.\,(n\,t+\varepsilon-\varpi)-\frac{m'}{2}\,.\,\boldsymbol{D}\,.\,n\,t\,.\,e'\,.\sin.\,(n\,t+\varepsilon-\varpi') \qquad [1015]$$

$$+m'\,.\,\Sigma\,.\,\frac{C^{(i)}\,.\,n^2}{\{i\,.\,(n-n')-n\}^2-n^2}\,.\,e\,.\cos.\,\{i\,.\,(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$+m'\,.\,\Sigma\,.\,\frac{D^{(i)}\,.\,n^2}{\{i\,.(n-n')-n\}^2-n^2}\,.\,e'\,.\cos.\,\{i\,.\,(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}.$$

$f_{\prime}$ and $f'_{\prime}$ being two arbitrary constant quantities. The expression of δr by means of δu, found in § 47 [947], will give* [1015′]

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observing that if any one of the terms of $\alpha\,Q$ be denoted by $K\,.\cos.\,(m\,t+\varepsilon)$, it will, by [871], furnish in $\delta\,u$ the term $\frac{K}{m^2-n^2}\,.\,\cos.\,(m\,t+\varepsilon)$; and by using the appropriate values of m, namely, $m=0$, $m=i\,.\,(n'-n)$, $m=i\,.\,(n'-n)+n=-\{i.(n-n')-n\}$, we shall get all the terms of $\delta\,u$ [1015], except those depending on the angles $n\,t+\varepsilon-\varpi$, $n\,t+\varepsilon-\varpi'$. These two angles depend on $m=n$, and by [871″], they will produce the terms depending on C, D, [1015], also terms similar to those depending on $f_{\prime}$, $f'_{\prime}$, which might be connected with the constant quantities c, c', [870].

* (698) The equation [947], neglecting e^2. and the higher powers of e, gives

$$\frac{\delta\,r}{a}=-\,\delta\,u-2\,\delta\,u\,.\,e\,.\,\cos.\,(n\,t+\varepsilon-\varpi), \qquad [1016a]$$

and by substituting, in $-2\,\delta\,u\,.\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)$, the terms of the value of $\delta\,u$, [1015], independent of e, we get

$$-2\,\delta\,u\,.\,e\,.\cos.\,(n\,t+\varepsilon-\varpi)=\left\{-\,4\,m'\,.\,a\,g-m'\,.\,a^2\,.\left(\frac{d\mathcal{A}^{(0)}}{d\,a}\right)\right\}.\,e\,.\,\cos.\,(n\,t+\varepsilon-\varpi)$$

$$+m'\,n^2\,.\,\Sigma\,.\,\frac{\left\{a^2.\left(\frac{d\mathcal{A}^{(i)}}{d\,a}\right)+\frac{2\,n}{n-n'}\,.\,a\,\mathcal{A}^{(i)}\right\}}{i^2\,.(n-n')^2-n^2}\,.\,e\,.\cos.\,\{i\,.\,(n'\,t-nt+\varepsilon'-\varepsilon)+n\,t+\varepsilon-\varpi\},$$

subtracting from this the value of $\delta\,u$, [1015] we shall obtain

$$-\,\delta\,u-2\,\delta\,u\,.\,e\,.\,\cos.\,(n\,t+\varepsilon-\varpi)=\frac{\delta\,r}{a},$$

[1016a], and by putting $f'_{\prime}=f'$, and $-f_{\prime}-4\,a\,g-a^2\,.\left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right)=-f$, it will become as in [1016].

[1016]

$$
\begin{aligned}
\frac{\delta r}{a} = & -2m'.ag - \frac{m'}{2}.a^2.\left(\frac{dA^{(0)}}{da}\right) \\
& + \frac{m'}{2}.n^2.\Sigma.\left\{\frac{a^2.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.aA^{(i)}}{i^2.(n-n')^2-n^2}\right\}.\cos.i.(n't-nt+\varepsilon'-\varepsilon) \\
& - m'.fe.\cos.(nt+\varepsilon-\varpi) - m'.f'e'.\cos.(nt+\varepsilon-\varpi') \\
& + \tfrac{1}{2}m'C.nt.e.\sin.(nt+\varepsilon-\varpi) + \tfrac{1}{2}m'D.nt.e'.\sin.(nt+\varepsilon-\varpi') \\
& + m'.n^2.\Sigma.\left\{\begin{array}{c}\left\{\frac{\left\{a^2.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.aA^{(i)}\right\}}{i^2.(n-n')^2-n^2} - \frac{C^{(i)}}{\{i.(n-n')-n\}^2-n^2}\right\} \\ \times e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}\end{array}\right\} \\
& - m'n^2.\Sigma.\frac{D^{(i)}}{\{i.(n-n')-n\}^2-n^2}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\};
\end{aligned}
$$

[1016′] f and f' being arbitrary quantities, depending on $f_{,}$ and $f'_{,}$.

This value of δr being substituted in the formula (Y) § 46 [931], will give δv, or the perturbations of the motion of the planet in longitude; but we ought to observe that as nt expresses the mean motion of m, the term [1016″] proportional to the time t, ought to disappear from the expression of δv. This condition will serve to determine the constant quantity g, and we shall find*

[1017]

$$g = -\tfrac{1}{3}a.\left(\frac{dA^{(0)}}{da}\right).$$

We might have dispensed with the arbitrary quantities f and f', in the value of δr, since they could be supposed to be included in the elements e, ϖ, of the elliptical† motion; but then the expression of δv would have contained

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* (699) The calculation of δv, from [931], is made in note 702, in which it will appear that the term, independent of sines and cosines, is $3m'.agnt + m'.a^2.\left(\frac{dA^{(0)}}{da}\right).nt$, [1021$f$], putting this equal to nothing, as in [1017′], we shall get g, [1017].

† (700) If in the elliptical value of $\frac{r}{a}$ [669], we neglect e^2, with the higher powers of e, and put for brevity $nt+\varepsilon=\tau$, we shall get

[1017a]

$$
\begin{aligned}
\frac{r}{a} &= 1 - e.\cos.(nt+\varepsilon-\varpi) = 1 - e.\cos.(\tau-\varpi) \\
&= 1 - e.\cos.\varpi.\cos.\tau - e.\sin.\varpi.\sin.\tau, \qquad \text{[24] Int.}
\end{aligned}
$$

terms, depending on the mean anomaly, which would not have been comprised in those of the elliptical motion; now it is more convenient to make the terms disappear from the expression of the longitude, in order to introduce [1017'] them in the expression of the radius vector; we shall therefore determine $f_{,}$ and $f'_{,}$ so as to satisfy this condition. This being premised, if we substitute for $a'.\left(\frac{d\mathcal{A}^{(i-1)}}{da'}\right)$ its value [1003], $-\mathcal{A}^{(i-1)}-a.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)$, we shall have [1013, &c.],*

In like manner, the terms of $\frac{\delta r}{a}$, [1016], depending on f and f', are

$-m'fe.(\cos.\varpi.\cos.\tau+\sin.\varpi.\sin.\tau)$, and $-m'f'e'.(\cos.\varpi'.\cos.\tau+\sin.\varpi'.\sin.\tau)$.

If we add these terms of $\frac{\delta r}{a}$ to $\frac{r}{a}$, [1017a], and put

$$e.\cos.\varpi+m'fe.\cos.\varpi+m'f'e'.\cos.\varpi'=e_{,}.\cos.\varpi_{,},$$
$$e.\sin.\varpi+m'fe.\sin.\varpi+m'f'e'.\sin.\varpi'=e_{,}.\sin.\varpi_{,},$$

it will become

$$\frac{r+\delta r}{a}=1-e_{,}.(\cos.\varpi_{,}.\cos.\tau+\sin.\varpi_{,}.\sin.\tau)=1-e_{,}.\cos.(\tau-\varpi_{,})$$
$$=1-e_{,}.\cos.(nt+\varepsilon-\varpi_{,}), \qquad [24]\text{ Int.}$$

which is of the same form as the equation of the ellipsis, [1017a], changing e, ϖ, into $e_{,}, \varpi_{,}$.

* (701) The value of C, [1018], is deduced from that in [1013], by substituting g, [1017), and reducing. The value of D, [1013], becomes, by substituting the values of $a'.\left(\frac{d\mathcal{A}^{(1)}}{da'}\right)$, and $a'.\left(\frac{dd\mathcal{A}^{(1)}}{da\,da'}\right)$, [1003], the same as in [1018]. Similar substitutions, in $D^{(i)}$ [1013], produce $D^{(i)}$ [1018], after making the usual reductions in connecting the coefficients of the similar terms. These values of C, D, $D^{(i)}$, and that of g, [1017], are to be substituted in [1016]; and if we put, in the coefficient of the angle

$$i.\{n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}, \quad [1016],$$

$$E^{(i)}=C^{(i)}-\left\{a^2.\left(\frac{d\mathcal{A}^{(i)}}{da}\right)+\frac{2n}{n-n'}.a\mathcal{A}^{(i)}\right\}.\frac{\{i.(n-n')-n\}^2-n^2}{i^2.(n-n')^2-n^2},$$

we shall get $\frac{\delta r}{a}$, [1020]. This value of $E^{(i)}$ may be reduced in the following manner. Put for brevity

$$\nu=n-n', \qquad T=n't-nt+\varepsilon'-\varepsilon, \qquad W=nt+\varepsilon-\varpi, \qquad W'=nt+\varepsilon-\varpi',$$ [1018a]

$$G=a^2.\left(\frac{d\mathcal{A}^{(i)}}{da}\right)+\frac{2n}{n-n'}.a\mathcal{A}^{(i)};$$

$$C = a^2 . \left(\frac{d\mathcal{A}^{(0)}}{da}\right) + \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(0)}}{da^2}\right);$$

$$D = a\mathcal{A}^{(1)} - a^2 . \left(\frac{d\mathcal{A}^{(1)}}{da}\right) - \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(1)}}{da^2}\right);$$

[1018] $$D^{(i)} = \frac{(i-1).(2i-1).n}{n-i.(n-n')} a . \mathcal{A}^{(i-1)} + \frac{\{i^2.(n-n')-n\}}{n-i.(n-n')} . a^2 . \left(\frac{d\mathcal{A}^{(i-1)}}{da}\right) - \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(i-1)}}{da^2}\right);$$

$$f = \tfrac{2}{3} . a^2 . \left(\frac{d\mathcal{A}^{(0)}}{da}\right) + \tfrac{1}{4} a^3 . \left(\frac{dd\mathcal{A}^{(0)}}{da^2}\right);$$

$$f' = \tfrac{1}{4} . \left\{ a . \mathcal{A}^{(1)} - a^2 . \left(\frac{d\mathcal{A}^{(1)}}{da}\right) - a^3 . \left(\frac{dd\mathcal{A}^{(1)}}{da^2}\right) \right\}.$$

---

substitute in $E^{(i)}$ the value of $C^{(i)}$, [1013], and it will become

[1018b] $$E^{(i)} = \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(i)}}{da^2}\right) + \frac{(2i+1)}{2} . a^2 . \left(\frac{d\mathcal{A}^{(i)}}{da}\right) + \frac{\{i\nu - 3n\}}{2.\{i\nu - n\}} . G + \frac{(i-1).n}{i\nu - n} . \left\{ a^2 . \left(\frac{d\mathcal{A}^{(i)}}{da}\right) + 2i.a\mathcal{A}^{(i)} \right\} - \frac{\{i\nu - n\}^2 - n^2}{i^2\nu^2 - n^2} . G;$$

connecting the terms depending on $d\mathcal{A}^{(i)}$, G, we shall get

$$E^{(i)} = \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(i)}}{da^2}\right) + \frac{\{2i^2\nu + i\nu - 3n\}}{2.(i\nu - n)} . a^2 . \left(\frac{d\mathcal{A}^{(i)}}{da}\right) + \frac{2i.(i-1).n}{i\nu - n} . a\mathcal{A}^{(i)} + \frac{-\tfrac{1}{2}i^2.\nu^2 + in.\nu - \tfrac{3}{2}.n^2}{i^2\nu^2 - n^2} . G,$$

observing that the first coefficient of G is reduced to the same denominator as that of the second, by multiplying the numerator and the denominator by $\tfrac{1}{2}i\nu + \tfrac{1}{2}n$. Now the above value of G gives $a^2 . \left(\frac{d\mathcal{A}^{(i)}}{da}\right) = G - \frac{2n}{\nu} . a\mathcal{A}^{(i)}$. This being substituted, we get

[1018c] $$E^{(i)} = \tfrac{1}{2} a^3 . \left(\frac{dd\mathcal{A}^{(i)}}{da^2}\right) + \frac{\{2i^2\nu + i\nu - 3n\}}{2.\{i\nu - n\}} . \left\{ G - \frac{2n}{\nu} . a\mathcal{A}^{(i)} \right\} + \frac{2i.(i-1).n}{i\nu - n} . a\mathcal{A}^{(i)} + \frac{-\tfrac{1}{2}i^2.\nu^2 + in.\nu - \tfrac{3}{2}.n^2}{i^2\nu^2 - n^2} . G.$$

The coefficient of $\frac{a\mathcal{A}^{(i)}}{i\nu - n}$, in this expression, taking the terms in the order in which they occur, is $-2i^2n - in + \frac{3n^2}{\nu} + 2i^2n - 2in = -3in + \frac{3n^2}{\nu} = -3n . \left(\frac{i\nu - n}{\nu}\right)$, which being multiplied by $\frac{a\mathcal{A}^{(i)}}{i\nu - n}$, produces the corresponding term of $E^{(i)}$, depending

Now putting

$$E^{(i)} = -\frac{3n}{n-n'} \cdot a A^{(i)} + \frac{\{i^2 \cdot (n-n') \cdot [n + i \cdot (n-n')] - 3n^2\}}{i^2 \cdot (n-n')^2 - n^2} \times \left\{ a^2 \cdot \left(\frac{d\mathcal{A}^{(i)}}{da}\right) + \frac{2n}{n-n'} \cdot a A^{(i)} \right\} + \tfrac{1}{2} a^3 \cdot \left(\frac{d\,d\mathcal{A}^{(i)}}{da^2}\right);$$

$$F^{(i)} = \frac{(i-1) \cdot n}{n-n'} \cdot a A^{(i)} + \frac{\left\{ \frac{in}{2} \cdot \{n + i \cdot (n-n')\} - 3n^2 \right\}}{i^2 \cdot (n-n')^2 - n^2} \times \left\{ a^2 \cdot \left(\frac{d\mathcal{A}^{(i)}}{da}\right) + \frac{2n}{n-n'} \cdot a A^{(i)} \right\} - \frac{2n^2 \cdot E^{(i)}}{n^2 - \{n - i \cdot (n-n')\}^2}; \qquad [1019]$$

$$G^{(i)} = \frac{(i-1) \cdot (2i-1) \cdot n a \cdot A^{(i-1)} + (i-1) \cdot n a^2 \cdot \left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)}{2 \cdot \{n - i \cdot (n-n')\}} - \frac{2n^2 \cdot D^{(i)}}{n^2 - \{n - i \cdot (n-n')\}^2};$$

we shall have*

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on $\mathcal{A}^{(i)}$, namely $-\frac{3n}{\mathrm{v}} \cdot a \mathcal{A}^{(i)}$. The coefficient of $\frac{G}{i^2 \mathrm{v}^2 - n^2}$, in this last expression of $E^{(i)}$, is $\{i^2 \mathrm{v} + \frac{1}{2} i \mathrm{v} - \frac{3}{2} n\} \cdot \{i \mathrm{v} + n\} - \frac{1}{2} i^2 \mathrm{v}^2 + i n \mathrm{v} - \frac{3}{2} n^2$, because the first term can be reduced to the same denominator as the second, by multiplying the numerator and denominator by $\frac{1}{2} \cdot \{i \mathrm{v} + n\}$. Performing the multiplications and reducing, it becomes $i^3 \mathrm{v}^2 + i^2 n \mathrm{v} - 3n^2 = i^2 \mathrm{v} \cdot (n + i \mathrm{v}) - 3n^2$, consequently the term of $E^{(i)}$, depending on G, is $\frac{i^2 \mathrm{v} \cdot (n + i \mathrm{v}) - 3n^2}{i^2 \mathrm{v}^2 - n^2} \cdot G$, and the complete value of $E^{(i)}$, becomes as in [1019]. The values of f, f', $F^{(i)}$, $G^{(i)}$, [1018, 1019], are computed by means of δv, in the following note.

* (702) The value of δv [1021] may be obtained from [931], by using the symbols [1018*a*], and substituting for $\frac{\delta r}{a}$, its value [1020]

$$\frac{r}{a} = 1 - e \cdot \cos.(nt + \varepsilon - \varpi) = (1 - e \cdot \cos. W), \qquad [1018d]$$

[1017*a*]; also $\int dR$ and $r \cdot \left(\frac{dR}{dr}\right)$, [1012*c*, *d*); then determining f and f', by making the coefficients of $\sin.(nt + \varepsilon - \varpi)$, and $\sin.(nt + \varepsilon - \varpi')$, equal to nothing, [1017']; observing that terms of the order e^2 are neglected, and $\mu = 1$. This calculation is rather long, but as the equation is of great importance, it will be proper to enter into a full explanation of the whole computation. The equation [931], with these conditions, becomes

Perturbations of the radius vector.

[1020]

$$\begin{aligned}\frac{\delta r}{a} = {} & \frac{m'}{6}.a^2.\left(\frac{d A^{(0)}}{d a}\right)+\frac{m'.n^2}{2}.\Sigma.\frac{\left\{a^2.\left(\frac{d A^{(i)}}{d a}\right)+\frac{2 n}{n-n'}.aA^{(i)}\right\}}{i^2.(n-n')^2-n^2}.\cos.i.(n't-nt+\varepsilon'-\varepsilon) \\ & -m'.fe.\cos.(nt+\varepsilon-\varpi)-m'.f'e'.\cos.(nt+\varepsilon-\varpi') \\ & +\tfrac{1}{2}m'.C.nt.e.\sin.(nt+\varepsilon-\varpi)+\tfrac{1}{2}m'.D.nt.e'.\sin.(nt+\varepsilon-\varpi') \\ & +n^2.m'.\Sigma.\left\{\begin{array}{l}\dfrac{E^{(i)}}{n^2-\{n-i.(n-n')\}^2}.e.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \\ +\dfrac{D^{(i)}}{n^2-\{n-i.(n-n')\}^2}.e'.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}\end{array}\right\};\end{aligned}$$

[1020a] $\delta v=\frac{2r.d\delta r}{a^2.ndt}+\frac{dr.\delta r}{a^2.ndt}+3a.\int ndt.\int \mathrm{d}R+2a\int ndt.r.\left(\frac{dR}{dr}\right)$. Each of the four terms of the second member may be computed in the following manner.

[1021a] The first term $\frac{2r.d\delta r}{a^2.ndt}=\frac{2a.\{1-e.\cos.W\}.d\delta r}{a^2.ndt}=\frac{2.d\delta r}{a.ndt}-\frac{2e.\cos.W.d\delta r}{a.ndt}$; and in the part depending on e, we may substitute the terms of the value of $d\,\delta\,r$, [1020], independent of e, e'; hence we shall get

$$\begin{aligned}\frac{-2e.\cos.W.d\delta r}{a.ndt} & =\left\{-\frac{2e.\cos.W}{ndt}\right\}.\frac{m'n^2}{2}.\Sigma.\frac{G}{i^2\nu^2-n^2}.i\nu.dt.\sin.iT \\ & =m'n.\Sigma.\frac{i\nu.G}{n^2-i^2\nu^2}.e.\sin.(iT+W), \quad [955b].\end{aligned}$$

Hence $\frac{2r.d\delta r}{a^2.ndt}=\frac{2.d\delta r}{a.ndt}+m'n.\Sigma.\frac{i\nu.G}{n^2-i^2\nu^2}.e.\sin.(iT+W)$. Now if in this, we substitute the value of $\frac{2.d\delta r}{a.ndt}$, deduced from [1020], it will give the following value of of $\frac{2r.d\delta r}{a^2.ndt}$,

[1021b]

$$\begin{aligned}\frac{2r.d\delta r}{a^2.ndt}= {} & m'n.\Sigma.\frac{i\nu.G}{i^2\nu^2-n^2}.\quad\sin.iT \\ & +2m'.fe.\sin.W+2m'.f'e'.\sin.W' \\ & +m'.Ce.\sin.W+m'.De'.\sin.W'+m'.C.nt.e.\cos.W+m'.D.nt.e'.\cos.W' \\ & +2m'n.\Sigma.\left\{\begin{array}{l}\dfrac{\{i\nu-n\}.E^{(i)}}{n^2-\{n-i\nu\}^2}.e.\sin.(iT+W) \\ +\dfrac{\{i\nu-n\}.D^{(i)}}{n^2-\{n-i\nu\}^2}.e'.\sin.(iT+W')\end{array}\right\} \\ & +m'n.\Sigma.\frac{i\nu.G}{n^2-i^2\nu^2}.e.\sin.(iT+W).\end{aligned}$$

The second term of δv, [1020a], is $\frac{dr.\delta r}{a^2.ndt}=\frac{(ae.ndt.\sin.W).\delta r}{a^2.ndt}=e.\frac{\delta r}{a}.\sin.W$, Perturbations in longitude.

$$\delta v=\frac{m'}{2}.\Sigma.\left\{\frac{n^2}{i.(n-n')^2}.aA^{(i)}+\frac{2n^3.\left\{a^2.\left(\frac{dA^{(i)}}{da}\right)+\frac{2n}{n-n'}.aA^{(i)}\right\}}{i.(n-n').\{i^2.(n-n')^2-n^2\}}\right\}.\sin.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$+m'.C.nt.e.\cos.(nt+\varepsilon-\varpi)+m'.D.nt.e'.\cos.(nt+\varepsilon-\varpi')$$

$$+nm'.\Sigma.\left\{\begin{array}{l}\frac{F^{(i)}}{n-i.(n-n')}.e.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \\ +\frac{G^{(i)}}{n-i.(n-n')}.e'.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi'\}\end{array}\right\}; \qquad [1021]$$

[1018*d*], and by substituting, for $\frac{\delta r}{a}$, the terms of [1020], independent of e, e', it becomes by using the formula [954*b*],

$$\frac{dr.\delta r}{a^2.n\,dt}=\frac{m'}{6}.a^2.e.\left(\frac{dA^{(0)}}{da}\right).\sin.W+\frac{m'n^2}{2}.\Sigma.\frac{G}{i^2\nu^2-n^2}.e.\sin.(iT+W). \qquad [1021c]$$

The third term of δv, [1020*a*], $3a.\int n\,dt.\int\mathrm{d}R$, is easily deduced from that of $2\int\mathrm{d}R$, [1012*c*], by multiplying it by $\frac{3a}{2}.n\,dt$, and again taking the integral. It will not be necessary to add any constant term, the arbitrary term ε, in the value of v, [669], being sufficient; hence

$$3a.\int n\,dt.\int\mathrm{d}R=3m'.agnt-\tfrac{3}{2}m'.\Sigma.\frac{n^2}{i\nu^2}.aA^{(i)}.\sin.iT-\tfrac{3}{2}m'.a^2.\left(\frac{dA^{(0)}}{da}\right).e.\sin.W$$

$$-\tfrac{3}{2}m'.\left\{aa'.\left(\frac{dA^{(1)}}{da'}\right)+2aA^{(1)}\right\}.e'.\sin.W'$$

$$+\tfrac{3}{2}m'.\Sigma.\frac{(i-1).n^2}{\{i\nu-n\}^2}.\left\{a^2.\left(\frac{dA^{(i)}}{da}\right)+2i.aA^{(i)}\right\}.e.\sin.(iT+W) \qquad [1021d]$$

$$+\tfrac{3}{2}m'.\Sigma.\frac{(i-1).n^2}{\{i\nu-n\}^2}.\left\{aa'.\left(\frac{dA^{(i-1)}}{da'}\right)-2.(i-1).aA^{(i-1)}\right\}.e'.\sin.(iT+W').$$

The fourth term of δv, [1020*a*], is $2a.\int n\,dt.r.\left(\frac{dR}{dr}\right)$, found by multiplying $r.\left(\frac{dR}{dr}\right)$, [1012*d*], by $2a.n\,dt$, and integrating

$$2a.\int n\,dt.r.\left(\frac{dR}{dr}\right)=-m'.\Sigma.\frac{n}{i\nu}.a^2.\left(\frac{dA^{(i)}}{da}\right).\sin.iT+m'a^2.\left(\frac{dA^{(0)}}{da}\right).nt$$

$$-m'.\left\{a^3.\left(\frac{ddA^{(0)}}{da^2}\right)+a^2.\left(\frac{dA^{(0)}}{da}\right)\right\}.e.\sin.W-m'.\left\{a^2a'.\left(\frac{ddA^{(1)}}{da\,da'}\right)+2a^2.\left(\frac{dA^{(1)}}{da}\right)\right\}.e'.\sin.W'$$

$$+m'.\Sigma.\frac{n}{i\nu-n}.\left\{a^3.\left(\frac{ddA^{(i)}}{da^2}\right)+a^2.\left(\frac{dA^{(i)}}{da}\right)+2i.a^2.\left(\frac{dA^{(i)}}{da}\right)\right\}.e.\sin.(iT+W) \qquad [1021e]$$

$$-m'.\Sigma.\frac{n}{n-i\nu}.\left\{a^2a'.\left(\frac{ddA^{(i-1)}}{da\,da'}\right)-2a^2.(i-1).\left(\frac{dA^{(i-1)}}{da}\right)\right\}.e'.\sin.(iT+W').$$

[1021'] The sign Σ includes, in these expressions, all the integral positive and negative values of i, the value $i=0$ being excepted [1012'].

Connecting together these four terms of δv, [1021b, c, d, e], we shall have the complete value of δv. The coefficients of these sines admit of various reductions. To obtain these, we shall compute each separately; first noting the terms, in the same order as they occur, in these expressions, [1021b, c, d, e], and then making the necessary reductions.

First, The term of δv, independent of cosines and sines, is

$$3\,m' . a\,g\,n\,t + m'\,a^2 . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) . n\,t,$$

[1021f] this is reduced to nothing, by putting $g = -\tfrac{1}{3}\,a . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right)$, [1017].

Second, The coefficient depending on $\sin. (n\,t + \varepsilon - \varpi)$, or $\sin. W$, in δv, [1021b, c, d, e] is

$$2\,m' . f\,e + m' . C\,e + \frac{m'}{6} . a^2\,e . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) - \tfrac{3}{2}\,m' . a^2\,e . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right)$$
$$- m' . \left\{ a^3 . \left(\frac{d\,d\,\mathcal{A}^{(0)}}{d\,a^2}\right) + a^2 . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) \right\} . e.$$

This is to be put equal to nothing, by [1017']. Dividing it by $-2\,m'\,e$, and transposing f, we get,

$$f = -\tfrac{1}{2}\,C - \tfrac{1}{12}\,a^2 . \left(\frac{d\mathcal{A}^{(0)}}{d\,a}\right) + \tfrac{3}{4}\,a^2 . \left(\frac{d\mathcal{A}^{(0)}}{d\,a}\right) + \tfrac{1}{2} . \left\{ a^3 . \left(\frac{d\,d\,\mathcal{A}^{(0)}}{d\,a^2}\right) + a^2 . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) \right\}.$$

Substituting the value of $-\tfrac{1}{2}\,C$, [1018], namely $-\tfrac{1}{2}\,a^2 . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) - \tfrac{1}{4}\,a^3 . \left(\frac{d^2\,\mathcal{A}^{(0)}}{d\,a^2}\right)$, and connecting like terms, we get $f = \tfrac{2}{3}\,a^2 . \left(\frac{d\,\mathcal{A}^{(0)}}{d\,a}\right) + \tfrac{1}{4}\,a^3 . \left(\frac{d^2\,\mathcal{A}^{(0)}}{d\,a^2}\right)$, as in [1018].

Third, The coefficient of $\sin. (n\,t + \varepsilon - \varpi')$, or $\sin. W'$, in δv, [1021b, c, d, e], is

$$2\,m' . f'\,e' + m' . D\,e' - \tfrac{3}{2}\,m' . \left\{ a\,a' . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a'}\right) + 2\,a\,\mathcal{A}^{(1)} \right\} . e'$$
$$- m' . \left\{ a^2\,a' . \left(\frac{d^2\,\mathcal{A}^{(1)}}{d\,a\,d\,a'}\right) + 2\,a^2 . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) \right\} . e'.$$

This being put equal to nothing, in the same manner as with the last coefficient, and then dividing by $-2\,m'\,e'$, also transposing f', we get

$$f' = -\tfrac{1}{2}\,D + \tfrac{3}{4} . \left\{ a\,a' . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a'}\right) + 2\,a\,\mathcal{A}^{(1)} \right\} + \tfrac{1}{2} . \left\{ a^2\,a' . \left(\frac{d^2\,\mathcal{A}^{(1)}}{d\,a\,d\,a'}\right) + 2\,a^2 . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) \right\}.$$

Substituting the value of D, [1018], of $a' . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a'}\right)$, and $a' . \left(\frac{d^2\,\mathcal{A}^{(1)}}{d\,a\,d\,a'}\right)$, [1003], it becomes, without reduction,

$$f' = -\tfrac{1}{2} . \left\{ a\,\mathcal{A}^{(1)} - a^2 . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) - \tfrac{1}{2}\,a^3 . \left(\frac{d^2\,\mathcal{A}^{(1)}}{d\,a^2}\right) \right\} + \tfrac{3}{4} . \left\{ a . \left[-\mathcal{A}^{(1)} - a . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) \right] + 2\,a\,\mathcal{A}^{(1)} \right\}$$
$$+ \tfrac{1}{2} . \left\{ a^2 . \left[-2 . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) - a . \left(\frac{d\,d\,\mathcal{A}^{(1)}}{d\,a^2}\right) \right] + 2\,a^2 . \left(\frac{d\,\mathcal{A}^{(1)}}{d\,a}\right) \right\},$$

We may here observe, that even when the series represented by

$$\Sigma\,.\,A^{(i)}\,.\,\cos.\,i\,.\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$$

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and by connecting the terms together, it becomes

$$f'=\tfrac{1}{4}\,.\left\{a\,A^{(1)}-a^2\,.\left(\frac{d\,A^{(1)}}{d\,a}\right)-a^3\,.\left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right)\right\}, \qquad [1021g]$$

as in [1018]. In the original, the sign of the last term was positive instead of negative. This was corrected afterwards by the author, in vol. iii, [4060].

Fourth, The coefficient of $\sin.\,i\,T$, in $\delta\,v$, [1021*b*, *c*, *d*, *e*], is

$$m'\,n\,.\,\Sigma\,.\,\frac{i\,\nu\,.\,G}{i^2\,\nu^2-n^2}-\tfrac{3}{2}\,m'\,.\,\Sigma\,.\,\frac{n^2}{i\,\nu^2}\,.\,a\,A^{(i)}-m'\,.\,\Sigma\,.\,\frac{n}{i\,\nu}\,.\,a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right),$$

substituting $a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)=G-\frac{2n}{\nu}\,.\,a\,A^{(i)}$, [1018*a*], it becomes

$$m'\,.\,\Sigma\,.\,\frac{i\,\nu\,n}{i^2\,\nu^2-n^2}\,.\,G-\tfrac{3}{2}\,m'\,.\,\Sigma\,.\,\frac{n^2}{i\,\nu^2}\,.\,a\,A^{(i)}-m'\,.\,\Sigma\,.\,\frac{n}{i\,\nu}\,.\left\{G-\frac{2\,n}{\nu}\,.\,a\,A^{(i)}\right\},$$

and by connecting the terms of $A^{(i)}$ and G,

$$\frac{m'}{2}\,.\,\Sigma\,.\left\{\frac{2\,i\,.\,\nu\,n}{i^2\,\nu^2-n^2}-\frac{2\,n}{i\,\nu}\right\}.\,G+\frac{m'}{2}\,.\,\Sigma\,.\left\{\frac{-3\,n^2+4\,n^2}{i\,\nu^2}\right\}.\,a\,A^{(i)},$$

or, by reduction $\frac{m'}{2}\,.\,\Sigma\,.\,\frac{2\,n^3\,.\,G}{i\,\nu\,.\,\{i^2\,\nu^2-n^2\}}+\frac{m'}{2}\,.\,\Sigma\,.\,\frac{n^2}{i\,\nu^2}\,.\,a\,A^{(i)}$, which is the same as the coefficient of $\sin.\,i\,.\,(n'\,t-n\,t+\varepsilon'-\varepsilon)$, in [1021].

Fifth, The terms of $\delta\,v$, [1021*b*], which contain t without the signs of cosine, agree with the second and third terms of the expression [1021].

Sixth, The terms of $\delta\,v$, depending on $\sin.\,(i\,T+W)$, [1021*b*, *c*, *d*, *e*], are $m'\,n\,e\,.\,\sin.\,(i\,T+W)$, multiplied by the following expression H, in which the sign Σ is to be prefixed to the terms of the second member, [1021*h*]

$$H=\frac{2\,.\,(i\,\nu-n)\,.\,E^{(i)}}{n^2-(n-i\,\nu)^2}+\frac{i\,\nu}{n^2-i^2\,\nu^2}\,.\,G+\frac{\tfrac{1}{2}\,n\,G}{i^2\,\nu^2-n^2}$$

$$+\frac{\tfrac{3}{2}\,.\,(i-1)\,.\,n}{(i\,\nu-n)^2}\,.\left\{a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)+2\,i\,.\,a\,A^{(i)}\right\}+\frac{1}{i\,\nu-n}\,.\left\{a^3\,.\left(\frac{d\,d\,A^{(i)}}{d\,a^2}\right)+(2\,i+1)\,.\,a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)\right\}.$$

This being multiplied by $n-i\,\nu$, and for $\frac{-2\,.\,(n-i\,\nu)^2}{n^2-(n-i\,\nu)^2}\,.\,E^{(i)}$, substituting its value $2\,E^{(i)}-\frac{2\,n^2}{n^2-(n-i\,\nu)^2}\,.\,E^{(i)}$, we shall get

$$(n-i\,\nu)\,.\,H=2\,E^{(i)}-\frac{2\,n^2\,.\,E^{(i)}}{n^2-(n-i\,\nu)^2}+\frac{i\,\nu}{n+i\,\nu}\,.\,G-\frac{\tfrac{1}{2}\,n\,G}{n+i\,\nu}$$

$$-\frac{\tfrac{3}{2}\,.\,(i-1)\,.\,n}{i\,\nu-n}\,.\left\{a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)+2\,i\,.\,a\,A^{(i)}\right\}-\left\{a^3\,.\left(\frac{d\,d\,A^{(i)}}{d\,a^2}\right)+(2\,i+1)\,.\,a^2\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)\right\}.$$

Substituting for $2\,E^{(i)}$, in the first term of the second member, its value given in [1018], which, by the symbols we have used, is converges slowly, the expressions of $\frac{\delta r}{a}$ and δv, become converging, by

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$$2\,E^{(i)} = -\frac{6n}{\nu}\,.\,a\,\mathcal{A}^{(i)} + \frac{\{2\,i^2\nu\,.\,(n+i\nu)-6\,n^2\}}{i^2\nu^2-n^2}\,.\,G + a^3\,.\left(\frac{d^2\,\mathcal{A}^{(i)}}{d\,a^2}\right),$$

we obtain,

$$(n-i\nu)\,.\,H = -\frac{6n}{\nu}\,.\,a\mathcal{A}^{(i)} + \left\{\frac{2\,i^2\nu\,.\,(n+i\nu)-6\,n^2}{i^2\nu^2-n^2} + \frac{i\nu}{n+i\nu} - \frac{\frac{1}{2}n}{n+i\nu}\right\}.\,G$$
$$-\frac{\frac{3}{2}\,.\,(i-1)\,.\,n}{i\nu-n}\,.\left\{a^2.\left(\frac{d\mathcal{A}^{(i)}}{da}\right) + 2\,i\,.\,a\,\mathcal{A}^{(i)}\right\} - (2\,i+1)\,.\,a^2\,.\left(\frac{d\mathcal{A}^{(i)}}{d\,a}\right) - \frac{2\,n^2\,.\,E^{(i)}}{n^2-(n-i\nu)^2},$$

and if in this we substitute, for $a^2\,.\left(\frac{d\mathcal{A}^{(i)}}{d\,a}\right)$, its value $G-\frac{2n}{\nu}\,.\,a\,\mathcal{A}^{(i)}$, [1018*a*], it will become

[1021*i*] $$(n-i\nu)\,.\,H = -\frac{6\,n}{\nu}\,.\,a\,\mathcal{A}^{(i)} + \left\{\frac{2i^2\nu.(n+i\nu)-6n^2}{i^2\nu^2-n^2} + \frac{i\nu}{n+i\nu} - \frac{\frac{1}{2}n}{n+i\nu} - \frac{\frac{3}{2}.(i-1)\,.\,n}{i\nu-n} - (2i+1)\right\}.\,G$$
$$+\frac{\frac{3}{2}.(i-1)\,.\,n}{i\nu-n}\,.\left\{\frac{2n}{\nu}\,.\,a\,\mathcal{A}^{(i)} - 2\,i\,.\,a\,\mathcal{A}^{(i)}\right\} + (2\,i+1)\,.\,\frac{2n}{\nu}\,.\,a\,\mathcal{A}^{(i)} - \frac{2\,n^2\,.\,E^{(i)}}{n^2-(n-i\nu)^2}.$$

The coefficient of $a\,\mathcal{A}^{(i)}$, in this expression is

$$\frac{n}{\nu}\,.\left\{-6+\frac{3\,.\,(i-1)\,.\,n}{i\nu-n}+2\,.\,(2i+1)\right\} - \frac{3i\,.\,(i-1)\,.\,n}{i\nu-n} = \frac{n}{\nu}\,.\left\{\frac{3\,.\,(i-1)\,.\,n}{i\nu-n}+2\,.\,(2i-2)\right\}$$
$$-\frac{3i\,.\,(i-1)\,.\,n}{i\nu-n} = (i-1)\,.\,\frac{n}{\nu}\,.\left\{\frac{3n}{i\nu-n}+4\right\} - \frac{3i.(i-1).n}{i\nu-n} = \frac{(i-1).n}{\nu}\,.\left\{\frac{3n}{i\nu-n}+4-\frac{3i\nu}{i\nu-n}\right\}$$
$$= \frac{(i-1).n}{\nu}\,.\left\{\frac{3n+4.(i\nu-n)-3i\nu}{i\nu-n}\right\} = \frac{(i-1)\,.\,n}{\nu}\,.\left\{\frac{i\nu-n}{i\nu-n}\right\} = \frac{(i-1)\,.\,n}{\nu},$$

[1021*k*] hence this term of $H\,.\,(n-i\nu)$, is $\frac{(i-1).n}{\nu}\,.\,a\,\mathcal{A}^{(i)}$. The coefficient of G, in [1021*i*], becomes, by putting the first term $\frac{2i^2\nu.(n+i\nu)}{i^2\nu^2-n^2} = \frac{2i^2\nu}{i\nu-n} = 2\,i+\frac{2in}{i\nu-n}$, and rejecting $2\,i-2\,i$, which destroy each other, $\frac{2in}{i\nu-n} - \frac{6\,n^2}{i^2\nu^2-n^2} + \frac{i\nu-\frac{1}{2}n}{i\nu+n} - \frac{\frac{3}{2}.(i-1)\,.\,n}{i\nu-n} - 1$; connecting the first and fourth terms, also the third and fifth terms, it becomes

$$\frac{\frac{1}{2}\,i\,n+\frac{3}{2}\,n}{i\nu-n} - \frac{\frac{3}{2}\,n}{i\nu+n} - \frac{6\,n^2}{i^2\nu^2-n^2},$$

by reducing all the terms to the denominator $i^2\nu^2-n^2$, it becomes $\frac{\frac{1}{2}in.(i\nu+n)-3n^2}{i^2\nu^2-n^2}$. This part of [1021*i*], being connected with that found in [1021*k*], gives

$$(n-i\nu)\,.\,H = \frac{(i-1).n}{\nu}\,.\,a\,\mathcal{A}^{(i)} + \frac{\frac{1}{2}in.(i\nu+n)-3n^2}{i^2\nu^2-n^2}\,.\,G - \frac{2\,n^2\,.\,E^{(i)}}{n^2-(n-i\nu)^2},$$

which is equal to $F^{(i)}$, [1019], therefore $H = \frac{F^{(i)}}{n-i\nu}$. This, by [1021*h*], is the coefficient of $m'ne\,.\,\sin.\,(i\,T+W)$, in δv. It agrees with [1021].

the divisors they acquire. This circumstance is the more important, as otherwise it might have been impossible to express analytically the reciprocal

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Seventh, The terms of δv, [1021*b, c, d, e*], depending on the angle $iT+W'$, are $m'ne'.\sin.(iT+W')$, multiplied by the following expression H', the sign Σ is to be prefixed as above, [1021*l*]

$$H'=\frac{2.(i\nu-n)}{n^2-(n-i\nu)^2}.D^{(i)}+\frac{\frac{3}{2}.(i-1).n}{(i\nu-n)^2}.\left\{aa'.\left(\frac{d\mathcal{A}^{(i-1)}}{da'}\right)-2.(i-1).a\mathcal{A}^{(i-1)}\right\}$$
$$+\frac{1}{i\nu-n}.\left\{a^2a'.\left(\frac{d^2\mathcal{A}^{(i-1)}}{da\,da'}\right)-2.(i-1).a^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)\right\},$$

multiplying this by $n-i\nu$, and then putting the coefficient of $D^{(i)}$, or $\frac{-2.(i\nu-n)^2}{n^2-(n-i\nu)^2}$, under the form $2-\frac{2n^2}{n^2-(n-i\nu)^2}$, substituting also the values of $\left(\frac{d\mathcal{A}^{(i-1)}}{da'}\right)$, $\left(\frac{d^2\mathcal{A}^{(i-1)}}{da\,da'}\right)$, [1003], we shall get

$$(n-i\nu).H'=2D^{(i)}-\frac{2n^2.D^{(i)}}{n^2-(n-i\nu)^2}+\frac{\frac{3}{2}.(i-1).n}{(i\nu-n)}.\left\{a^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)+(2i-1).a\mathcal{A}^{(i-1)}\right\}$$
$$+\left\{2ia^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)+a^3.\left(\frac{dd\mathcal{A}^{(i-1)}}{da^2}\right)\right\},$$

in which, for the first term of the second member $2D^{(i)}$, we must substitute its value [1018], which is

$$2D^{(i)}=\frac{2.(i-1).(2i-1).n}{n-i\nu}.a\mathcal{A}^{(i-1)}+\frac{(2i^2\nu-2n)}{n-i\nu}.a^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)-a^3.\left(\frac{dd\mathcal{A}^{(i-1)}}{da^2}\right),$$

and then, by connecting the similar terms, in the order in which they occur, we get

$$(n-i\nu).H'=\left\{\frac{2.(i-1).(2i-1).n}{n-i\nu}-\frac{\frac{3}{2}.(i-1).(2i-1).n}{n-i\nu}\right\}.a\mathcal{A}^{(i-1)}$$
$$+\left\{\frac{2i^2\nu-2n}{n-i\nu}-\frac{\frac{3}{2}.(i-1).n}{n-i\nu}+2i\right\}.a^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)-\frac{2n^2.D^{(i)}}{n^2-(n-i\nu)^2},$$

which is easily reduced to the form

$$(n-i\nu).H'=\frac{(i-1).(2i-1)}{2.(n-i\nu)}.na\mathcal{A}^{(i-1)}+\frac{(i-1)}{2.(n-i\nu)}.na^2.\left(\frac{d\mathcal{A}^{(i-1)}}{da}\right)-\frac{2n^2.D^{(i)}}{n^2-(n-i\nu)^2}=G^{(i)},$$

[1019], hence we get $H'=\frac{G^{(i)}}{n-i\nu}$, and, by [1021*l*], this is to be multiplied by $m'ne'.\sin.(iT+W')$, and the sign Σ prefixed, to obtain the corresponding term of δv, depending on the angle $iT+W'$, which will therefore be

$$nm'.\Sigma.\frac{G^{(i)}}{n-i\nu}.e'.\sin.(iT+W'),$$

as in [1021]. Thus we have proved the correctness of the expressions [1018—1022].

perturbations of those planets,* in which the ratio of their distances from the sun differs but little from unity.

These expressions may be reduced to other forms, which will be useful in the course of the work, by putting

[1022]
$$h = e\,.\,\sin.\varpi\,; \qquad h' = e'\,.\,\sin.\varpi'\,;$$
$$l = e\,.\,\cos.\varpi\,; \qquad l' = e'\,.\,\cos.\varpi'\,;$$

whence we shall get†

$$\frac{\delta r}{a} = \frac{m'}{6}\,.\,a^2.\left(\frac{d\,A^{(0)}}{d\,a}\right) + \frac{m'\,n^2}{2}\,.\Sigma.\,\frac{\left\{a^2.\left(\frac{d\,A^{(i)}}{d\,a}\right) + \frac{2\,n}{n-n'}.aA^{(i)}\right\}}{i^2.\,(n-n')^2 - n^2}.\cos.i.(n't - nt + \varepsilon' - \varepsilon)$$

[1023]
$$-m'\,.\,(hf + h'f')\,.\,\sin.\,(n\,t+\varepsilon) - m'\,.\,(lf + l'f')\,.\,\cos.\,(n\,t+\varepsilon)$$
$$+\frac{m'}{2}.\{l.C + l'.D\}.nt.\sin.(nt+\varepsilon) - \frac{m'}{2}.\{h.C + h'.D\}.nt.\cos.(nt+\varepsilon)$$
$$+n^2.m'.\Sigma\left\{\begin{array}{l} \frac{\{h\,E^{(i)} + h'.D^{(i)}\}}{n^2 - \{n - i.(n-n')\}^2}.\sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon\} \\ + \frac{\{l\,E^{(i)} + l'\,D^{(i)}\}}{n^2 - \{n - i.(n-n')\}^2}.\cos.\{i.(n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon\} \end{array}\right\};$$

Second forms of the perturbations in longitude and latitude.

$$\delta v = \frac{m'}{2}.\Sigma.\left\{\frac{n^2}{i.(n-n')^2}.aA^{(i)} + \frac{2n^3.\left\{a^2.\left(\frac{d\,A^{(i)}}{d\,a}\right) + \frac{2\,n}{n-n'}.aA^{(i)}\right\}}{i.(n-n').\{i^2.(n-n')^2 - n^2\}}\right\}.\sin.i.(n't - nt + \varepsilon' - \varepsilon)$$

[1024]
$$+m'.\{h.C + h'.D\}.nt.\sin.(nt+\varepsilon) + m'.\{l.C + l'.D\}.nt.\cos.(nt+\varepsilon)$$
$$+n\,m'.\Sigma.\left\{\begin{array}{l} \frac{\{l\,F^{(i)} + l'\,G^{(i)}\}}{n - i.(n-n')}.\sin.\{i.(n't - nt + \varepsilon' - \varepsilon) + n\,t + \varepsilon\} \\ - \frac{\{h\,F^{(i)} + h'G^{(i)}\}}{n - i.(n-n')}.\cos.\{i.(n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon\} \end{array}\right\}.$$

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* (702a) This will easily be perceived, by examining the terms of $\frac{\delta r}{a}$, $d\,v$, [1020, 1021]; it being evident, that as i increases, the divisors of the form $i.(n-n')$, will increase, and most commonly also, the divisor $i^2.(n-n')^2 - n^2$, &c.

† (703) By [22, 24] Int. we have $\cos.(H-\varpi) = \cos.H.\cos.\varpi + \sin.H.\sin.\varpi$; $\sin.(H-\varpi) = \sin.H.\cos.\varpi - \cos.H.\sin.\varpi$. Multiplying these by e, and substituting the values [1022], we shall get

[1023a] $e.\cos.(H-\varpi) = l.\cos.H + h.\sin.H$. $\qquad e.\sin.(H-\varpi) = l.\sin.H - h.\cos.H$.

In like manner $e'.\cos.(H-\varpi') = l'.\cos.H + h'.\sin.H$, and

$e'.\sin.(H-\varpi') = l'.\sin.H - h'.\cos.H$.

These values being substituted in [1020, 1021], we shall get, [1023, 1024].

Connecting these expressions of δr and δv with the values of r and v [669], [1024′]
in the elliptical motion, we shall have the whole values of the radius vector of m and its motion in longitude.

51. We shall now consider the motion of m in latitude. For this purpose we shall resume the formula (Z') § 47 [948]. If we neglect the product of [1024″]
the inclination by the excentricities of the orbits, it will become

$$0=\frac{d^2.\delta u'}{dt^2}+n^2.\delta u'-\frac{1}{a^2}.\left(\frac{dR}{dz}\right); \qquad [1025]$$

the expression of R § 48 [957] gives, by taking for the fixed plane the orbit of m at the commencement of the motion,* [1025′]

$$\left(\frac{dR}{dz}\right)=\frac{m'z'}{a'^3}-\frac{m'z'}{2}.\Sigma.B^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon). \qquad [1026]$$

The value of i comprises all integral positive and negative numbers, including also $i=0$ [954″]. Let γ be the tangent of the orbit of m' upon the primitive [1026′]
orbit of m, and Π the longitude of the ascending node of the first of these

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It has been remarked, by M. Plana, that the constant part of $\frac{\delta r}{a}$, [1020, 1023], represented by $\frac{m'}{6}.a^2.\left(\frac{d\mathcal{A}^{(0)}}{da}\right)$, does not express the *whole* variation of the mean distance a, arising from the disturbing force; or, in other words, the whole difference between the values of a, in the primitive orbit, and in the disturbed orbit; because the use of the constant quantity g, and the finding nt from observation, [1016″, 1021f], produce in the value of n, some part of the effect of the disturbing force; and as a is found from the equation $n^2a^3=1$, [1013a], it will also introduce into the assumed value of a, some part of the effect of this disturbing force. This subject is discussed by M. Plana, in vol. ii, page 326, of the Memoirs of the Astronomical Society of London, and in the same paper, he has also made several important and interesting remarks on other parts of the Mécanique Céleste.

* (704) The terms of the second member of [1026], are produced by the terms $\frac{mzz'}{a'^3}+\frac{m'.(z'-z)^2}{4}.\Sigma.B^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$, of R, [957]; rejecting the terms containing z, after taking the differentials, because $m'z$ is of the order of the square of the disturbing forces, [926′]. The other parts of R do not produce any term in $\left(\frac{dR}{dz}\right)$.

orbits, upon the second; we shall have very nearly*

[1027] $$z' = a' . \gamma . \sin. (n't + \varepsilon' - \Pi);$$

which gives

[1028] $$\left(\frac{dR}{dz}\right) = \frac{m'}{a'^2} . \gamma . \sin. (n't + \varepsilon' - \Pi) - \frac{m'}{2} . a' . B^{(1)} . \gamma . \sin. (nt + \varepsilon - \Pi)$$
$$- \frac{m'}{2} . a' . \Sigma . B^{(i-1)} . \gamma . \sin. \{i . (n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon - \Pi\}.$$

The value of i, in this and in the following expressions, includes all integral positive or negative numbers, excepting $i = 0$. The differential equation in $\delta u'$ will therefore become, by multiplying the value of $\left(\frac{dR}{dz}\right)$ by $n^2 a^3 = 1$, [1013a],† [1028]

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* (705) In the figure, page 351, if C be the place of the sun, D that of the planet m, B its projection on the fixed plane, we shall have $BD = z$, and by [678′, 679′], tang. $BCD = s$, $CB = r_{,}$. Then, in the rectangular triangle CBD, we shall get
[1026a] $BD = CB . \text{tang.} BCD$, or $z = r_{,} s$. If the orbit be but little inclined to the fixed
[1027a] plane, we shall have $r_{,}$ very nearly equal to r, [680], and z will become $z = rs$, as in
[1027b] [957^{vi}]. Substituting the value of s, [679], we shall find $z = r . \text{tang.} \varphi . \sin. (v_{,} - \theta)$.
Accenting the letters we shall get the corresponding expression for the planet m',

$$z' = r' . \text{tang.} \varphi' . \sin. (v'_{,} - \theta');$$

and if the orbit be nearly circular, we shall have r' nearly equal to a'; also $\text{tang.} \varphi' = \gamma$, [669″, 1026′], hence $z' = a' \gamma . \sin. (v'_{,} - \theta')$. But $v'_{,} - \theta'$ is nearly equal to $n't + \varepsilon' - \theta'$, [669′] or $n't + \varepsilon' - \Pi$, [1026′], hence $z' = a' \gamma . \sin. (n't + \varepsilon' - \Pi)$, as in [1027]. This being substituted in [1026], we shall get, by using [954b],

$$\left(\frac{dR}{dz}\right) = \frac{m'}{a'^2} . \gamma . \sin. (n't + \varepsilon' - \Pi) - \frac{m'}{2} . a' . \Sigma . B^{(i)} . \gamma . \sin. \{i . (n't - nt + \varepsilon' - \varepsilon) + n't + \varepsilon' - \Pi\},$$

and by changing i into $i - 1$, as in note 693,

$$\left(\frac{dR}{dz}\right) = \frac{m'}{a'^2} . \gamma . \sin. (n't + \varepsilon' - \Pi) - \frac{m'}{2} . a' . \Sigma . B^{(i-1)} . \gamma . \sin. \{i . (n't - nt + \varepsilon' - \varepsilon) + nt + \varepsilon - \Pi\},$$

in which $B^{(i-1)}$ includes the term depending on $i = 0$; if we wish to exclude this we must bring the term, depending on $B^{(-1)}$, from under the sign Σ, and then we shall obtain the expression [1028], observing that $B^{(-1)} = B^{(1)}$, [956′].

† (706) Multiplying the term $\frac{1}{a^2} . \left(\frac{dR}{dz}\right)$, [1025], by $n^2 a^3 = 1$, [1013a], it becomes $n^2 a . \left(\frac{dR}{dz}\right)$, and by using the value of $\left(\frac{dR}{dz}\right)$, [1028], the equation [1025] will take the form [1029].

$$0=\frac{d\,d.\delta u'}{d t^2}+n^2.\delta u'-m'.n^2.\frac{a}{a'^2}.\gamma.\sin.(n't+\varepsilon'-\Pi)$$

$$+\frac{m'.n^2}{2}.a a'.B^{(1)}.\gamma.\sin.(n t+\varepsilon-\Pi)$$ [1029]

$$+\frac{m'.n^2}{2}.a a'.\Sigma.B^{(i-1)}\gamma\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+n t+\varepsilon-\Pi\};$$

hence, by taking the integral, and observing that by § 47 [948] $\delta s=-a.\delta u'$,* [1029']

Perturbations in latitude.

$$\delta s=-\frac{m'.n^2}{n^2-n'^2}.\frac{a^2}{a'^2}.\gamma.\sin.(n't+\varepsilon'-\Pi)$$

$$-\frac{m'.a^2a'}{4}.B^{(1)}.n t.\gamma.\cos.(n t+\varepsilon-\Pi)$$ [1030]

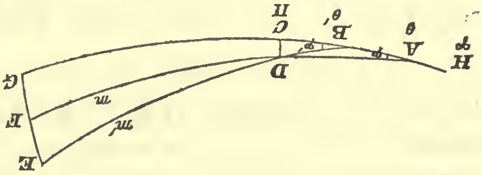
$$+\frac{m'.n^2.a^2a'}{2}.\Sigma.\frac{B^{(i-1)}}{n^2-\{n-i.(n-n')\}^2}.\gamma.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\Pi\}.$$

To obtain the latitude of m above a fixed plane, but little inclined to that of its primitive orbit,† we shall put φ for the inclination of this orbit to the fixed [1030']

---

* (707) Put $y=\delta u'$, and $a=n$, in the differential equation, [865], and it will become of the same form as [1029]. Its integral [870—871''], will give $\delta u'$, which being multiplied by $-a$, gives $-a\,\delta u'=\delta s$, [1029'], and the result will be as in [1030]; no constant terms c, c', being added, because they are included in p, q, &c., mentioned in the general value of s, [1034].

† (708) Suppose a spherical surface to be described about the body M, as a centre, with a radius equal to unity, cutting the fixed plane, in the arc $HABCG$; the primitive orbit of m, in ADF; and the orbit of m', in BDE. From the point D, let fall the arc DC, perpendicular to HG; and from any other point E, corresponding to the time t, let fall the perpendicular EFG. Then taking H for the origin of the longitudes, we shall have $HA=\theta$, $HB=\theta'$, $HC=\Pi$, nearly. $HG=nt+\varepsilon$, $AG=nt+\varepsilon-\theta$, and $CG=nt+\varepsilon-\Pi$, nearly; $FAG=\varphi$, $EBG=\varphi'$, tang. $EDF=\gamma$. Then in the right angled spherical triangles, AGF, BGE, we shall have



$$\text{tang.}\,FG=\text{tang.}\,FAG.\sin.AG=\text{tang.}\,\varphi.\sin.(nt+\varepsilon-\theta),$$ [1030a]

$$\text{tang.}\,EG=\text{tang.}\,EBG.\sin.BG=\text{tang.}\,\varphi'.\sin.(nt+\varepsilon-\theta'),$$

plane, θ for the longitude of its ascending node upon the same plane; then this latitude would be obtained with sufficient exactness, by adding δs to
[1030''] the following quantity, $\text{tang.}\,\varphi\,.\,\text{sin.}\,(v-\theta)$, or $\text{tang.}\,\varphi\,.\,\text{sin.}\,(nt+\varepsilon-\theta)$, neglecting the excentricity of the orbit [669]. We shall also put φ' and θ', in the orbit of m', to correspond to φ and θ in the orbit of m. If m should be supposed to move in the primitive orbit of m', the tangent of its latitude
[1030'''] would be $\text{tang.}\,\varphi'\,.\,\text{sin.}\,(nt+\varepsilon-\theta')$; it would be $\text{tang.}\,\varphi\,.\,\text{sin.}\,(nt+\varepsilon-\theta)$, if m should continue to move in its primitive orbit. The difference of these
[1030$^{\text{iv}}$] two tangents is nearly equal to the tangent of the latitude of m above the plane of its primitive orbit, supposing it should move in the plane of the primitive orbit of m'; therefore we shall have

[1031] $$\text{tang.}\,\varphi'\,.\,\text{sin.}\,(nt+\varepsilon-\theta')-\text{tang.}\,\varphi\,.\,\text{sin.}\,(nt+\varepsilon-\theta)=\gamma\,.\,\text{sin.}\,(nt+\varepsilon-\Pi).$$

Putting

[1032] $$\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta=p\,; \qquad \text{tang.}\,\varphi'\,.\,\text{sin.}\,\theta'=p'\,;$$
$$\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta=q\,; \qquad \text{tang.}\,\varphi'\,.\,\text{cos.}\,\theta'=q'\,;$$

we shall have*

[1033] $$\gamma\,.\,\text{sin.}\,\Pi=p'-p\,; \qquad \gamma\,.\,\text{cos.}\,\Pi=q'-q\,;$$

The first subtracted from the second, gives $\text{tang.}\,EG-\text{tang.}\,FG$, which, by [30] Int., is equal to $\text{tang.}\,(EG-FG)\,.\,\{1+\text{tang.}\,EG\,.\,\text{tang.}\,FG\}$, or simply

$$\text{tang.}\,(EG-FG)=\text{tang.}\,EF,$$

neglecting terms of the third order in EG, FG, hence,

$$\text{tang.}\,EF=\text{tang.}\,\varphi'\,.\,\text{sin.}\,(nt+\varepsilon-\theta')-\text{tang.}\,\varphi\,.\,\text{sin.}\,(nt+\varepsilon-\theta),$$

and this would represent, very nearly, the tangent of the latitude of m, above the plane of its primitive orbit, supposing it should move in the orbit of m'. Now this same tangent corresponding to the angle $nt+\varepsilon$, is, by [1027], equal to $\gamma\,.\,\text{sin.}\,(nt+\varepsilon-\Pi)$. Putting these two expressions equal to each other we shall obtain [1031].

* (709) Put for brevity $nt+\varepsilon=\tau$, and the expression [1031] will become

$$\text{tang.}\,\varphi'\,.\,\text{sin.}\,(\tau-\theta')-\text{tang.}\,\varphi\,.\,\text{sin.}\,(\tau-\theta)=\gamma\,.\,\text{sin.}\,(\tau-\Pi).$$

Developing the sines of $\tau-\theta'$, $\tau-\theta$, $\tau-\Pi$, by [22] Int., we shall get

$$\text{tang.}\,\varphi'\,.\,\{\text{sin.}\,\tau\,.\,\text{cos.}\,\theta'-\text{cos.}\,\tau\,.\,\text{sin.}\,\theta'\}-\text{tang.}\,\varphi\,.\,\{\text{sin.}\,\tau\,.\,\text{cos.}\,\theta-\text{cos.}\,\tau\,.\,\text{sin.}\,\theta\}$$
$$=\gamma\,.\,\{\text{sin.}\,\tau\,.\,\text{cos.}\,\Pi-\text{cos.}\,\tau\,.\,\text{sin.}\,\Pi\},$$

therefore if we put s equal to the latitude of m above the fixed plane, we shall have nearly* [1033']

Formula for the latitude.

$$
\begin{aligned}
s = {} & q \,.\, \sin.(nt+\varepsilon) - p \,.\, \cos.(nt+\varepsilon) \\
& - \frac{m'.a^2a'}{4} \,.\, (p'-p) \,.\, B^{(1)} \,.\, nt \,.\, \sin.(nt+\varepsilon) \\
& - \frac{m'.a^2a'}{4} \,.\, (q'-q) \,.\, B^{(1)} \,.\, nt \,.\, \cos.(nt+\varepsilon) \\
& - \frac{m'.n^2}{n^2-n'^2} \,.\, \frac{a^2}{a'^2} \,.\, \{(q'-q) \,.\, \sin.(n't+\varepsilon') - (p'-p) \,.\, \cos.(n't+\varepsilon')\} \\
& + \frac{m'.n^2.a^2a'}{2} .\Sigma. \left\{ \begin{array}{l} \dfrac{(q'-q) \,.\, B^{(i-1)}}{n^2-\{n-i.(n-n')\}^2} . \sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\} \\ \dfrac{-(p'-p) \,.\, B^{(i-1)}}{n^2-\{n-i.(n-n')\}^2} . \cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\} \end{array} \right\} .
\end{aligned}
\qquad [1034]
$$

52. We shall now collect together the formulas which we have here computed. Putting (r) and (v) for the parts of the radius vector, and the longitude v, upon the orbit, depending on the elliptical motion; we shall have† [1034']

$$r = (r) + \delta r \,; \qquad\qquad v = (v) + \delta v. \qquad [1035]$$

---

and by substituting the values [1032], it will become

$$q' \,.\, \sin.\tau - p' \,.\, \cos.\tau - q \,.\, \sin.\tau + p \,.\, \cos.\tau = \gamma \,.\, \cos.\Pi \,.\, \sin.\tau - \gamma \,.\, \sin.\Pi \,.\, \cos.\tau,$$

and, as this ought to exist, for all values of τ, the coefficients of $\sin.\tau$, $\cos.\tau$, in each member of the equation, must be equal to each other; hence we obtain the two equations [1033].

* (710) If the body m should continue to move in the primitive orbit ADF, in the figure page 563, its latitude FG would be, as in note 708, nearly equal to

$$\text{tang.}\,\varphi \,.\, \sin.(nt+\varepsilon-\theta), \qquad [1034a]$$

which being developed, as in the last note, is $q \,.\, \sin.(nt+\varepsilon) - p \,.\, \cos.(nt+\varepsilon)$. These are the two first terms of s, [1034]; the other terms are deduced from the value of δs, [1030], by similar developments, relative to Π, substituting the values [1033]. Thus

$$
\begin{aligned}
\gamma \,.\, \sin.(n't+\varepsilon'-\Pi) &= \gamma \,.\, \cos.\Pi \,.\, \sin.(n't+\varepsilon') - \gamma \,.\, \sin.\Pi \,.\, \cos.(n't+\varepsilon') \\
&= (q'-q) \,.\, \sin.(n't+\varepsilon') - (p'-p) \,.\, \cos.(n't+\varepsilon'),
\end{aligned}
$$

and so on for the other terms.

† (711) The values (r), (v), are given in [669], δr and δv in [1023, 1024].

The preceding value of s [1034] will be the latitude of m above the fixed
[1035'] plane; but it will be more accurate to use, instead of its two first terms,
which are independent of m', the value of the latitude, which would take
place if m did not quit the plane of its primitive orbit.* *These expressions
contain the whole theory of the planets, when we neglect the squares and
products of the excentricities and inclinations of the orbits, which can generally*
[1035''] *be done.* They have besides the advantage of being under a very simple
form, in which we can easily perceive the law of their different terms.

Sometimes it will be necessary to include terms depending on the squares
and products of the inclinations and excentricities, and even of higher
powers and products. We may determine these terms by the preceding
[1035'''] analysis: the consideration which renders them necessary will always
facilitate their computation. The approximations in which these are noticed,
will introduce other terms, depending on new arguments; they will also
reproduce the arguments, given by the preceding approximations, but with
smaller coefficients, according to the following law, which is easy to deduce
[1035$^{\text{iv}}$] from the development of R in a series, in § 48; *an argument which in the
successive approximations, is found for the first time among quantities of an
order* r, *is reproduced only by quantities of the orders* r+2, r+4, &c.†

Hence it follows, that the coefficients of the terms, of the form $t.{}^{\text{sin.}}_{\text{cos.}}(nt+\varepsilon)$,

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* (711a) These two terms express the tangent of the latitude, which was taken for the latitude in [1034a], it is therefore more exact to use the latitude itself.

† (712) Comparing the values of r, v, [952, 953], with those in [659, 668], altered as in [669], it will be perceived that the elliptical values of r, v, $u_{\prime}$, $v_{\prime}$, and therefore of r', v', $u'_{\prime}$, $v'_{\prime}$, possess the property mentioned in [1035$^{\text{iv}}$], relative to the successive terms of the series. This law would not be affected by reductions similar to those in [675, 676'], and a little attention will also show, that z, z', [1027], are affected in like manner. Therefore all the terms of R, [957], possess this property, and the same must evidently take place with $2\int \mathrm{d}R + r.\left(\frac{dR}{dr}\right)$, $r.\left(\frac{dR}{dr}\right)$, and $\left(\frac{dR}{dz}\right)$. Hence it follows that δr, δv, δs, [930, 931, 932], are formed in a similar manner, consequently $(r)+\delta r$, $(v)+\delta v$, $(s)+\delta s$, or the complete values of r, v, s, must each be expressed by a series, whose successive terms, depending on the same angle, have the same property as in [1035$^{\text{iv}}$].

which enter into the expressions of r, v, s, [1023, 1024, 1030, 1035], are [1035^{v}]
correct as far as quantities of the third order; that is, the approximation in which we shall notice the squares and products of the excentricities and inclinations of the orbits, will add nothing to these values; they have, therefore, all the precision that is necessary. This is the more important, [1035^{vi}]
because the secular variations of the orbits depend on these coefficients.

The various terms of the perturbations of r, v, s, are comprised in the form

$$k\,.\,{\textstyle\overset{\text{sin.}}{\text{cos.}}}\{i\,.\,(n't-nt+\varepsilon'-\varepsilon)+\mathrm{r}\,nt+\mathrm{r}\,\varepsilon\},\qquad [1036]$$

r being a whole number, or nothing; and k a function of the excentricities and inclinations of the orbits, of the order r, or of a higher order;* hence [1036']
we may judge of the order of any term depending on a given angle.

It is evident that the action of the bodies m'', m''', &c., produces in r, v, s, some additional terms, similar to those resulting from the action of m'; and [1036'']
by neglecting the square of the disturbing force, the sums of all these terms

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* (713) From the remarks [957^{ix}], it appears that the elliptical values of r, v, $u_{,}$, $v_{,}$, &c., have the property mentioned in [1036']. The formula [961] shows also that R has the same property; for by putting

$$i'n't-int+i'\varepsilon'-i\varepsilon=T',\qquad g\varpi+g'\varpi'+g''\theta+g'''\theta'=G,$$
$$H'=H\,.\,e^{g}\,.\,e'^{g'}\,.\,(\text{tang.}\,\tfrac{1}{2}\,\varphi)^{g''}\,.\,(\text{tang.}\,\tfrac{1}{2}\,\varphi')^{g'''},\qquad [1036a]$$

the formula [961] becomes

$$H'\,.\,\text{cos.}\,(T'-G)=H'\,.\,(\text{sin.}\,G\,.\,\text{sin.}\,T'+\text{cos.}\,G\,.\,\text{cos.}\,T'),$$

[24] Int.; and by putting $H'\,.\,\text{sin.}\,G=k$, $H'\,.\,\text{cos}\,G=k'$, it changes into
$k\,.\,\text{sin.}\,T'+k'\,.\,\text{cos.}\,T'$. Now if we put $T=n't-nt+\varepsilon'-\varepsilon$, and $\mathrm{r}=i'-i$, [1036b]
the value of T', [1036a], will be

$$T'=i'n't-int+i'\varepsilon'-i\varepsilon=i'\,.\,(n't-nt+\varepsilon'-\varepsilon)+(i'-i)\,.\,(nt+\varepsilon)=i'\,T+\mathrm{r}\,.\,(nt+\varepsilon),$$

and the expression [1036b] will become

$$k\,.\,\text{sin.}\,(i'\,T+\mathrm{r}\,nt+\mathrm{r}\,\varepsilon)+k'\,.\,\text{cos.}\,(i'\,T+\mathrm{r}\,nt+\mathrm{r}\,\varepsilon),$$

which is of the same form as in [1036], and this term of R is, by [961'], of the order r, or of a higher order. Lastly, as the value of R, and the elliptical values of r, v, &c., satisfy the above condition, it is evident from the equations [930, 931, 932], that δr, δv, δs, must also be subject to the same condition.

will give the complete values of r, v, s. This follows from the nature of the [1036'''] formulas (X'), (Y), and (Z'), [946, 931, 948], which are linear with respect to quantities depending on the disturbing force.*

Lastly, we shall obtain the perturbations of m', produced by the action of [1036iv] m, by changing, in the preceding formulas, a, n, h, l, ε, ϖ, p, q, and m', into a', n', h', l', ε', ϖ', p', q', and m, and the contrary.

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* (713a) That is, R and its differential are found only in the first power in these equations.

CHAPTER VII.

ON THE SECULAR INEQUALITIES OF THE MOTIONS OF THE HEAVENLY BODIES.

53. The forces which disturb the elliptical motion, introduce into the expressions of r, $\frac{dv}{dt}$, and s, of the preceding chapter, the time t, out [1036v] of the signs of *sine* and *cosine*, or under the form of arcs of a circle; and as these arcs increase indefinitely, they will finally render the expressions defective. It is therefore necessary to make these arcs disappear, by reducing the series which contain them to the original functions, from which they were produced by development. We have given, for this purpose, in Chapter V, a general method, from which it follows, that these arcs arise from the variations of the elements of the elliptical motion, which then become functions of the time. As these variations are produced in a very [1036vi] slow manner, they have been called by the name of *secular equations*. The Secular equations. theory of these equations is one of the most interesting points in the system of the world; and we shall here explain the subject with all the fulness its importance requires.

We have, by the preceding chapter,*

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* (714) The expression [1037] is found by adding the values of r, [669], and δr, [1023], putting $m'S$ for all the terms of δr, which do not contain nt without the signs of *sine* and *cosine*; developing also the term $-ae.\cos.(nt+\varepsilon-\varpi)$, of the expression [669], as in [1023*a*], by which means it is reduced to the form

$$-ah.\sin.(nt+\varepsilon)-al.\cos.(nt+\varepsilon). \qquad [1037a]$$

The other terms, depending on quantities of the order e^2, e^3, &c., [669], might be developed in a similar manner, and would produce, in [1037], quantities of the order h^2, l^2, &c.; but such terms are neglected in the present calculation, as is observed in [1051″]. The value

[1037]
$$r=a.\left\{\begin{array}{l} 1-h.\sin.(nt+\varepsilon)-l.\cos.(nt+\varepsilon)-\&c. \\ +\frac{m'}{2}.\{l.C+l'.D\}.nt.\sin.(nt+\varepsilon) \\ -\frac{m'}{2}.\{h.C+h'.D\}.nt.\cos.(nt+\varepsilon)+m'.S \end{array}\right\};$$

[1038]
$$\begin{array}{l}\frac{dv}{dt}=n+2nh.\sin.(nt+\varepsilon)+2nl.\cos.(nt+\varepsilon)+\&c. \\ \qquad -m'.\{l.C+l'.D\}.n^2t.\sin.(nt+\varepsilon) \\ \qquad +m'.\{h.C+h'.D\}.n^2t.\cos.(nt+\varepsilon)+m'.T;\end{array}$$

[1039]
$$\begin{array}{l}s=q.\sin.(nt+\varepsilon)-p.\cos.(nt+\varepsilon)+\&c. \\ \qquad -\frac{m'}{4}.a^2a'.(p'-p).B^{(1)}.nt.\sin.(nt+\varepsilon) \\ \qquad -\frac{m'}{4}.a^2a'.(q'-q).B^{(1)}.nt.\cos.(nt+\varepsilon)+m'.\chi;\end{array}$$

[1039′] S, T, and χ, being periodical functions of the time t. We shall first consider the expression of $\frac{dv}{dt}$, and compare it to that of y, § 43 [877]. As the arbitrary constant quantity n is multiplied by t,* under the periodical signs, in

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$\frac{dv}{dt}$, [1038], is found, by adding v, [669], to δv, [1024], taking the differential of the sum relative to dt, dividing it by dt and putting $m'T$ for all the terms arising from δv, except those containing nt without the signs of *sine* and *cosine*; developing also, as before, the term $2ne.\cos.(nt+\varepsilon-\varpi)$, of $\frac{dv}{dt}$, deduced from [669], so as to put it under the form
[1037b] $2nh.\sin.(nt+\varepsilon)+2nl.\cos.(nt+\varepsilon)$. The value s is the same as that of [1034], putting $m'\chi$ for all the terms of δs, independent of the arcs of a circle. The reason of using $\frac{dv}{dt}$, instead of v, in [1038], is to render the second member free from t, without the sign of sine and cosine, except in the terms depending on C, D, arising from the disturbing forces; by which means, it becomes of a form that is directly comparable with the value of y [877].

* (715) The value of y, [887], being compared with $\frac{dv}{dt}$, [1038], gives X and Y, as in [1041], Z, &c., being nothing. Now by [889] we have $\left(\frac{dX}{d\theta}\right)=X'+tX''$, &c., the arbitrary quantities n, h, l, ε, being considered as functions of θ. Therefore from the value of X, [1041], we must find $\left(\frac{dX}{d\theta}\right)$, considering n, h, l, ε, variable, and we shall obtain X', X'', [1042].

the value of $\frac{dv}{dt}$, we must use the following equations, computed in § 43 [892]:

$$\begin{aligned} 0 &= X' + \theta \,.\, X'' - Y ; \\ 0 &= Y' + \theta \,.\, Y'' + X'' - 2\,Z ; \\ &\text{\&c.} \end{aligned}$$ [1040]

We must now find what X, X', X'', Y, &c., become in this case; and if we compare the expression of $\frac{dv}{dt}$ [1038], with that of y, in the article just quoted, [877], we shall get

$$\begin{aligned} X &= n + 2\,n\,h \,.\, \sin.\,(n\,t+\varepsilon) + 2\,n\,l \,.\cos.\,(nt+\varepsilon) + m'\,T ; \\ Y &= m' \,.\, n^2 . \{h \,.\, C + h' \,.\, D\} . \cos.\,(nt+\varepsilon) - m' . n^2 . \{l . C + l' . D\} . \sin.(nt+\varepsilon). \end{aligned}$$ [1041]

If we neglect the product of the partial differentials of the constant quantities by the disturbing masses, which may be done, because these differentials are of the same order as the masses, we shall have by § 43 [889], [1041′]

$$\begin{aligned} X' = {} & \left(\frac{dn}{d\theta}\right) . \{1 + 2\,h \,.\, \sin.\,(nt+\varepsilon) + 2\,l \,.\, \cos.\,(nt+\varepsilon)\} \\ & + 2\,n \,.\, \left(\frac{d\varepsilon}{d\theta}\right) . \{h \,.\, \cos.\,(nt+\varepsilon) - l \,.\, \sin.\,(nt+\varepsilon)\} \\ & + 2\,n \,.\, \left(\frac{dh}{d\theta}\right) . \sin.\,(nt+\varepsilon) + 2n . \left(\frac{dl}{d\theta}\right) . \cos.\,(n\,t+\varepsilon) ; \\ X'' = {} & 2n \,.\, \left(\frac{dn}{d\theta}\right) . \{h \,.\, \cos.\,(n\,t+\varepsilon) - l \,.\, \sin.\,(n\,t+\varepsilon)\}. \end{aligned}$$ [1042]

Hence the equation $0 = X' + \theta \,.\, X'' - Y$, will become

$$\begin{aligned} 0 = {} & \left(\frac{dn}{d\theta}\right) . \{1 + 2\,h \,.\, \sin.\,(n\,t+\varepsilon) + 2\,l \,.\, \cos.\,(n\,t+\varepsilon)\} \\ & + 2\,n \,.\, \left(\frac{dh}{d\theta}\right) . \sin.\,(n\,t+\varepsilon) + 2\,n \,.\, \left(\frac{dl}{d\theta}\right) . \cos.\,(n\,t+\varepsilon) \\ & + 2\,n \,.\, \left\{ \theta \,.\, \left(\frac{dn}{d\theta}\right) + \left(\frac{d\varepsilon}{d\theta}\right) \right\} . \{h \,.\, \cos.\,(n\,t+\varepsilon) - l \,.\, \sin.\,(nt+\varepsilon)\} \\ & - m' . n^2 . \{h . C + h' . D\} . \cos.\,(nt+\varepsilon) + m' . n^2 . \{l . C + l' . D\} . \sin.\,(nt+\varepsilon). \end{aligned}$$ [1043]

The coefficients of the different sines and cosines being put separately equal to nothing,* we shall have

$$0=\left(\frac{dn}{d\theta}\right);$$

[1044] $$0=\left(\frac{dh}{d\theta}\right)-l.\left(\frac{d\varepsilon}{d\theta}\right)+\frac{m'.n}{2}.\{l.C+l'.D\};$$

$$0=\left(\frac{dl}{d\theta}\right)+h.\left(\frac{d\varepsilon}{d\theta}\right)-\frac{m'.n}{2}.\{h.C+h'.D\}.$$

[1044'] If we integrate these equations, and in their integrals change θ into t, we shall have, by § 43 [885''], the values of the arbitrary quantities, in functions of t, and we may then efface the arcs of a circle from the values of $\frac{dv}{dt}$, and r; but instead of this change, we may, in the first instance, write t for θ, in these differential equations.† The first of these equations shows, that n is constant; and as the arbitrary quantity a in the expression of r depends on

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* (716) The equation [887] is identical, as appears from [887', 879'], therefore [892] and [1043], which were deduced from it, must also be identical; consequently the coefficient of the different sines and cosines must be nothing, as well as the term independent of those sines and cosines. Now this last term is $\left(\frac{dn}{d\theta}\right)$, which, being put equal to nothing, gives the first equation, [1044]. This being substituted in [1043], and the whole divided by $2n$, gives

$$0=\left(\frac{dh}{d\theta}\right).\sin.(nt+\varepsilon)+\left(\frac{dl}{d\theta}\right).\cos.(nt+\varepsilon)+\left(\frac{d\varepsilon}{d\theta}\right).\left\{h.\cos.(nt+\varepsilon)-l.\sin.(nt+\varepsilon)\right\}$$
$$-\frac{m'n}{2}.(h\,C+h'D).\cos.(nt+\varepsilon)+\frac{m'n}{2}.(l\,C+l'D).\sin.(nt+\varepsilon).$$

The coefficient of $\sin.(nt+\varepsilon)$, being put equal to nothing, gives the second equation [1044], and the third equation is found by putting the coefficient of $\cos.(nt+\varepsilon)$ equal to nothing.

† (717) This may be done because C, D, n, which occur in the second and third of the equations [1044] are constant, these terms being functions of a, a', as is evident from the values of C, D, [1018], which are functions of a, a', [954]; and n [1013a] is equal to $a^{-\frac{3}{2}}$, which is constant [1044''].

it, by means of the equation $n^2=\frac{1}{a^3}$, [1013*a*], *a* will also be constant. The other two equations are not sufficient for the determination of h, l, ε. We [1044′] may obtain another equation, observing that the expression of $\frac{dv}{dt}$ gives by integration $\int n\,dt$ for the value of the mean longitude of m;* now we have supposed this longitude equal to $nt+\varepsilon$ [952′]; therefore we shall have $nt+\varepsilon=\int n\,dt$, which gives [1044, 1045′], [1044‴]

$$t\,.\frac{dn}{dt}+\frac{d\varepsilon}{dt}=0\,; \qquad [1045]$$

and as $\frac{dn}{dt}=0$, we shall also have $\frac{d\varepsilon}{dt}=0$. Thus the two arbitrary [1045′] quantities n and ε are constant; the arbitrary quantities h and l will therefore be determined by means of the differential equations

[1046]

$$\frac{dh}{dt}=-\frac{m'.n}{2}.\{l.C+l'.D\}\,; \qquad (1)$$

$$\frac{dl}{dt}=\frac{m'.n}{2}.\{h.C+h'.D\}. \qquad (2)$$

The consideration of the expression of $\frac{dv}{dt}$ having enabled us to determine the values of n, a, h, l, and ε; we see *à priori*, that the differential equations, between the same quantities, which would result from the expression of r, [1046′] must agree with the preceding. This may be easily proved *à posteriori*, by applying to this expression the method of § 43.†

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* (718) This is evident from the equation [1038]. The differential of [1044‴] gives [1045].

† (719) Putting $a=n^{-\frac{2}{3}}$, in [1037], and comparing the resulting expression of r with that of y, [877], we shall obtain values of X, Y, of the following forms,

[1046*a*]

$$X=n^{-\frac{2}{3}}.\{1-h.\sin.(nt+\varepsilon)-l.\cos.(nt+\varepsilon)-\&c.\}+m'S\,;$$

$$Y=\tfrac{1}{2}m'.n^{-\frac{2}{3}}.\{n.(l\,C+l'\,D).\sin.(nt+\varepsilon)-n.(h\,C+h'\,D).\cos.(nt+\varepsilon)\},$$

which correspond to the equations [1041]. From these we may deduce other expressions analogous to those in [1042—1044]. A very slight attention makes it evident that the

We shall now consider the expression of s [1039]. Comparing it with that of y, in the article before mentioned, we shall find,*

[1047]
$$X = q\,.\,\text{sin.}\,(n\,t+\varepsilon) - p\,.\,\text{cos.}\,(n\,t+\varepsilon) + m'\,.\,\chi\,;$$
$$Y = \frac{m'\,.\,n}{4}\,.\,a^2\,a'\,.\,B^{(1)}\,.\,(p-p')\,.\,\text{sin.}\,(n\,t+\varepsilon)$$
$$+ \frac{m'\,.\,n}{4}\,.\,a^2\,a'\,.\,B^{(1)}\,.\,(q-q')\,.\,\text{cos.}\,(n\,t+\varepsilon)\,;$$

n and ε being constant, as has been already shown [1045′]. Hence we shall have, by § 43,

[1048]
$$X' = \left(\frac{d\,q}{d\,\theta}\right).\,\text{sin.}\,(n\,t+\varepsilon) - \left(\frac{d\,p}{d\,\theta}\right).\,\text{cos.}\,(n\,t+\varepsilon)\,;$$
$$X'' = 0.$$

The equation $0 = X' + \theta\,.\,X'' - Y$ [1040], will by this means become

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equations [1046a] will produce, in the terms independent of $\overset{\text{sin.}}{\text{cos.}}\,(n\,t+\varepsilon)$, an equation $\left(\frac{d n}{d\,\theta}\right) = 0$, like the first of [1044], which gives n constant, consequently a constant, and ε is then constant as in [1045′]. Supposing now n, a, ε, to be constant in the value of X, and putting $\left(\frac{dX}{d\,\theta}\right) = X' + t\,X''$, as in [889], we obtain

$$X' = n^{-\frac{2}{3}}\,.\left\{-\left(\frac{dh}{d\,\theta}\right).\,\text{sin.}\,(n\,t+\varepsilon) - \left(\frac{d\,l}{d\,\theta}\right).\,\text{cos.}\,(n\,t+\varepsilon) - \&\text{c.}\right\}, \qquad X'' = 0,$$

and the equation $0 = X' + \theta\,X'' - Y$, [1040], becomes $0 = X' - Y$, or by dividing by $n^{-\frac{2}{3}}$,

$$0 = \left\{-\left(\frac{dh}{d\,\theta}\right) - \tfrac{1}{2}\,m'\,n\,.\,(l\,C + l'\,D)\right\}.\,\text{sin.}\,(n\,t+\varepsilon)$$
$$+ \left\{-\left(\frac{d\,l}{d\,\theta}\right) + \tfrac{1}{2}\,m'\,n\,.\,(h\,C + h'\,D)\right\}.\,\text{cos.}\,(n\,t+\varepsilon),$$

from which we get the equations [1046], by putting the coefficients of sin. $(n\,t+\varepsilon)$, and cos. $(n\,t+\varepsilon)$, separately equal to nothing and changing θ into t.

* (720) The equations [1047, 1048, 1049], are deduced from s, [1039], in the same manner as [1041, 1042, 1043], were deduced from $\frac{d\,v}{d\,t}$, [1038].

$$\begin{aligned}0=\left(\frac{dq}{d\theta}\right).\sin.(nt+\varepsilon)&-\left(\frac{dp}{d\theta}\right).\cos.(nt+\varepsilon)\\ &-\frac{m'.n}{4}.a^2a'.B^{(1)}.(p-p').\sin.(nt+\varepsilon) \qquad [1049]\\ &-\frac{m'.n}{4}.a^2a'.B^{(1)}.(q-q').\cos.(nt+\varepsilon).\end{aligned}$$

Hence, by comparing the coefficients of the similar cosines and sines, and changing θ into t, to obtain directly p and q in functions of t, we shall get

$$\frac{dp}{dt}=-\frac{m'.n}{4}.a^2a'.B^{(1)}.(q-q') \qquad (3)$$
[1050]
$$\frac{dq}{dt}=\frac{m'.n}{4}.a^2a'.B^{(1)}.(p-p'). \qquad (4)$$

When p and q have been found, from these equations, we must substitute them in the preceding expression of s [1039]; then rejecting the terms which contain the arcs of a circle, we shall have

$$s=q.\sin.(nt+\varepsilon)-p.\cos.(nt+\varepsilon)+m'.\chi. \qquad [1051]$$

Mean motion and transverse axis invariable.

54. The equation $\frac{dn}{dt}=0$, just found [1045'], is of great importance in the theory of the system of the world, because it shows that the mean motions of the heavenly bodies, and the transverse axes of their orbits, are [1051'] unchangeable; but this equation is correct only in quantities of the order $m'.h$ inclusively. If the quantities of the order $m'.h^2$, and of the higher [1051''] orders, produce in $\frac{dv}{dt}$, a term of the form $2kt$; k being a function of the elements of the orbits of m and m'; it would produce, in the expression of v, the term kt^2, which, by affecting the longitude of m, in proportion to the square of the time, would become at length extremely sensible. We should then no longer have $\frac{dn}{dt}=0$; but instead of this equation we should, by [1051'''] the preceding article, have $\frac{dn}{dt}=2k$;* it is therefore very important, to

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* (721) If $\frac{dv}{dt}$, in the formula [1038], should contain the term $2kt$, the value of Y, [1041], would be increased by $2k$; X, X', X'', [1041, 1042], being unchanged. This

ascertain whether there exists any term of the form kt^2 in the expression [1051$^{\text{iv}}$] of v. We shall now proceed to demonstrate, that if we only notice the first power of the disturbing masses, however far we may carry on the approximations, relative to the powers and products of the excentricities, [1051$^{\text{v}}$] and the inclinations of the orbits; the expresssion of v will not contain similar terms. We shall resume, for this purpose, the formula (X), § 46 [930],

[1052]
$$\delta r = \frac{\left\{\begin{array}{l} a\,.\cos.v\,.\int n\,dt\,.r\,.\sin.v\,.\left\{2\int \mathrm{d}R + r\,.\left(\frac{dR}{dr}\right)\right\} \\ -a\,.\sin.v\,.\int n\,dt\,.r\,.\cos.v\,.\left\{2\int \mathrm{d}R + r\,.\left(\frac{dR}{dr}\right)\right\}\end{array}\right\}}{\mu\,.\sqrt{1-e^2}}.$$

We shall consider the part of δr, which contains the terms multiplied by t^2, or, for greater generality, the terms which being multiplied by the sine or [1052′] cosine of an angle $\alpha t+\beta$, in which α is very small, have at the same time α^2 for a divisor. It is evident that by supposing $\alpha=0$, there will result a term multiplied by t^2; therefore this second case includes the first.* The

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would introduce in the equation $0=X'+\theta''.X-Y$, [1043], the term $-2k$, which would make the first of the equations [1044] become $0=\frac{dn}{d\theta}-2k$, and by changing θ into t, [1044′], it would become $0=\frac{dn}{dt}-2k$, or $\frac{dn}{dt}=2k$, as above.

* (722) To illustrate this, suppose that δr contains a term, depending on the double [1052a] integral of the expression $b\,.\,dt^2\,.\sin.(\alpha t+\beta)$, and let the integrals be taken so as to vanish when $t=0$. The first integral will be $-\frac{b}{\alpha}.dt\,.\cos.(\alpha t+\beta)+\frac{b}{\alpha}.dt\,.\cos.\beta$,

[1052b] and the second $-\frac{b}{\alpha^2}.\sin.(\alpha t+\beta)+\frac{b}{\alpha}.t\,.\cos.\beta+\frac{b}{\alpha^2}.\sin.\beta$. If we now develop $\sin.(\alpha t+\beta)$, according to the powers of αt, by means of the formula [678a], in which τ is changed into β, and α into αt, we shall get

$$\sin.(\alpha t+\beta)=\left(1-\frac{\alpha^2t^2}{2}+\&c.\right).\sin.\beta+\left(\alpha t-\frac{\alpha^3t^3}{6}+\&c.\right).\cos.\beta,$$

hence

$$-\frac{b}{\alpha^2}.\sin.(\alpha t+\beta)=-\frac{b}{\alpha^2}.\sin.\beta+\tfrac{1}{2}b\,t^2.\sin.\beta-\frac{b}{\alpha}.t.\cos.\beta+\frac{b\,\alpha}{6}.t^3.\cos.\beta+\&c.$$

terms which have α^2 for a divisor, must evidently arise from a double integration; they cannot therefore be produced except by the part of δr which contains the double sign* of integration $\int$. We shall first examine the term [1052] [1052'']

$$\frac{2a\,.\cos.v\,.\int n\,dt\,.(r\,.\sin.v\,.\int \mathrm{d}R)}{\mu\,.\sqrt{1-e^2}}. \qquad [1053]$$

If we fix the origin of the angle v at the perihelion, we shall have, in the elliptical orbit, by § 20 [603], [1053']

$$r=\frac{a\,.(1-e^2)}{1+e\,.\cos.v}; \qquad [1054]$$

consequently†

$$\cos.v=\frac{a\,.(1-e^2)-r}{er}=\frac{a\,.(1-e^2)}{er}-\frac{1}{e}; \qquad [1055]$$

Substituting this in the preceding integral [1052*b*], it will become

$$\tfrac{1}{2}\,b\,t^2\,.\sin.\beta+\frac{b\,\alpha}{6}\,.\,t^3\,.\cos.\beta+\&c.,$$

and by putting $\alpha=0$, it changes into $\frac{1}{2}\,b\,t^2$; which is the same as would be produced from the double integral of $b\,dt^2\,.\sin.(\alpha t+\beta)$, putting $\alpha=0$, in the first instance, by which means it would become $b\,dt^2\,.\sin.\beta$, whose integral, taken twice, would produce the term $\frac{1}{2}\,b\,t^2\,.\sin.\beta$, as above.

* (723) By neglecting the square of the disturbing force, the expression of R, [957], becomes of the form $m'\,\Sigma\,.\,\mathcal{N}\,.\cos.(pt+p')$, $\mathcal{N}$, p, p', being wholly composed of the elliptical values of the elements of the orbits of m, m', which, by [659, 668, 675], contain no terms where t is without the signs of *sine* and *cosine*. Substituting this value of R, in δr, [1052], and still neglecting the square of the disturbing force, we may consider $r\,.\sin.v$, $r\,.\cos.v$, and $n\,dt$, as containing only the elliptical values [659, 668], without any term of the form $\mathcal{A}\,t$, so that there is nothing but a constant term of the form $\mathcal{A}'\,dt$ in $\mathrm{d}R$, which can produce in δr terms of the form $\mathcal{A}''\,t^2$. Similar results follow from δv, [931].

† (724) The formula [1055] is the same as [716']. If we take its differential, considering r, v, as variable, and multiply it by $-r^2$, we shall get the formula [1056]. Now from $r^2\,dv=h\,dt$, [585], $h=\sqrt{\mu a\,.(1-e^2)}$, [599], and $\sqrt{\mu}=n\,a^{\frac{3}{2}}$, [605'], we get $r^2\,dv=dt\,.\sqrt{\mu a\,.(1-e^2)}=a^2\,.\,n\,dt\,.\sqrt{1-e^2}$, [1057]. Putting this value of

and by taking its differential,

[1056] $$r^2 . dv . \sin. v = \frac{a.(1-e^2)}{e} . dr ;$$

but by § 19,

[1057] $$r^2 . dv = dt.\sqrt{\mu a.(1-e^2)} = a^2.n dt.\sqrt{1-e^2} ;$$

therefore we shall have

[1058] $$\frac{a.ndt.r.\sin.v}{\sqrt{1-e^2}} = \frac{rdr}{e}.$$

The term $\frac{2a.\cos.v.\int ndt.\{r.\sin.v\int dR\}}{\mu.\sqrt{1-e^2}}$, will therefore become

[1059] $$\frac{2\cos.v}{\mu.e}.\int\{rdr.\int dR\}, \qquad \text{or} \qquad \frac{\cos.v}{\mu.e}.\{r^2.\int dR - \int r^2.dR\}.$$

As this last function does not contain any double integral, it is evident that
[1059′] there cannot result from it any term which has α^2 for a divisor.

We shall now consider the term

[1060] $$-\frac{2a.\sin.v.\int ndt.\{r.\cos.v.\int dR\}}{\mu.\sqrt{1-e^2}},$$

of the expression of δr [1052]. Substituting for cos. v, the preceding value in terms of r [1055], it becomes

[1061] $$\frac{2a.\sin.v.\int ndt.\{r-a.(1-e^2)\}.\int dR}{\mu e.\sqrt{1-e^2}}.$$

Now we have, by § 22 [659, 669],

[1062] $$r = a.\{1+\tfrac{1}{2}e^2+e.\chi'\},$$

[1062′] χ' being an infinite series of cosines of the angle $nt+\varepsilon$ and its multiples ; therefore we shall have

[1063] $$\frac{\int ndt}{e}.\{r-a.(1-e^2)\}.\int dR = a.\int ndt.\{\tfrac{3}{2}e+\chi'\}.\int dR.$$

$r^2 dv$ in [1056], and multiplying by $\frac{r}{a.(1-e^2)}$, we shall get [1058]. This being substituted in [1053], it becomes $\frac{\cos.v}{\mu e}.\int\{2rdr.\int dR\}$. Integrating it by parts, relative to r, we get $\frac{\cos.v}{\mu e}.\{r^2.\int dR - \int r^2.dR\}$, as in [1059].

If we put the integral of $\int \chi' . n\, d\, t = \chi''$, we shall have* [1063′]

$$a . \int n\, dt . \{\tfrac{3}{2} e + \chi'\} . \int \mathrm{d} R = \tfrac{3}{2} a\, e . \int n\, d\, t . \int \mathrm{d}\, R + a \chi'' . \int \mathrm{d}\, R - a . \int \chi''\, \mathrm{d}\, R. \qquad [1064]$$

As these two last terms do not contain the double sign of integration, they cannot produce any term having α^2 for a divisor; noticing therefore only terms of this kind, we shall have [1060—1064]† [1064′]

$$-\frac{2a . \sin. v . \int n\, dt . \{r . \cos. v . \int \mathrm{d} R\}}{\mu . \sqrt{1-e^2}} = \frac{3a^2 e . \sin. v . \int n\, dt . \int \mathrm{d} R}{\mu . \sqrt{1-e^2}} = \frac{d\, r}{n\, d\, t} . \frac{3\, a}{\mu} . \int n\, dt . \int \mathrm{d} R; \qquad [1065]$$

and the radius r will become

$$(r) + \left(\frac{d\, r}{n\, d\, t}\right) . \frac{3\, a}{\mu} . \int n\, d\, t . \int \mathrm{d}\, R; \qquad [1066]$$

(r) and $\left(\frac{d\, r}{n\, d\, t}\right)$ being the expressions of r and $\frac{d\, r}{n\, d\, t}$, relative to the [1066′] elliptical motion. *Therefore if we notice only the part of the perturbations divided by* α^2, *in the expression of the radius vector, it will be only necessary to increase the mean longitude*‡ $n\, t + \varepsilon$ *by the quantity* $\frac{3\, a}{\mu} . \int n\, dt . \int \mathrm{d} R$, *in* [1066″] *the expression of that radius relative to the motion in an ellipsis.*

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* (725) Substituting in [1063] the value $\chi' . n\, d\, t = d\, \chi''$, [1063′], it becomes $\frac{3}{2} a\, e . \int n\, d\, t . \int \mathrm{d}\, R + a . \int d\, \chi'' . \int \mathrm{d}\, R$, and if we integrate by parts the term $a . \int d\, \chi'' . \int \mathrm{d}\, R$, it becomes $a\, \chi'' . \int \mathrm{d}\, R - a . \int \chi'' . \mathrm{d}\, R$, as in [1064]. Neither χ', [1062′], nor χ'', [1063′], contain t, without the sign of sine and cosine, noticing the terms as in [1051ᵛ].

† (726) The last member of [1065] is deduced from the second member by substituting for $3\, a^2\, e . \sin. v$, its value deduced from [1058], namely $\frac{3 dr . a . \sqrt{1-e^2}}{n dt}$.

‡ (727) From [1053′] we have $\varpi = 0$, and the elliptical value of r, [669], becomes a function of $n\, t + \varepsilon$, which we shall denote by $(r) = \varphi . (n\, t + \varepsilon)$, and we shall suppose that (r) becomes $(r) + \delta\, r$, by increasing the angle $n\, t$ by the small quantity $\delta\, T$, so that $(r) + \delta\, r = \varphi . (n\, t + \varepsilon + \delta\, T)$. The second member of this expression being developed, by the formula [617], will be

$$\varphi . (n\, t + \varepsilon) + \delta\, T . \frac{d . \varphi (n t + \varepsilon)}{n dt} + \&c., \qquad \text{or} \qquad (r) + \delta\, T . \left(\frac{dr}{n\, dt}\right) + \&c.$$

and if we neglect the second and higher powers of $\delta\, T$, it will give $\delta\, r = \delta\, T . \left(\frac{dr}{n\, dt}\right)$. Putting this increment equal to that in [1066], $\left(\frac{d\, r}{n\, d\, t}\right) . \frac{3a}{\mu} . \int n\, d\, t . \int \mathrm{d}\, R$, we shall get

We shall now examine into the manner of noticing this part of the perturbations in the expression of the longitude v. The formula (Y) § 46 [1066'''] [931] gives, by substituting $\frac{3a}{\mu}\cdot\frac{dr}{n\,dt}\cdot\int n\,dt\,.\int \mathrm{d}R$ [1066] for δr, and noticing only the terms divided by α^2,*

[1067] $$\delta v=\frac{\left\{\frac{2r\,.\,ddr+dr^2}{a^2\,.\,n^2\,dt^2}+1\right\}}{\sqrt{1-e^2}}\cdot\frac{3a}{\mu}\cdot\int n\,dt\,.\int \mathrm{d}R\,;$$

now by what precedes,†

[1068] $$dr=\frac{ae\,.\,n\,dt\,.\,\sin.\,v}{\sqrt{1-e^2}}\,;\qquad r^2\,dv=a^2\,.\,n\,dt\,.\sqrt{1-e^2}\,;$$

hence it is easy to obtain, by substituting for $\cos.\,v$ the preceding value in terms of r,‡

[1069] $$\frac{\frac{2r\,.\,ddr+dr^2}{a^2\,.\,n^2\,dt^2}+1}{\sqrt{1-e^2}}=\frac{dv}{n\,dt}\,;$$

$\delta T=\frac{3a}{\mu}\cdot\int n\,dt\,.\int \mathrm{d}R$. Therefore, if in the elliptical value of (r), we increase the mean longitude $nt+\varepsilon$ by $\frac{3a}{\mu}\cdot\int n\,dt\,.\int \mathrm{d}R$, we shall obtain the value of r, in which, terms having the divisor α^2 are noticed.

[*] (728) Put for brevity, $\frac{3a}{\mu}\cdot\int n\,dt\,.\int \mathrm{d}R=W$, then $\delta r=\frac{dr}{n\,dt}\cdot W$, [1066], this gives $d\,\delta r=\frac{ddr}{n\,dt}\cdot W$, the term $d\,W$ being neglected because it does not contain the double integral. These values of $d\,\delta r$, δr, being substituted, in [931], we shall get the two first terms of δv, [1067], the third term of [931] being like the third of [1067].

[†] (729) The equation [1058] gives the value of dr, and the value of $r^2\,dv$, is as in [1057].

[‡] (730) The value of dr, [1068] gives $ddr=\frac{ae\,.\,ndt\,.\,dv\,.\cos.\,v}{\sqrt{1-e^2}}$, which by substituting $dv=\frac{a^2\,.\,n\,dt\,.\sqrt{1-e^2}}{r^2}$, deduced from the second of the equations [1068], gives $ddr=\frac{a^3\,e\,n^2\,.\,dt^2\,.\cos.\,v}{r^2}$, hence $\frac{2r\,.\,ddr}{a^2\,n^2\,dt^2}=\frac{2\,ae\,.\cos.\,v}{r}$, and the value of dr,

noticing therefore only the part of the perturbations, which has the divisor α^2, the longitude v will become

$$(v)+\left(\frac{dv}{n\,dt}\right)\cdot\frac{3a}{\mu}\cdot\int n\,dt\cdot\int \mathrm{d}R\,; \qquad [1070]$$

(v) and $\left(\frac{dv}{n\,dt}\right)$ being the parts of v and $\frac{dv}{n\,dt}$, relative to the elliptical motion. Therefore in order to notice this part of the perturbations, in the expression of the longitude of m, we ought to follow the same rule which we have given [1066″] for the similar terms of the radius vector; that is, *we* [1070′] *must increase the mean longitude* $nt+\varepsilon$ *by the quantity* $\frac{3a}{\mu}\cdot\int n\,dt\cdot\int \mathrm{d}R$, *in the elliptical expression of the true longitude.* [1070″]

The constant part of the expression of $\left(\frac{dv}{n\,dt}\right)$, being developed in a series of cosines of the angle $nt+\varepsilon$ and its multiples, is reduced to unity, as we [1070‴] have seen in § 22;* hence there arises, in the expression of the longitude, the term $\frac{3a}{\mu}\cdot\int ndt\cdot\int \mathrm{d}R$. If $\mathrm{d}R$ should contain a constant term $km'.ndt$, [1070^iv] it would produce, in the expression of the longitude v, the term $\frac{3}{2}\cdot\frac{am'}{\mu}\cdot kn^2t^2$.

[1068], gives $\frac{dr^2}{a^2n^2dt^2}=\frac{e^2.\sin.^2v}{1-e^2}$. Substituting these in the first member of [1069], we shall get $\frac{2ae.\cos.v}{r.\sqrt{1-e^2}}+\frac{e^2.\sin.^2v}{(1-e^2)^{\frac{3}{2}}}+\frac{1}{(1-e^2)^{\frac{1}{2}}}$. The two last terms, reduced to the [1034a] denominator $(1-ee)^{\frac{3}{2}}$, are $\frac{e^2.\sin.^2v+1-e^2}{(1-e^2)^{\frac{3}{2}}}=\frac{1-e^2.\cos.^2v}{(1-e^2)^{\frac{3}{2}}}=\frac{(1-e.\cos.v).(1+e.\cos.v)}{(1-e^2)^{\frac{3}{2}}}$, substituting $1+e.\cos.v=\frac{a.(1-e^2)}{r}$, [1054], it becomes $\frac{(1-e.\cos.v).a}{\sqrt{1-e^2}.r}$, connecting this with the first term $\frac{2ae.\cos.v}{r.\sqrt{1-e^2}}$, the sum will be

$$\frac{a}{r.\sqrt{1-e^2}}.(1+e.\cos.v)=\frac{a}{r.\sqrt{1-e^2}}.\frac{a.(1-e^2)}{r}=\frac{a^2.\sqrt{1-e^2}}{r^2},$$

and this, by means of the second equation [1068], becomes $\frac{dv}{n\,dt}$, as in [1069]. Substituting this in [1067], it produces the last term of [1070].

* (731) This follows from the value of v, [668 or 669].

The investigation of the existence of such terms in the longitude v, is therefore reduced to the examination whether $\mathrm{d}R$ contains a constant term.

When the orbits have but little excentricity, and are inclined to each other
[1070$^{\rm v}$] by very small angles, we have seen in § 48 [957], that R may always be
reduced to an infinite series of sines and cosines of angles, increasing in
proportion to the time t. We may represent them, in general, by the
[1070$^{\rm vi}$] term $km'.\cos.\{i'n't+int+A\}$, i' and i being integral numbers, positive
or negative, including $i=0$. The differential of this term, taken only with
[1070$^{\rm vii}$] respect to the mean motion of m, is $-ik.m'.ndt.\sin.\{i'n't+int+A\}$; which
is the part of $\mathrm{d}R$ relative to this term. This cannot be constant, unless we
[1070$^{\rm viii}$] have $i'n'+in=0$; *which requires that the mean motions of the bodies m
and m', should be commensurable with each other*; and as this is not the case
[1070$^{\rm ix}$] in the solar system, it must follow, that the value of $\mathrm{d}R$ does not contain
any constant term; hence, if we take into consideration only the first power
of the disturbing masses, the mean motions of the heavenly bodies will be
[1070$^{\rm x}$] uniform; or, in symbols, $\frac{dn}{dt}=0$.* The value of a being connected with
that of n, by means of the equation $n^2=\frac{\mu}{a^3}$ [605'], it follows, that if we
[1070$^{\rm xi}$] neglect the periodical quantities, the great axes of the orbits will be
constant.

Equations of a long period.

If the mean motions of the bodies m and m', without being exactly
commensurable, are however very nearly so; there will exist, in the theory
[1070$^{\rm xii}$] of their motions, some equations of a long period, which may become very
sensible, on account of the smallness of the divisor α^2. We shall see hereafter
that this is the case with Jupiter and Saturn. The preceding analysis will

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* (732) Another demonstration of this proposition is given in § 65, [1197'''], and in
the supplement to the third volume, it is proved, that the same is true even when the
approximation is carried on to terms of the order of the square of the disturbing masses,
and Poisson, who first extended the demonstration to terms of the second order of the
[1070a] masses, has also proved that the proposition is true for terms of the third power of the
masses, arising from those of the second order in the disturbed planet, as will be mentioned
hereafter, in the notes upon this supplement.

give, in a very simple manner, the part of the perturbations which depends [1070$^{\text{xiii}}$] on this divisor; since, from what has been said, it will only be necessary to vary the mean longitude $nt+\varepsilon$, or $\int n dt$, by the quantity $\frac{3a}{\mu}.\int n dt.\int \mathrm{d}R$, [1066″, 1070″]; which amounts to the same thing as to increase n, in the integral $\int n\,d\,t$, by the quantity $\frac{3\,a\,n}{\mu}.\int \mathrm{d}\,R$. Now if we consider the [1070$^{\text{xiv}}$] orbit of m as a variable ellipsis, we shall have $n^2=\frac{\mu}{a^3}$ [605′]; and the preceding variation of n will produce in the semi-transverse axis a, the variation* $-\frac{2a^2.\int \mathrm{d}R}{\mu}$. [1070$^{\text{xv}}$]

If in the value of $\frac{d\,v}{dt}$, we carry on the approximation to quantities of the order of the squares of the disturbing masses, we shall obtain some terms [1070$^{\text{xvi}}$] proportional to the time; but, by considering with attention the differential equations of the motions of the bodies m, m', &c., we shall easily perceive that these terms are also multiplied by quantities of the order of the squares and products of the excentricities and inclinations of the orbits. However, [1070$^{\text{xvii}}$] as every thing which affects the mean motion may at length become very sensible; we shall, in the course of the work, *notice these terms, and we shall find that they produce the secular equations observed in the motion of the moon.*† [1070$^{\text{xviii}}$]

Terms of the order of the square of the masses.

55. We shall now resume the equations (1) and (2) § 53, [1046], and shall suppose

$$(0,1)=-\frac{m'.n.C}{2}; \qquad \boxed{0,1}=\frac{m'.n.D}{2}; \qquad [1071]$$

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* (734) By [605′] we have $n=\mu^{\frac{1}{2}}\,a^{-\frac{3}{2}}$, the differential of its logarithm is

$$\frac{dn}{n}=-\tfrac{3}{2}.\frac{da}{a}.$$

Substituting for $d\,n$ its value [1070$^{\text{xiv}}$], $\frac{3an}{\mu}.\int \mathrm{d}\,R$, and multiplying by $-\tfrac{2}{3}.a$, we shall [1070b] get $d\,a=\frac{-2\,a^2}{\mu}.\int \mathrm{d}\,R$, as above.

† (735) In Book vii, § 23, [5543].

they will become

[1072]
$$\frac{dh}{dt} = (0,1) \,.\, l - \boxed{0,1} \,.\, l' ;$$
$$\frac{dl}{dt} = -(0,1) \,.\, h + \boxed{0,1} \,.\, h'.$$

The expressions of $(0,1)$ and $\boxed{0,1}$ may be determined, very easily, in the following manner. Substituting for C and D their values found in § 50, [1018], we shall have

[1073]
$$(0,1) = -\frac{m' \,.\, n}{2} \,.\, \left\{ a^2 . \left(\frac{d A^{(0)}}{d a}\right) + \tfrac{1}{2}\, a^3 \,.\, \left(\frac{d\, d A^{(0)}}{d a^2}\right) \right\};$$
$$\boxed{0,1} = \frac{m' \,.\, n}{2} \,.\, \left\{ a\, A^{(1)} - a^2 \,.\, \left(\frac{d A^{(1)}}{d a}\right) - \tfrac{1}{2}\, a^3 \,.\, \left(\frac{d\, d A^{(1)}}{d a^2}\right) \right\}.$$

Now by § 49,*

[1074]
$$a^2 . \left(\frac{d A^{(0)}}{d a}\right) + \tfrac{1}{2}\, a^3 \,.\, \left(\frac{d\, d A^{(0)}}{d a^2}\right) = -\alpha^2 \,.\, \frac{d\, b_{\frac{1}{2}}^{(0)}}{d \alpha} - \tfrac{1}{2}\, \alpha^3 \,.\, \frac{d\, d\, b_{\frac{1}{2}}^{(0)}}{d \alpha^2} ;$$

we shall easily obtain, by the same article, $\frac{d\, b_{\frac{1}{2}}^{(0)}}{d\alpha}$, $\frac{d\, d\, b_{\frac{1}{2}}^{(0)}}{d\alpha^2}$, in functions of $b_{\frac{1}{2}}^{(0)}$, and $b_{\frac{1}{2}}^{(1)}$; and these quantities are given, in linear functions of $b_{-\frac{1}{2}}^{(0)}$, and $b_{-\frac{1}{2}}^{(1)}$; we shall thus find†

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* (736) Putting $i = 0$, we shall get in [999],

$$a^2 \,.\, \left(\frac{d A^{(0)}}{d a}\right) = -\frac{a^2}{a'^2} \,.\, \frac{d\, b_{\frac{1}{2}}^{(0)}}{d \alpha} = -\alpha^2 \,.\, \frac{d\, b_{\frac{1}{2}}^{(0)}}{d \alpha},$$

and in the first of the equations [1001], $\tfrac{1}{2}\, a^3 \,.\, \left(\frac{d\, d A^{(0)}}{d a^2}\right) = -\tfrac{1}{2}\, \alpha^3 \,.\, \frac{d\, d b_{\frac{1}{2}}^{(0)}}{d \alpha^2}$. The sum of these is as in [1074].

† (737) Putting in [981], $s = \tfrac{1}{2}$, and successively $i = 0$, and $i = 1$, we get

[1074a]
$$\frac{d b_{\frac{1}{2}}^{(0)}}{d \alpha} = \frac{\alpha}{1-\alpha^2} \,.\, b_{\frac{1}{2}}^{(0)} - \frac{1}{1-\alpha^2} \,.\, b_{\frac{1}{2}}^{(1)} ; \qquad \frac{d\, b_{\frac{1}{2}}^{(1)}}{d \alpha} = \frac{1+2\alpha^2}{\alpha \,.\, (1-\alpha^2)} \,.\, b_{\frac{1}{2}}^{(1)} - \frac{3}{1-\alpha^2} \,.\, b_{\frac{1}{2}}^{(2)} ;$$

also putting $i = 2$, $s = \tfrac{1}{2}$, in [966], we find $b_{\frac{1}{2}}^{(2)} = \frac{(1+\alpha^2) \,.\, b_{\frac{1}{2}}^{(1)} - \frac{1}{2}\, \alpha \,.\, b_{\frac{1}{2}}^{(0)}}{\frac{3}{2}\, \alpha}$, which, being substituted in $d\, b_{\frac{1}{2}}^{(1)}$, gives

[1074b]
$$\frac{d\, b_{\frac{1}{2}}^{(1)}}{d \alpha} = \frac{1+2\,\alpha^2}{\alpha \,.\, (1-\alpha^2)} \,.\, b_{\frac{1}{2}}^{(1)} - \frac{3}{1-\alpha^2} \,.\, \left\{ \frac{(1+\alpha^2) \,.\, b_{\frac{1}{2}}^{(1)} - \frac{1}{2}\, \alpha \,.\, b_{\frac{1}{2}}^{(0)}}{\frac{3}{2}\, \alpha} \right\} = \frac{-1}{\alpha \,.\, (1-\alpha^2)} \,.\, b_{\frac{1}{2}}^{(1)} + \frac{b_{\frac{1}{2}}^{(0)}}{1-\alpha^2} .$$

Putting $i = 0$, and $s = \tfrac{1}{2}$, in [982], we obtain

$$\frac{d\, d b_{\frac{1}{2}}^{(0)}}{d \alpha^2} = \frac{\alpha}{1-\alpha^2} \,.\, \frac{d\, b_{\frac{1}{2}}^{(0)}}{d \alpha} + \frac{1+\alpha^2}{(1-\alpha^2)^2} \,.\, b_{\frac{1}{2}}^{(0)} - \frac{1}{1-\alpha^2} \,.\, \frac{d b_{\frac{1}{2}}^{(1)}}{d \alpha} - \frac{2\, \alpha}{(1-\alpha^2)^2} \,.\, b_{\frac{1}{2}}^{(1)}.$$

$$a^2 . \left(\frac{d A^{(0)}}{d a}\right) + \tfrac{1}{2} a^3 . \left(\frac{d d A^{(0)}}{d a^2}\right) = \frac{3 \alpha^2 . b^{(1)}_{-\frac{1}{2}}}{2 . (1 - \alpha^2)^2} ; \qquad [1075]$$

therefore

$$(0 , 1) = - \frac{3 m' . n . \alpha^2 . b^{(1)}_{-\frac{1}{2}}}{4 . (1 - \alpha^2)^2} . \qquad [1076]$$

Put

$$(a^2 - 2 a a' . \cos. \theta + a'^2)^{\frac{1}{2}} = (a , a') + (a , a')' . \cos. \theta + (a , a')'' . \cos. 2 \theta + \&c., \qquad [1077]$$

we shall have, by § 49*

$$(a , a') = \tfrac{1}{2} a' . b^{(0)}_{-\frac{1}{2}} , \qquad (a , a')' = a' . b^{(1)}_{-\frac{1}{2}} , \qquad \&c. ; \qquad [1078]$$

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which, by substituting $\frac{d b^{(0)}_{\frac{1}{2}}}{d \alpha}$, [1074*a*], and $\frac{d b^{(1)}_{\frac{1}{2}}}{d \alpha}$, [1074*b*], becomes

$$\frac{d d b^{(0)}_{\frac{1}{2}}}{d \alpha^2} = \frac{\alpha}{1 - \alpha^2} . \left\{ \frac{\alpha b^{(0)}_{\frac{1}{2}}}{1 - \alpha^2} - \frac{b^{(1)}_{\frac{1}{2}}}{1 - \alpha^2} \right\} + \frac{1 + \alpha^2}{(1 - \alpha^2)^2} . b^{(0)}_{\frac{1}{2}} - \frac{1}{(1 - \alpha^2)} . \left\{ \frac{- b^{(1)}_{\frac{1}{2}}}{\alpha . (1 - \alpha^2)} + \frac{b^{(0)}_{\frac{1}{2}}}{(1 - \alpha^2)} \right\} - \frac{2 \alpha b^{(1)}_{\frac{1}{2}}}{(1 - \alpha^2)^2} .$$

and this, by reduction, becomes

$$\frac{d d b^{(0)}_{\frac{1}{2}}}{d \alpha^2} = \frac{2 \alpha^2}{(1 - \alpha^2)^2} . b^{(0)}_{\frac{1}{2}} + \frac{1 - 3 \alpha^2}{(1 - \alpha^2)^2 . \alpha} . b^{(1)}_{\frac{1}{2}} .$$

This value, and that of $\frac{d b^{(0)}_{\frac{1}{2}}}{d \alpha}$, [1074*a*], being substituted in the second member of [1074], we get $\left\{ - \frac{\alpha^3}{1 - \alpha^2} - \frac{\alpha^5}{(1 - \alpha^2)^2} \right\} . b^{(0)}_{\frac{1}{2}} + \left\{ \frac{\alpha^2}{1 - \alpha^2} - \frac{\frac{1}{2} \alpha^2 . (1 - 3 \alpha^2)}{(1 - \alpha^2)^2} \right\} . b^{(1)}_{\frac{1}{2}}$, or by reduction $\frac{- \alpha^3 b^{(0)}_{\frac{1}{2}}}{(1 - \alpha^2)^2} + \frac{\frac{1}{2} \alpha^2 . (1 + \alpha^2)}{(1 - \alpha^2)^2} . b^{(1)}_{\frac{1}{2}}$; and, by substituting $b^{(0)}_{\frac{1}{2}}$, $b^{(1)}_{\frac{1}{2}}$, [990, 991], the denominator will be $(1 - \alpha^2)^4$, and the numerator,

$$- \alpha^3 . \{ (1 + \alpha^2) . b^{(0)}_{-\frac{1}{2}} + 6 \alpha . b^{(1)}_{-\frac{1}{2}} \} + \tfrac{1}{2} \alpha^2 . (1 + \alpha^2) . \{ 2 \alpha b^{(0)}_{-\frac{1}{2}} + 3 . (1 + \alpha^2) . b^{(1)}_{-\frac{1}{2}} \} .$$

The coefficient of $b^{(0)}_{-\frac{1}{2}}$, in this numerator, is evidently equal to nothing, and that of $b^{(1)}_{-\frac{1}{2}}$, is

$$- 6 \alpha^4 + \tfrac{3}{2} \alpha^2 . (1 + \alpha^2)^2 = \tfrac{3}{2} \alpha^2 . \{ (1 + \alpha^2)^2 - 4 \alpha^2 \} = \tfrac{3}{2} \alpha^2 . (1 - \alpha^2)^2 ;$$

therefore this second member of [1074] will be $\frac{\frac{3}{2} \alpha^2 . (1 - \alpha^2)^2 . b^{(1)}_{-\frac{1}{2}}}{(1 - \alpha^2)^4}$, and by rejecting $(1 - \alpha^2)^2$, common to the numerator and denominator it becomes like the second member of [1075]. Substituting this in (0, 1), [1073], we shall get [1076].

* (738) Putting $s = - \frac{1}{2}$, in [964], and multiplying by a', using $a = a' \alpha$, [963^{iv}], [1074*c*] we shall get

$$\{ a^2 - 2 a a' . \cos. \theta + a'^2 \}^{-\frac{1}{2}} = a' . \{ \tfrac{1}{2} b^{(0)}_{-\frac{1}{2}} + b^{(1)}_{-\frac{1}{2}} . \cos. \theta + b^{(2)}_{-\frac{1}{2}} . \cos. 2 \theta + \&c. \}$$

therefore we shall find

[1079] $$(0\,,1)=-\frac{3\,m'.\,n\,a^2\,a'.\,(a\,,a')'}{4\,.\,(a'^2-a^2)^2}\,.$$

Now we have, by § 49,*

[1080] $$a\,A^{(1)}-a^2.\left(\frac{d\,A^{(1)}}{d\,a}\right)-\tfrac{1}{2}\,a^3.\left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right)=-\alpha\,.\left\{b_{\frac{1}{2}}^{(1)}-\alpha\,.\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}-\tfrac{1}{2}\,\alpha^2.\frac{d\,d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2}\right\}.$$

Substituting for $b_{\frac{1}{2}}^{(1)}$ and its differentials, their values in $b_{-\frac{1}{2}}^{(0)}$, and $b_{-\frac{1}{2}}^{(1)}$, we shall find the preceding function to be equal to†

[1081] $$-\frac{3\,\alpha\,.\,\{(1+\alpha^2)\,.\,b_{-\frac{1}{2}}^{(1)}+\tfrac{1}{2}\,\alpha\,.\,b_{-\frac{1}{2}}^{(0)}\}}{(1-\alpha^2)^2}\;;$$

Comparing this with [1077], we shall obtain the equations [1078]. Multiplying the value of (0, 1), [1076], by $\frac{a'^4}{a'^4}$, and substituting $a'\,\alpha=a$, it becomes,

$$(0,1)=-\frac{3\,m'\,.\,n\,a'^2\,a^2\,.\,b_{-\frac{1}{2}}^{(1)}}{4.(a'^2-a^2)^2}\,,$$

and this, by substituting $a'\,.\,b_{-\frac{1}{2}}^{(1)}=(a,\,a')'$, [1078], becomes as in [1079].

* (739) From [997] we get $a\,A^{(1)}=\frac{a^2}{a'^2}-\frac{a}{a'}\,.\,b_{\frac{1}{2}}^{(1)}=\alpha^2-\alpha\,.\,b_{\frac{1}{2}}^{(1)}$; from [1000], we find $-a^2.\left(\frac{d\,A^{(1)}}{d\alpha}\right)=-\frac{a^2}{a'^2}+\frac{a^2}{a'^2}.\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}=-\alpha^2+\alpha^2.\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}$, and from [1001] we get $-\tfrac{1}{2}\,a^3.\left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right)=+\tfrac{1}{2}\,\alpha^3.\frac{d\,d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2}$; the sum of these three expressions will be as in [1080].

† (740) Putting $i=1$, and $s=\frac{1}{2}$, in [982], it becomes

$$\frac{d\,d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha^2}=\frac{1+2\,\alpha^2}{\alpha\,.(1-\alpha^2)}.\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}+\left\{\frac{3.(1+\alpha^2)}{(1-\alpha^2)^2}-\frac{1}{\alpha^2}\right\}.\,b_{\frac{1}{2}}^{(1)}-\frac{3}{(1-\alpha^2)}.\frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha}-\frac{6\,\alpha}{(1-\alpha^2)^2}.\,b_{\frac{1}{2}}^{(2)}.$$

Substituting this in the second member of [1080], which for brevity we shall call S, we shall get

$$S=\left\{-\alpha+\tfrac{1}{2}\,\alpha^3.\left[\frac{3.(1+\alpha^2)}{(1-\alpha^2)^2}-\frac{1}{\alpha^2}\right]\right\}.\,b_{\frac{1}{2}}^{(1)}+\left\{\alpha^2+\tfrac{1}{2}\,\alpha^2.\frac{(1+2\alpha^2)}{1-\alpha^2}\right\}.\frac{d b_{\frac{1}{2}}^{(1)}}{d\,\alpha}$$
$$-\frac{3\,\alpha^3}{2.(1-\alpha^2)}.\frac{d\,b_{\frac{1}{2}}^{(2)}}{d\,\alpha}-\frac{3\,\alpha^4}{(1-\alpha^2)^2}.\,b_{\frac{1}{2}}^{(2)},$$

and by reduction

$$S=\frac{3.(-1+3\,\alpha^2).\,\alpha}{2.(1-\alpha^2)^2}.\,b_{\frac{1}{2}}^{(1)}+\frac{3\,\alpha^2}{2.(1-\alpha^2)}.\frac{d\,b_{\frac{1}{2}}^{(1)}}{d\,\alpha}-\frac{3\,\alpha^3}{2.(1-\alpha^2)}.\frac{d b_{\frac{1}{2}}^{(2)}}{d\alpha}-\frac{3\,\alpha^4}{(1-\alpha^2)^2}.\,b_{\frac{1}{2}}^{(2)}.$$

therefore

$$\boxed{0\,,1} = -\frac{3\,\alpha\,.\,m'\,.\,n\,.\{(1+\alpha^2)\,.\,b^{(1)}_{-\frac{1}{2}}+\frac{1}{2}\,\alpha\,.\,b^{(0)}_{-\frac{1}{2}}\}}{2\,.\,(1-\alpha^2)^2}\,; \qquad [1082]$$

Now putting $s=\frac{1}{2}$, and $i=2$, in [981] we shall get

$$\frac{d\,b^{(2)}_{\frac{1}{2}}}{d\alpha}=\frac{2+3\,\alpha^2}{\alpha\,.(1-\alpha^2)}\,.\,b^{(2)}_{\frac{1}{2}}-\frac{5}{1-\alpha^2}\,.\,b^{(3)}_{\frac{1}{2}}.$$

substituting this and $\frac{d\,b^{(1)}_{\frac{1}{2}}}{d\,\alpha}$, [1074$a$], in S, it becomes

$$S=\frac{3\,\alpha\,.(-1+3\,\alpha^2)}{2.(1-\alpha^2)^2}\,.\,b^{(1)}_{\frac{1}{2}}+\frac{3\,\alpha^2}{2.(1-\alpha^2)}\,.\left\{\frac{1+2\,\alpha^2}{\alpha\,.(1-\alpha^2)}\,.\,b^{(1)}_{\frac{1}{2}}-\frac{3}{1-\alpha^2}\,.\,b^{(2)}_{\frac{1}{2}}\right\}$$
$$+\frac{3\,\alpha^3}{2.(1-\alpha^2)}\,.\left\{-\frac{(2+3\,\alpha^2)}{\alpha.(1-\alpha^2)}\,.\,b^{(2)}_{\frac{1}{2}}+\frac{5}{1-\alpha^2}\,.\,b^{(3)}_{\frac{1}{2}}\right\}-\frac{3\,\alpha^4}{(1-\alpha^2)^2}\,.\,b^{(2)}_{\frac{1}{2}};$$

or, by reduction, $S=\frac{15\,\alpha^3}{2.(1-\alpha^2)^2}\,.\,b^{(1)}_{\frac{1}{2}}-\frac{15\,\alpha^2.(1+\alpha^2)}{2\,.(1-\alpha^2)^2}\,.\,b^{(2)}_{\frac{1}{2}}+\frac{15\,\alpha^3}{2.(1-\alpha^2)^2}\,.\,b^{(3)}_{\frac{1}{2}}$, hence we get

$$\frac{2\,.(1-\alpha^2)^2.\,S}{\alpha}=15\,\alpha^2\,.\,b^{(1)}_{\frac{1}{2}}-15\,\alpha\,.\,(1+\alpha^2)\,.\,b^{(2)}_{\frac{1}{2}}+15\,\alpha^2\,.\,b^{(3)}_{\frac{1}{2}}. \qquad [1080a]$$

Now putting $i=3$, and $s=\frac{1}{2}$, in [966], we shall find

$$b^{(3)}_{\frac{1}{2}}=\frac{2\,.\,(1+\alpha^2)\,.\,b^{(2)}_{\frac{1}{2}}-\frac{3}{2}\,\alpha\,.\,b^{(1)}_{\frac{1}{2}}}{\frac{5}{2}\,\alpha}, \quad \text{and} \quad 15\,\alpha^2\,.\,b^{(3)}_{\frac{1}{2}}=12\,\alpha\,.\,(1+\alpha^2)\,.\,b^{(2)}_{\frac{1}{2}}-9\,\alpha^2\,.\,b^{(1)}_{\frac{1}{2}},$$

hence [1080a] becomes $\frac{2\,.(1-\alpha^2)^2.\,S}{\alpha}=6\,\alpha^2\,.\,b^{(1)}_{\frac{1}{2}}-3\,\alpha\,.\,(1+\alpha^2)\,.\,b^{(2)}_{\frac{1}{2}}$. Again, the same formula [966], by putting $i=2$, and $s=\frac{1}{2}$, gives $b^{(2)}_{\frac{1}{2}}=\frac{(1+\alpha^2)\,.\,b^{(1)}_{\frac{1}{2}}-\frac{1}{2}\,\alpha\,.\,b^{(0)}_{\frac{1}{2}}}{\frac{3}{2}\,\alpha}$,

or $-3\,\alpha\,.\,(1+\alpha^2)\,.\,b^{(2)}_{\frac{1}{2}}=-(1+\alpha^2)\,.\,\{2\,.\,(1+\alpha^2)\,.\,b^{(1)}_{\frac{1}{2}}-\alpha\,.\,b^{(0)}_{\frac{1}{2}}\}$; hence

$$\frac{2.(1-\alpha^2)^2.\,S}{\alpha}=6\,\alpha^2\,.\,b^{(1)}_{\frac{1}{2}}-(1+\alpha^2)\,.\,\{2\,.\,(1+\alpha^2)\,.\,b^{(1)}_{\frac{1}{2}}-\alpha\,.\,b^{(0)}_{\frac{1}{2}}\}$$
$$=(-2+2\,\alpha^2-2\,\alpha^4).\,b^{(1)}_{\frac{1}{2}}+\alpha\,.\,(1+\alpha^2)\,.\,b^{(0)}_{\frac{1}{2}}.$$

This, substituting the values of $b^{(0)}_{\frac{1}{2}}$, $b^{(1)}_{\frac{1}{2}}$, [990, 991], and multiplying by $(1-\alpha^2)^2$ gives

$$\frac{2.(1-\alpha^2)^4.\,S}{\alpha}=(-2+2\,\alpha^2-2\,\alpha^4)\,.\,\{2\,\alpha\,.\,b^{(0)}_{-\frac{1}{2}}+3\,.\,(1+\alpha^2)\,.\,b^{(1)}_{-\frac{1}{2}}\}$$
$$+\alpha\,.\,(1+\alpha^2)\,.\,\{(1+\alpha^2)\,.\,b^{(0)}_{-\frac{1}{2}}+6\,\alpha\,.\,b^{(1)}_{-\frac{1}{2}}\},$$

or by reduction $=-3\,.\,(1-\alpha^2)^2\,.\,\alpha\,b^{(0)}_{-\frac{1}{2}}-6\,.\,(1+\alpha^2)\,.\,(1-\alpha^2)^2\,.\,b^{(1)}_{-\frac{1}{2}}$. Multiplying this equation by $\frac{\alpha}{2.(1-\alpha^2)^4}$ we get $S=\frac{-\frac{3}{2}\,\alpha^2\,.\,b^{(0)}_{-\frac{1}{2}}-3\,.\,(1+\alpha^2)\,.\,\alpha\,.\,b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2}$, as in [1081]. Substituting this in [1073] we shall get [1082]. Multiplying the numerator and denominator of [1082] by a'^4, substituting also the values of $b^{(0)}_{-\frac{1}{2}}$, $b^{(1)}_{-\frac{1}{2}}$, [1078], and $a'\alpha=a$, [1074c], we shall obtain the formula [1083].

or

[1083] $$\boxed{0,1} = -\frac{3m'.an.\{(a^2+a'^2).(a,a')'+aa'.(a,a')\}}{2.(a'^2-a^2)^2};$$

therefore we shall obtain, in this manner, very simple expressions of $(0,1)$ and $\boxed{0,1}$; and it is easy to prove, from the values of $b_{-\frac{1}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)}$, given in series, in § 49, that these expressions will be positive, if n be positive, and negative if n be negative.* [1083']

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* (741) The second of the formulas [992] gives $b_{-\frac{1}{2}}^{(1)} = -\frac{1}{3}.(1-\alpha^2)^2.b_{\frac{3}{2}}^{(1)}$, and by putting $s=\frac{3}{2}$, in the second formula [988], we obtain $b_{\frac{3}{2}}^{(1)}$ which, being substituted, gives

$$b_{-\frac{1}{2}}^{(1)} = -\frac{1}{3}.(1-\alpha^2)^2.2\alpha.\left\{\frac{3}{2}+\frac{3}{2}.\frac{3.5}{2.4}.\alpha^2+\frac{3.5}{2.4}.\frac{3.5.7}{2.4.6}.\alpha^4+\&c.\right\},$$

and as every term of the infinite series $\frac{3}{2}+\frac{3}{2}.\frac{3.5}{2.4}.\alpha^2+\&c.$, is positive, its sum must be positive, hence $b_{-\frac{1}{2}}^{(1)}$ is negative, consequently $-\frac{3m'.\alpha^2.b_{-\frac{1}{2}}^{(1)}}{4.(1-\alpha^2)^2}$, must be positive, therefore the expression $(0,1)$, [1076], must have the same sign as n. Again if we substitute the values $b_{-\frac{1}{2}}^{(0)} = (1-\alpha^2)^2.b_{\frac{3}{2}}^{(0)}$, $b_{-\frac{1}{2}}^{(1)} = -\frac{1}{3}.(1-\alpha^2)^2.b_{\frac{3}{2}}^{(1)}$, [992] in [1081], it becomes $\alpha.\{(1+\alpha^2).b_{\frac{3}{2}}^{(1)}-\frac{3}{2}\alpha\, b_{\frac{3}{2}}^{(0)}\}$; and putting $s=\frac{3}{2}$, in [988], we shall get [1081a]

[1081b] $$b_{\frac{3}{2}}^{(0)} = 2.\left\{1+\left(\frac{3}{2}\right)^2.\alpha^2+\left(\frac{3.5}{2.4}\right)^2.\alpha^4+\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^6+\&c.\right\},$$

$$b_{\frac{3}{2}}^{(1)} = 2\alpha.\left\{\frac{3}{2}+\frac{3}{2}.\frac{3.5}{2.4}.\alpha^2+\frac{3.5}{2.4}.\frac{3.5.7}{2.4.6}.\alpha^4+\&c.\right\}$$

[1081c] $$=\left\{\frac{3}{1}.\alpha+\frac{5}{2}.\left(\frac{3}{2}\right)^2.\alpha^3+\frac{7}{3}.\left(\frac{3.5}{2.4}\right)^2.\alpha^5+\frac{9}{4}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^7+\&c.\right\}$$

$$=\frac{3}{1}.\frac{2^2}{3^2}.\left(\frac{3}{2}\right)^2.\alpha+\frac{5}{2}.\frac{4^2}{5^2}.\left(\frac{3.5}{2.4}\right)^2.\alpha^3+\frac{7}{3}.\frac{6^2}{7^2}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^5+\&c.$$

[1081d] $$=\frac{4}{3}.\left(\frac{3}{2}\right)^2.\alpha+\frac{8}{5}.\left(\frac{3.5}{2.4}\right)^2.\alpha^3+\frac{12}{7}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^5+\&c.$$

Multiplying [1081*b*, *c*] respectively by $-\frac{3}{2}\alpha^2$, and α, we shall get

$$-\frac{3}{2}.\alpha^2 b_{\frac{3}{2}}^{(0)} = -\frac{3}{1}.\alpha^2-\frac{3}{1}.\left(\frac{3}{2}\right)^2.\alpha^4-\frac{3}{1}.\left(\frac{3.5}{2.4}\right)^2.\alpha^6-\frac{3}{1}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^8-\&c.$$

$$\alpha b_{\frac{3}{2}}^{(1)} = \frac{3}{1}.\alpha^2+\frac{5}{2}.\left(\frac{3}{2}\right)^2.\alpha^4+\frac{7}{3}.\left(\frac{3.5}{2.4}\right)^2.\alpha^6+\frac{9}{4}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^8+\&c.,$$

whose sum is

[1081e] $$\alpha b_{\frac{3}{2}}^{(1)}-\frac{3}{2}.\alpha^2 b_{\frac{3}{2}}^{(0)} = -\frac{1}{2}.\left(\frac{3}{2}\right)^2.\alpha^4-\frac{2}{3}.\left(\frac{3.5}{2.4}\right)^2.\alpha^6-\frac{3}{4}.\left(\frac{3.5.7}{2.4.6}\right)^2.\alpha^8-\&c.$$

Put $(0,2)$ and $\boxed{0,2}$, for what $(0,1)$ and $\boxed{0,1}$ become by changing a', m', [1083''] into a'', m''. Also $(0,3)$ and $\boxed{0,3}$, for what the same quantities become by changing a', m', into a''', m'''; and in the same manner for others. Also let [1083'''] h'', l'', h''', l''', &c., be the values of h and l, relative to the bodies m'', m''', &c.; we shall have, by the combined actions of the different bodies, m', m'', [1083^iv] m''', &c., on m,*

$$\frac{dh}{dt}=\{(0,1)+(0,2)+(0,3)+\&c.\}.l-\boxed{0,1}.l'-\boxed{0,2}.l''-\&c.;$$
$$\frac{dl}{dt}=-\{(0,1)+(0,2)+(0,3)+\&c.\}.h+\boxed{0,1}.h'+\boxed{0,2}.h''+\&c. \qquad [1084]$$

It is evident that $\frac{dh'}{dt}$, $\frac{dl'}{dt}$, $\frac{dh''}{dt}$, $\frac{dl''}{dt}$, &c., will be determined by similar expressions to those of $\frac{dh}{dt}$, and $\frac{dl}{dt}$, and it is easy to deduce them from [1084], by changing successively what relates to m, into the corresponding terms of m', m'', &c., and the contrary. Suppose therefore†

$$(1,0)\,,\ \boxed{1,0}, \qquad (1,2)\,,\ \boxed{1,2}, \qquad \&c., \qquad [1085]$$

Now multiplying [1081d] by α^3, we shall get

$$\alpha^3\, b_{\frac{3}{2}}^{(1)}=\frac{4}{3}\cdot\left(\frac{3}{2}\right)^2.\,\alpha^4+\frac{8}{5}\cdot\left(\frac{3.5}{2.4}\right)^2.\,\alpha^6+\frac{12}{7}\cdot\left(\frac{3.5.7}{2.4.6}\right)^2.\,\alpha^8+\&c.$$

Adding this to [1081e] we shall get the value of the required function [1081a],

$$\alpha.\left\{(1+\alpha^2).\,b_{\frac{3}{2}}^{(1)}-\frac{3}{2}.\,\alpha b_{\frac{3}{2}}^{(0)}\right\}=\frac{1.5}{2.3}\cdot\left(\frac{3}{2}\right)^2.\,\alpha^4+\frac{2.7}{3.5}\cdot\left(\frac{3.5}{2.4}\right)^2.\,\alpha^6+\frac{3.9}{4.7}\cdot\left(\frac{3.5.7}{2.4.6}\right)^2.\,\alpha^8+\&c. \qquad [1083a]$$

in which the law of continuation is manifest, and every term is positive, consequently the whole expression [1081] must be positive. Hence it follows that the value of $\boxed{0,1}$, [1082, 1083], must have the same sign as n.

* (742) The planet m' produces in $\frac{dh}{dt}$ the terms $(0,1).l-\boxed{0,1}.l'$, [1072]. In like manner m'' must produce $(0,2).l-\boxed{0,2}.l''$, m''' must produce $(0,3).l-\boxed{0,3}.l'''$, &c. The sum of all these gives the complete value of $\frac{dh}{dt}$, as in [1084]. $\frac{dl}{dt}$ is deduced in the same manner from [1072].

† (743) In all these expressions $\boxed{1,0}$, $\boxed{1,2}$, &c., $(0,1)$, $(0,2)$, &c., the first figure denotes the number of accents on the mass of the *disturbed* planet, and the other that of the [1083b] *disturbing* planet.

to be what

[1086] $$(0,1)\,,\ \boxed{0,1}\,, \qquad (0,2)\,,\ \boxed{0,2}\,, \qquad \&c.,$$

become, when what relates to m is changed into the corresponding terms of m', and the contrary; suppose also

[1087] $$(2,0)\,,\ \boxed{2,0}\,, \qquad (2,1)\,,\ \boxed{2,1}\,, \qquad \&c.,$$

to be what

[1088] $$(0,2)\,,\ \boxed{0,2}\,, \qquad (0,1)\,,\ \boxed{0,1}\,, \qquad \&c.,$$

become when what relates to m, is changed into the corresponding terms of m'', and the contrary, and so on for the other bodies. The preceding differential equations, referred respectively to the bodies m, m', m'', &c., will give, to determine h, l, h', l', h'', l'', &c., the following system of equations,*

System of linear equations of the first degree, to find the excentricities.

[1089] $$\left.\begin{array}{l}
\frac{dh}{dt}=\{(0,1)+(0,2)+(0,3)+\&c.\}.l-\boxed{0,1}.l'-\boxed{0,2}.l''-\boxed{0,3}.l'''-\&c. \\
\frac{dl}{dt}=-\{(0,1)+(0,2)+(0,3)+\&c.\}.h+\boxed{0,1}.h'+\boxed{0,2}.h''+\boxed{0,3}.h'''+\&c. \\
\frac{dh'}{dt}=\{(1,0)+(1,2)+(1,3)+\&c.\}.l'-\boxed{1,0}.l-\boxed{1,2}.l''-\boxed{1,3}.l'''-\&c. \\
\frac{dl'}{dt}=-\{(1,0)+(1,2)+(1,3)+\&c.\}.h'+\boxed{1,0}.h+\boxed{1,2}.h''+\boxed{1,3}.h'''+\&c. \\
\frac{dh''}{dt}=\{(2,0)+(2,1)+(2,3)+\&c.\}.l''-\boxed{2,0}.l-\boxed{2,1}.l'-\boxed{2,3}.l'''-\&c. \\
\frac{dl''}{dt}=-\{(2,0)+(2,1)+(2,3)+\&c.\}.h''+\boxed{2,0}.h+\boxed{2,1}.h'+\boxed{2,3}.h'''+\&c. \\
\&c.
\end{array}\right\}\cdot (A)$$

The quantities $(0\,,1)$ and $(1\,,0)$, $\boxed{0,1}$ and $\boxed{1,0}$, have some remarkable relations with each other, which facilitate their computation, and will be useful hereafter. We have, by what precedes, [1079],

[1089′] $$(0\,,1)=-\frac{3\,m'.\,n\,a^2.\,a'.\,(a\,,a')'}{4.\,(a'^2-a^2)^2}\,.$$

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* (744) The two first of these equations are as in [1084], in which the disturbed planet is m, the accent on which is considered as nothing. Now if in these we change 0 into 1, and 1 into 0, in the expressions (0,1), (0,2), &c., $\boxed{0,1}$, &c., we shall obtain, as was observed in the last note, the third and fourth equations [1089], corresponding to the disturbed planet m', and so on for the others. It may be observed that these equations give h, l, &c., exact,
[1089a] except in terms of the order $m'\,e^3$, or m'^2.

If in this expression of $(0,1)$, we change m' into m, n into n', a into a', and the contrary, we shall have the expression of $(1,0)$, which will therefore be

$$(1,0) = -\frac{3m.n'a'^2.a.(a',a)'}{4.(a'^2-a^2)^2}; \qquad [1090]$$

but we have $(a,a')' = (a',a)'$, since both of these quantities result from the development of the function $(a^2 - 2aa'.\cos.\theta + a'^2)^{\frac{1}{2}}$, in a series [1077] arranged according to the cosines of the angle θ and its multiples; therefore we shall have* [1090']

$$(0,1)\,m.n'a' = (1,0).m'.na; \qquad [1091]$$

now by neglecting the masses m, m', &c., in comparison with M, we shall have

$$n^2 = \frac{M}{a^3}; \qquad n'^2 = \frac{M}{a'^3}; \qquad \text{\&c.}; \qquad [1092]$$

therefore

$$(0,1).m.\sqrt{a} = (1,0).m'.\sqrt{a'}; \qquad [1093]$$

from which equation we may easily compute $(1,0)$, when $(0,1)$ shall be determined. We shall find in the same manner

$$\boxed{0,1}.m.\sqrt{a} = \boxed{1,0}.m'.\sqrt{a'}. \qquad [1094]$$

These two equations will also take place when n and n' have contrary signs; that is when the two bodies revolve in opposite directions; but in this case we must prefix the sign of n to the radical $\sqrt{a}$, and the sign of n' to the radical $\sqrt{a'}$.† [1094']

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* (745) This appears from the equation [1077], the first member of which is not altered by changing a into a', and a' into a. The values $(0,1)$, $(1,0)$, [1089', 1090], being substituted in both members of [1091], they become identical, therefore this equation is correct. Now by [1013'], neglecting m in comparison with M, we get $M = n^2a^3$, and in like manner $M = n'^2a'^3$, hence $n^2a^3 = n'^2a'^3$, or $na.\sqrt{a} = n'a'.\sqrt{a'}$. [1091a] Multiplying this by [1091] and dividing the products by $nn'.aa'$, we shall get [1093]. In like manner we may find [1094] from [1083, 1085, 1086]. The formulas [1095] are merely the generalization of [1093, 1094], applying them to other bodies.

† (746) The radicals $\sqrt{a}$, $\sqrt{a'}$, &c., were introduced into the formulas [1093, &c.] by means of $\sqrt{M}$ deduced from [1091a], which is

$$\sqrt{M} = na.\sqrt{a} = n'a'.\sqrt{a'} = n''a''.\sqrt{a''} = \text{\&c.}$$

From the two preceding equations, we can evidently deduce the following:

[1095] $(0,2).m.\sqrt{a}=(2,0).m''.\sqrt{a''}$; $\boxed{0,2}.m.\sqrt{a}=\boxed{2,0}.m''.\sqrt{a''}$; &c.
$(1,2).m'.\sqrt{a'}=(2,1).m''.\sqrt{a''}$; $\boxed{1,2}.m'.\sqrt{a'}=\boxed{2,1}.m''.\sqrt{a''}$; &c.

56. Now to find the integrals of the equations (A) [1089] of the preceding article, we shall put

[1096]
$$\begin{aligned} h &= N.\sin.(gt+\beta); & l &= N.\cos.(gt+\beta); \\ h' &= N'.\sin.(gt+\beta); & l' &= N'.\cos.(gt+\beta); \\ &\text{\&c.} \end{aligned}$$

Substituting these values in the equations (A) [1089], we shall have*

[1097]
$$\left.\begin{aligned} Ng &= \{(0,1)+(0,2)+\text{\&c.}\}.N-\boxed{0,1}.N'-\boxed{0,2}.N''-\text{\&c.} \\ N'g &= \{(1,0)+(1,2)+\text{\&c.}\}.N'-\boxed{1,0}.N-\boxed{1,2}.N''-\text{\&c.} \\ N''g &= \{(2,0)+(2,1)+\text{\&c.}\}.N''-\boxed{2,0}.N-\boxed{2,1}.N'-\text{\&c.} \\ &\text{\&c.} \end{aligned}\right\}; \qquad (B)$$

Supposing therefore, $\sqrt{M}$ and the semi axes a, a', a'', &c., to be positive, it will follow from these equations that the expressions $n.\sqrt{a}$, $n'.\sqrt{a'}$, $n''.\sqrt{a''}$, &c., must also be positive, and $\sqrt{a}$ must have the same sign as n; $\sqrt{a'}$ the same sign as n', &c. Therefore if we suppose n to be positive when the motion of m is direct, we must also put $\sqrt{a}$ positive. In like manner we must put n' and $\sqrt{a'}$ positive, if the motion of m' be direct, but if its motion be retrograde, n' must evidently be negative, therefore $\sqrt{a'}$ must also be negative to preserve the same sign in the quantity $n'.\sqrt{a'}$. This change of signs of the quantities $\sqrt{a}$, $\sqrt{a'}$, &c. when the corresponding values n, n', &c. become negative, is a very important consideration, since it will be shown, in § 57, that the permanency of the solar system depends on these radicals having the same sign.

* (747) The assumed values of h, l, h', l', &c. [1096] being substituted in the equations [1089], produce the equations [1097]. The two first of the equations [1089], produce separately, the first of the equations [1097]. The two next produce the second of [1097], and so on; consequently the number of equations [1097] is just half that of [1089], so that there will be as many equations [1097] as there are bodies m, m', m'', &c., which number is i, the number of the equations [1089], being $2\,i$. To show, by a simple example the use of these equations, we shall suppose that there are only two of the bodies m, m', and the equations [1097] will become [1097a]

$$\mathcal{N}g=(0,1).\mathcal{N}-\boxed{0,1}.\mathcal{N}'; \qquad \mathcal{N}'g=(1,0).\mathcal{N}'-\boxed{1,0}.\mathcal{N}.$$

If we suppose the number of bodies m, m' m'', &c., to be equal to i, the number of these equations will be i; and by eliminating the constant quantities N, N', &c., we shall have a final equation in g of the degree i,* which may be obtained in the following manner. [1097′]

The first gives $\mathcal{N}'=\frac{(0,1)-g}{\boxed{0,1}}\,.\,\mathcal{N}$, which being substituted in the second, put under the form $0=\{(1,0)-g\}\,.\,\mathcal{N}'-\boxed{1,0}\,.\,\mathcal{N}$, becomes divisible by $\mathcal{N}$, and gives

$$0=\frac{\{(1,0)-g\}\,.\,\{(0,1)-g\}}{\boxed{0,1}}-\boxed{1,0},$$

which is of the second degree in g, and furnishes two values, which we shall denote by g, g_1. The first of these being substituted in $\mathcal{N}'$, gives its value $\mathcal{N}'=\frac{(0,1)-g}{\boxed{0,1}}\,.\,\mathcal{N}$, and if we assume another arbitrary term $\mathcal{N}_1$ instead of $\mathcal{N}$, to correspond to g_1, the corresponding value of $\mathcal{N}'$, which we shall denote by $\mathcal{N}_1'$, will be $\mathcal{N}_1'=\frac{(0,1)-g_1}{\boxed{0,1}}\,.\,\mathcal{N}_1$. Hence in addition to the values of h, h', l, l', in [1096], we may also put $h=\mathcal{N}_1\,.\,\sin.(g_1t+\beta_1)$, $l=\mathcal{N}_1\,.\,\cos.(g_1t+\beta_1)$, $h'=\mathcal{N}_1'.\sin.(g_1t+\beta_1)$, $l'=\mathcal{N}_1'.\cos.(g_1t+\beta_1)$, and as the equations [1089] are linear in h, l, h', l', we may take the sums of these two values of h, or l, &c.; by which means we shall have $h=\mathcal{N}.\sin.(gt+\beta)+\mathcal{N}_1\,.\,\sin.(g_1t+\beta_1)$, $l=\mathcal{N}\,.\,\cos.(gt+\beta)+\mathcal{N}_1\,.\,\cos.(g_1t+\beta_1)$, $h'=\mathcal{N}'.\sin.(gt+\beta)+\mathcal{N}_1'.\sin.(g_1t+\beta_1)$, $l_1=\mathcal{N}'.\cos.(gt+\beta)+\mathcal{N}_1'.\cos.(g_1t+\beta_1)$, which satisfy the four equations [1089], and contain four arbitrary constant quantities, $\mathcal{N}$, $\mathcal{N}_1$, β, β_1. They must therefore be the complete integrals of the proposed equations. [1097b]

* (748) It was proved in the last note that when there are two bodies m, m', the resulting equation in g will be of the second degree. If there be three bodies, m, m', m'', the first of the equations [1097], gives $\mathcal{N}'=(Ag+B)\,.\,\mathcal{N}+C\mathcal{N}''$, A, B, C, being coefficients depending on $(0,1)$, $(0,2)$, &c., and independent of g, $\mathcal{N}$, $\mathcal{N}'$, $\mathcal{N}''$. This value being substituted in the second and third of the equations [1097], they will become of the forms

$$0=(A'g^2+B'g+C')\,.\,\mathcal{N}+(D'g+E')\,.\,\mathcal{N}'';$$
$$0=(A''g+B'')\,.\,\mathcal{N}''+(C''g+D'')\,.\,\mathcal{N};$$ [1098a]

A', B', &c., A'', B'', &c., being like A, B, &c., independent of g. The value of $\mathcal{N}''$, deduced from the first of these equations, being substituted in the second, it becomes of the form

Let the following function be represented by φ, that is*

[1098]
$$\begin{aligned}\varphi = {} & N^2 . m . \sqrt{a} . \{ g - (0,1) - (0,2) - \&c. \} \\ & + N'^2 . m' . \sqrt{a'} . \{ g - (1,0) - (1,2) - \&c. \} \\ & + \&c. \\ & + 2\, N . m . \sqrt{a} . \{ \boxed{0,1} . N' + \boxed{0,2} . N'' + \&c. \} \\ & + 2\, N' . m' . \sqrt{a'} . \{ \boxed{1,2} . N'' + \boxed{1,3} . N''' + \&c. \} \\ & + 2\, N'' . m'' . \sqrt{a''} . \{ \boxed{2,3} . N''' + \boxed{2,4} . N'''' + \&c. \} \\ & + \&c\end{aligned}$$

The equations B [1097], are reduced to the following forms, by means of the conditions mentioned in the preceding article,†

[1099]
$$\left(\frac{d\varphi}{dN}\right) = 0 ; \qquad \left(\frac{d\varphi}{dN'}\right) = 0 ; \qquad \left(\frac{d\varphi}{dN''}\right) = 0 ; \qquad \&c.$$

$$0 = -\frac{(A''g + B'').(A'g^2 + B'g + C').N}{D'g + E'} + (C''g + D'').N.$$

Dividing this by the common factor N, and reducing, we obtain an equation in g, of the third degree of the form

$$0 = A''''g^3 + B''''g^2 + C''''g + D'''',$$

having three roots, g, g_1, g_2, which is the number required to give the complete integral in this case of three bodies; and it is evident, from a little consideration, that if the number of bodies be i, the number of the equations [1097] will also be i, and they will produce by elimination an equation in g of the degree i, which will give the number of arbitrary constant quantities necessary to obtain the complete integrals of those equations; this agrees with [1097′]. We may observe that all the quantities N, N', &c., being supposed to be of the same order, it will follow from either of the equations [1097], that g is of the same order as [1097c] the quantities $(0,1)$, $(0,2)$, &c., $\boxed{0,1}$, &c., which are of the same order as the disturbing forces by [1079, 1083, &c.]

* (749) This function φ is so formed that the coefficients of $N^2.m.\sqrt{a}$, $N'^2.m'.\sqrt{a'}$ &c., are the same as the coefficients of N, N', &c., in the equations [1097], the other terms are so formed that the coefficient of any term, as $2\,N^{(e)}$, contains only the terms $N^{(e+1)}$, $N^{(e+2)}$, &c., whose indices $(e+1)$, $(e+2)$, &c., exceed that of $2\,N^{(e)}$.

† (750) The function φ, [1098] gives

$$\begin{aligned}\left(\frac{d\varphi}{dN}\right) &= 2Nm.\sqrt{a}.\{g - (0,1) - (0,2) - \&c.\} + 2m.\sqrt{a}.\{\boxed{0,1}.N' + \boxed{0,2}.N'' + \&c.\} \\ &= 2m.\sqrt{a}.\{Ng - [(0,1) + (0,2) + \&c.].N + \boxed{0,1}.N' + \boxed{0,2}.N'' + \&c.\}.\end{aligned}$$

Supposing therefore N, N', N'', &c., to be variable quantities, φ will be a maximum. Moreover, since φ [1098] is a homogeneous function of these variable quantities, and of the second degree, we shall have* [1099']

$$N\cdot\left(\frac{d\varphi}{dN}\right)+N'\cdot\left(\frac{d\varphi}{dN'}\right)+\&\text{c.}=2\varphi\,; \qquad [1100]$$

therefore, in consequence of the preceding equations, we shall have $\varphi=0$. [1100']

We can now determine the maximum of the function φ in the following manner. Take first the differential of this function relative to N, and substitute in φ the value of N deduced from the equation $\left(\frac{d\varphi}{dN}\right)=0$, which [1100'']

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and as the part between the braces is, by the first of the equations [1097], equal to nothing, we shall have $\left(\frac{d\varphi}{dN}\right)=0$. The same value of φ, [1098], gives

$$\left(\frac{d\varphi}{dN'}\right)=2N'm'\cdot\sqrt{a'}\cdot\{g-(1,0)-(1,2)-\&\text{c.}\}$$
$$+2Nm\cdot\sqrt{a}\cdot\boxed{0,1}+2m'\cdot\sqrt{a'}\cdot\{\boxed{1,2}\cdot N''+\boxed{1,3}\cdot N'''+\&\text{c.}\},$$

but from [1094] we have $2Nm\cdot\sqrt{a}\cdot\boxed{0,1}=2Nm'\cdot\sqrt{a'}\cdot\boxed{1,0}$, hence

$$\left(\frac{d\varphi}{dN'}\right)=2m'\cdot\sqrt{a'}\cdot\{N'g-[(1,0)+(1,2)+\&\text{c.}]\cdot N'+\boxed{1,0}\cdot N+\boxed{1,2}\cdot N''+\&\text{c.}\};$$

and as the part between the braces is, by the second of the equations [1097], equal to nothing, we shall have $\left(\frac{d\varphi}{dN'}\right)=0$. Again, the expression [1098] gives

$$\left(\frac{d\varphi}{dN''}\right)=2N''m''\cdot\sqrt{a''}\cdot\{g-(2,0)-(2,1)-(2;3)+\&\text{c.}\}$$
$$+2Nm\cdot\sqrt{a}\cdot\boxed{0,2}+2N'm'\cdot\sqrt{a'}\cdot\boxed{1,2}+2m''\cdot\sqrt{a''}\cdot\{\boxed{2,3}\cdot N'''+\&\text{c.}\}.$$

But from [1095] we get

$2Nm\cdot\sqrt{a}\cdot\boxed{0,2}=2Nm''\cdot\sqrt{a''}\cdot\boxed{2,0}$, and $2N'm'\cdot\sqrt{a'}\cdot\boxed{1,2}=2N'm''\cdot\sqrt{a''}\cdot\boxed{2,1}$;

these being substituted we get

$$\left(\frac{d\varphi}{dN''}\right)=2m''\cdot\sqrt{a''}\cdot\{N''g-[(2,0)+(2,1)+\&\text{c.}]\cdot N''+\boxed{2,0}\cdot N+\&\text{c.}\},$$

which, by means of the third of the equations [1097], becomes $\left(\frac{d\varphi}{dN''}\right)=0$, and so on for the rest. These equations [1099] are evidently the same as the well known expressions for finding the maximum value of φ, as is observed in [1099'].

* (751) This follows from the theorem [1001a], changing $A^{(i)}$, a, a', &c., m, into φ, N, N', &c., and 2. Substituting the values [1099] in [1100], we shall get $\varphi=0$,

value will be a linear function of the quantities N', N'', &c.; we shall by this means have a rational, integral and homogeneous function of the second degree in N', N'', &c.; let this function be $\varphi^{(1)}$. Taking its differential relative to N', and substituting in $\varphi^{(1)}$ the value of N' deduced from the

[1100‴] equation $\left(\frac{d\,\varphi^{(1)}}{d\,N'}\right)=0$; we shall have a homogeneous function of the second

[1100ⁱᵛ] degree in N'', N''', &c.; let this function be $\varphi^{(2)}$. Continuing in this manner, we shall finally obtain a function $\varphi^{(i-1)}$ of the second degree in $N^{(i-1)}$, which will therefore be of the form $(N^{(i-1)})^2.k$; k being a function of g and constant quantities. If we put the differential of $\varphi^{(i-1)}$, taken relatively to $N^{(i-1)}$, equal to nothing, we shall find $k=0$; from which we

[1100ᵛ] shall get an equation in g, of the degree i, whose different roots will give as many different systems of indeterminate quantities N, N', N'', &c. The indeterminate quantity $N^{(i-1)}$ will be the arbitrary constant quantity of each system, and we shall immediately obtain the ratio of the other arbitrary quantities N, N', &c., of the system to this, by means of the preceding equations, taken in an inverse order, namely

[1101] $$\left(\frac{d\,\varphi^{(i-2)}}{d\,N^{(i-2)}}\right)=0\,; \qquad \left(\frac{d\,\varphi^{(i-3)}}{d\,N^{(i-3)}}\right)=0\,; \qquad \&c.$$

[1101′] Let g, g_1, g_2, &c., be the i roots of the equation in g; also N, N', N'', &c., the system of indeterminate quantities, relative to the root g; N_1, N_1', N_1'', &c., the system relative to the root g_1, and so on; then we shall have, by the known theory of linear partial differential equations,*

Solution of the system of linear equations [1089] of the first degree.

$$h = N\,.\,\sin.\,(g\,t+\beta)+N_1\,.\,\sin.\,(g_1\,t+\beta_1)+N_2\,.\,\sin.\,(g_2\,t+\beta_2)+\&c.\,;$$

$$h' = N'\,.\,\sin.\,(g\,t+\beta)+N_1'\,.\,\sin.\,(g_1\,t+\beta_1)+N_2'\,.\,\sin.\,(g_2\,t+\beta_2)+\&c.\,;$$

[1102] $$h'' = N''\,.\,\sin.\,(g\,t+\beta)+N_1''\,.\,\sin.\,(g_1\,t+\beta_1)+N_2''\,.\,\sin.\,(g_2\,t+\beta_2)+\&c.\,;$$

&c.;

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† (751*a*) Each of the values of g, namely, g, g_1, g_2, &c., will furnish a system of values of h, h', &c., l, l', &c., similar to [1096], which will satisfy the differential equations [1089], and as these differential equations are linear, it is evident, as in the example [1097*b*], that the sums of all the corresponding values, found as in [1102, 1102*a*], will also satisfy the same differential equations; moreover, these sums contain the requisite number of arbitrary constant quantities, [1102″], they will therefore represent the complete integrals of the equations [1089].

β, β_1, β_2, &c., being arbitrary constant quantities. Changing the sines into [1102']
cosines in these values of h, h', h'', &c., we shall have the values of l, l', l'',
&c.* These values contain twice as many arbitrary quantities as there are
roots g, g_1, g_2, &c.; for each system of indeterminate quantities contains
one arbitrary term, and there are also i arbitrary terms β, β_1, β_2, &c.; [1102'']
consequently these values will be the complete integrals of the equations (A)
[1089] of the preceding article.

It now remains to determine the constant quantities N, N_1, &c., N', N_1',
&c., β, β_1, &c. These quantities are not given directly by observation, but [1102''']
may be deduced from the excentricities of the orbits e, e', &c., and the
longitudes of the perihelia ϖ, ϖ', &c., at a known epoch, which give the
corresponding values of h, h', &c., l, l', &c.,† whence the former values may
easily be obtained. For this purpose we shall observe, that if we multiply
the first, third, fifth, &c., of the differential equations (A) [1089] of the [1102$^{\text{iv}}$]
preceding article, by $N.m.\sqrt{a}$, $N'.m'.\sqrt{a'}$, &c., we shall have, by means
of the equations (B) [1097], and of the relations, found in the preceding
article, between (0,1) and (1,0), (0,2) and (2,0), &c.,‡

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* (752) These are

$$l = N.\cos.(g\,t + \beta) + N_1.\cos.(g_1\,t + \beta_1) + N_2.\cos.(g\,t_2 + \beta_2) + \&c.,$$
$$l' = N'.\cos.(g\,t + \beta) + N_1'.\cos.(g_1\,t + \beta_1) + N_2'.\cos.(g\,t_2 + \beta_2) + \&c. \qquad [1102a]$$
&c.

A little consideration will show that in these values of h, h', &c., l, l', &c., terms of the [1102b]
order $m'e^3$, m'^2, are neglected; quantities of this kind having been neglected in [1089].
Also g, g_1, g_2, &c., are of the order m', m'', &c., [1097c]. In [1097'] it was observed that there are i equations [1097], and $2\,i$ equations [1089]. The equations [1097] furnish i arbitrary constant quantities N, N_1, N_2, &c., also i quantities β, β', β'', &c., making in all $2i$ quantities, being the number necessary for the complete integration of the $2\,i$ equations [1089] of the first degree.

† (753) h, h', &c., l, l', &c., are deduced from e, e', &c., ϖ, ϖ', &c., by means of the equations [1022].

‡ (754) The first member of the expression arises from the sum of the products thus found, without any reduction. The products to be added together in the second member are

$$N m.\sqrt{a}.\{[(0,1) + (0,2) + \&c.].l - \boxed{0,1}.l' - \boxed{0,2}.l'' - \&c.\},$$
$$N' m'.\sqrt{a'}.\{[(1,0) + (1,2) + \&c.].l' - \boxed{1,0}.l - \boxed{1,2}.l'' - \&c.\},$$
&c.

[1103]
$$N\,.\,\frac{dh}{dt}\,.\,m\,.\,\sqrt{a}+N'\,.\,\frac{dh'}{dt}\,.\,m'\,.\,\sqrt{a'}+N''\,.\,\frac{dh''}{dt}\,.\,m''\,.\,\sqrt{a''}+\&c.$$
$$=g.\{N.l.m\,.\,\sqrt{a}+N'\,.\,l'\,.\,m'\,.\,\sqrt{a'}+N''\,.\,l''\,.\,m''\,.\,\sqrt{a''}+\&c.\}$$

If we substitute in this equation the above values of h, h', &c., l, l', &c., we shall have, by comparing the coefficients of the same cosines,*

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which, by substituting the values of $(0,1)+(0,2)+\&c.$, $(1,0)+(1,2)+\&c.$, deduced from [1097], become

$$m\,.\,\sqrt{a}\,.\,l\,.\,\{Ng+\boxed{0,1}\,.\,N'+\boxed{0,2}\,.\,N''+\&c.\}-Nm\,.\,\sqrt{a}\,.\,\{\boxed{0,1}\,.\,l'+\boxed{0,2}\,.\,l''+\&c.\},$$
$$m'\,.\,\sqrt{a'}\,.\,l'\,.\{N'g+\boxed{1,0}\,.N+\boxed{1,2}\,.N''+\boxed{1,3}\,.N'''+\&c.\}$$
$$-N'm'\,.\,\sqrt{a'}\,.\,\{\boxed{1,0}\,.\,l+\boxed{1,2}\,.\,l''+\boxed{1,3}\,.\,l'''+\&c.\},$$
$$m''\,.\,\sqrt{a''}.l''\,.\,\{N''g+\boxed{2,0}\,.N+\boxed{2,1}\,.\,N'+\boxed{2,3}\,.N'''+\&c.\}$$
$$-N''m''\,.\,\sqrt{a''}.\,\{\boxed{2,0}\,.\,l+\boxed{2,1}\,.\,l'+\boxed{2,3}\,.\,l'''+\&c.\},$$
&c.

adding these products, then connecting all the terms multiplied by g together, and afterwards those by N, N', &c., we shall get

$$g\,.\,\{N\,.\,l\,.\,m\,.\,\sqrt{a}+N'\,.\,l'\,.\,m'\,.\,\sqrt{a'}+\&c.\}$$
$$+N.\{-m.\sqrt{a}.\left[\boxed{0,1}.l'+\boxed{0,2}.l''+\boxed{0,3}.l'''+\&c.\right]+m'.l'.\sqrt{a'}.\boxed{1,0}+m''.l'.\sqrt{a''}.\boxed{2,0}+\&c.\}$$
$$+N'.\{-m'.\sqrt{a'}.\left[\boxed{1,0}.l+\boxed{1,2}.l''+\boxed{1,3}.l'''+\&c.\right]+m.l.\sqrt{a}.\boxed{0,1}+m''.l''.\sqrt{a''}.\boxed{2,1}+\&c.\}$$
$$+\&c.$$
$$=g\,.\{N\,.\,l\,.\,m\,.\,\sqrt{a}+N'\,.\,l'\,.\,m'\,.\,\sqrt{a'}+N''\,.\,l''\,.\,m''\,.\,\sqrt{a''}+\&c.\}$$
$$+N.\left\{\begin{array}{l}l'.\,(m'.\sqrt{a'}.\boxed{1,0}-m\,.\,\sqrt{a}.\boxed{0,1})+l''.\,(m''.\,\sqrt{a''}.\,\boxed{2,0}-m\,.\,\sqrt{a}\,.\,\boxed{0,2}) \\ \qquad\qquad +l'''.\,(m'''.\sqrt{a'''}.\boxed{3,0}-m\,.\,\sqrt{a}\,.\,\boxed{0,3})+\&c.\end{array}\right\}$$
$$+N'.\left\{\begin{array}{l}l\,.\,(m\,.\,\sqrt{a}.\boxed{0,1}-m'\,.\,\sqrt{a'}\,.\boxed{1,0})+l''\,.\,(m''\,.\sqrt{a''}\,.\,\boxed{2,1}-m'.\sqrt{a'}\,.\,\boxed{1,2}) \\ \qquad\qquad +l'''.(m'''.\sqrt{a'''}.\,\boxed{3,1}-m'.\sqrt{a'}\,.\,\boxed{1,3})+\&c.\end{array}\right\}$$
$$+\&c.$$

each of the factors of l, l', l'', &c. in the terms of the second third, &c. lines of this expression becomes nothing, by means of [1094, 1095], and the whole is reduced to the first line, which is the same as the second member of [1103].

[1105a] * (755) We shall, for brevity, put the values [1102, 1102a], under these forms,

$$h=\Sigma\,.\,N_n\,.\,\text{sin.}\,(g_n\,t+\beta_n),\qquad h'=\Sigma\,.\,N_n^{(1)}\,.\,\text{sin.}\,(g_n\,t+\beta_n),$$
$$h''=\Sigma\,.\,N_n^{(2)}\,.\,\text{sin.}\,(g_n\,t+\beta_n),$$

$$0 = N . N_1 . m . \sqrt{a} + N' . N_1' . m' . \sqrt{a'} + N'' . N_1'' . m'' . \sqrt{a''} + \&c.;$$
$$0 = N . N_2 . m . \sqrt{a} + N' . N_2' . m' . \sqrt{a'} + N'' . N_2'' . m'' . \sqrt{a''} + \&c.; \qquad [1104]$$
&c.

This being premised, if we multiply the preceding values of h, h', &c., [1102], respectively by $N . m . \sqrt{a}$, $N' . m' . \sqrt{a'}$, &c., we shall have, by means of these last equations,*

$$N.m.h.\sqrt{a} + N'.m'.h'.\sqrt{a'} + N''.m''.h''.\sqrt{a''} + \&c.$$
$$= \{N^2\,m . \sqrt{a} + N'^2 . m' . \sqrt{a'} + N''^2 . m'' . \sqrt{a''} + \&c.\} . \sin.(gt + \beta). \qquad [1105]$$

We shall likewise have

$$N.m.l.\sqrt{a} + N'.m'.l'.\sqrt{a'} + N''.m''.l''.\sqrt{a''} + \&c.$$
$$= \{N^2 . m . \sqrt{a} + N'^2 . m' . \sqrt{a'} + N''^2 . m'' . \sqrt{a''} + \&c.\} . \cos.(gt + \beta). \qquad [1106]$$

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and generally $h^{(s)} = \Sigma . N_n^{(s)} . \sin.(g_n t + \beta_n)$. In like manner $l^{(s)} = \Sigma . N_n^{(s)} . \cos.(g_n t + \beta_n)$ These being substituted in [1103], it becomes

$$N m . \sqrt{a} . \Sigma . N_n\, g_n . \cos.(g_n t + \beta_n) + N' m' . \sqrt{a'} . \Sigma . N_n'\, g_n . \cos.(g_n t + \beta_n) + \&c.$$
$$= g . \{N m . \sqrt{a} . \Sigma . N_n . \cos.(g_n t + \beta_n) + N' m' . \sqrt{a'} . \Sigma . N_n' . \cos.(g_n t + \beta_n) + \&c.\}$$

Transposing all to the first member it becomes

$$0 = N m . \sqrt{a} . \Sigma . (g_n - g) . N_n . \cos.(g_n t + \beta_n) + N' m' . \sqrt{a'} . \Sigma . (g_n - g) . N_n' . \cos.(g_n t + \beta_n) + \&c.$$

Putting now successively $n = 1$, $n = 2$, $n = 3$, &c., we shall obtain the coefficients of $\cos.(g_1 t + \beta_1)$, $\cos.(g_2 t + \beta_2)$, &c., which being put equal to nothing, and divided respectively by $g_1 - g$, $g_2 - g$, &c., give the equations [1104].

* (756) The first member of the sum of these equations gives, without reduction, the first member of [1105], the coefficients of $\sin.(g_1 t + \beta_1)$, $\sin.(g_2 t + \beta_2)$, &c., in the second member, are respectively equal to the second members of the equations [1104], therefore they are equal to nothing. The coefficient of the remaining term $\sin.(gt + \beta)$, becomes like that in the second member of [1105]. The equation [1106] is obtained in like manner from the expressions [1102*a*]; or more simply, by changing β, β_1, β_2, &c., into $\beta + \frac{1}{2}\pi$, $\beta_1 + \frac{1}{2}\pi$, $\beta_2 + \frac{1}{2}\pi$, &c., respectively, $\frac{1}{2}\pi$ being a right angle. For the values of h, h', &c., [1102], by this means become l, l', &c., [1102*a*]. These changes being made in [1105], we shall get [1106]. If in these equations we suppose the values h, h', &c., l, l', &c. to correspond to the time $t = 0$, the terms $\sin.(gt + \beta)$, $\cos.(gt + \beta)$ will become simply $\sin.\beta$, $\cos.\beta$, and then dividing the expression [1105] by [1106], we shall obtain tang. β, [1107].

Fixing the origin of the time t, at the epoch for which the values of h, l, h', l', &c., are supposed to be known; the two preceding equations will give

[1107] $$\text{tang.}\,\beta=\frac{N.h.m.\sqrt{a}+N'.h'.m'.\sqrt{a'}+N''.h''.m''.\sqrt{a''}+\&c.}{N.l.m.\sqrt{a}+N'.l'.m'.\sqrt{a'}+N''.l''.m''.\sqrt{a''}+\&c.}$$

This expression of tang. β contains no indeterminate quantity; for although
the constant quantities N, N', N'', &c., depend on the indeterminate quantity
[1107'] $N^{(i-1)}$ [1100^{v}]; yet as the ratios of these indeterminate quantities are known
by what precedes, it will disappear from the expression of tang. β.* Having
thus found β, we can find $N^{(i-1)}$, by means of one of the two equations
which give the value of tang. β; hence we may obtain the system of
[1107''] indeterminate quantities N, N', N'', &c., relative to the root g. Changing
in the preceding expressions this root successively into g_1, g_2, g_3, &c., we
shall get the values of the arbitrary quantities, corresponding to each of
these roots.

[1107'''] If we substitute these values in the expressions of h, l, h', l', &c. [1102, 1102a], we may deduce from them the values of the excentricities e, e', &c., of the orbits, and the longitudes ϖ, ϖ', &c., of their perihelia, by means of the equations†

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* (757) The first of the equations [1101] is linear in $N^{(i-1)}$, $N^{(i-2)}$, and gives $N^{(i-2)}$, by a simple equation, of the form $N^{(i-2)}=a.N^{(i-1)}$. The second of the equations [1101] is linear in $N^{(i-1)}$, $N^{(i-2)}$, $N^{(i-3)}$, and gives $N^{(i-3)}=a'\,N^{(i-2)}+b'\,N^{(i-1)}$, and by means of the value of $N^{(i-2)}$, found by the preceding equation, it becomes $N^{(i-3)}=a''N^{(i-1)}$. The third of the equations [1101], by similar reductions, gives $N^{(i-4)}=a'''\,N^{(i-1)}$, &c., the terms b', a, a', a'', &c., being independent of N, N', &c. Substituting these values of $N^{(i-2)}$, $N^{(i-3)}$, &c., in [1107], the numerator and denominator will become divisible by $N^{(i-1)}$, and we shall get tang. β, independent of that quantity. Having found β, we may substitute it, and also the preceding values of $N^{(i-2)}$, $N^{(i-3)}$, in [1105] or [1106]; then putting $t=0$, we shall obtain the value of $N^{(i-1)}$.

† (758) These are easily deduced from [1022], $h=e.\sin.\varpi$, $l=e.\cos.\varpi$, &c. The sum of the squares of h, l, being evidently equal to e^2, and the former divided by the latter gives tang. ϖ.

$$e^2 = h^2 + l^2\,; \qquad e'^2 = h'^2 + l'^2\,; \qquad \&c.\,;$$
$$\text{tang.}\,\varpi = \frac{h}{l}\,; \qquad \text{tang.}\,\varpi' = \frac{h'}{l'}\,; \qquad \&c.\,;$$ [1108]

hence we shall have*

$$e^2 = N^2 + N_1^2 + N_2^2 + \&c. + 2\,N N_1\,.\,\text{cos.}\{(g_1 - g)\,.\,t + \beta_1 - \beta\}$$
$$+ 2 N N_2.\text{cos.}\{(g_2 - g).t + \beta_2 - \beta\} + 2 N_1 N_2.\text{cos.}\{(g_2 - g_1).t + \beta_2 - \beta_1\} + \&c.$$ [1109]

This quantity is always less than $(N + N_1 + N_2 + \&c.)^2$, when the roots g, g_1, &c., are all real and unequal, taking the quantities N, N_1, N_2, &c., positive. We shall also have† [1109′]

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* (758a) From h, l, [1102, 1102a], we get

$$h^2 + l^2 = \{N\,.\,\text{sin.}\,(g\,t + \beta) + N_1\,.\,\text{sin.}\,(g_1\,t + \beta_1) + \&c.\}^2$$
$$+ \{N\,.\,\text{cos.}\,(g\,t + \beta) + N_1\,.\,\text{cos.}\,(g_1\,t + \beta_1) + \&c.\}^2$$
$$= N^2\,.\,\{\text{sin.}^2\,(g\,t + \beta) + \text{cos.}^2\,(g\,t + \beta)\}$$
$$+ 2\,N N_1\,.\,\{\text{sin.}\,(g\,t + \beta)\,.\,\text{sin.}\,(g_1\,t + \beta_1) + \text{cos.}\,(g\,t + \beta)\,.\,\text{cos.}\,(g_1\,t + \beta_1)\} + \&c.$$
$$= N^2 + 2\,N N_1\,.\,\text{cos.}\,\{(g_1 - g)\,.\,t + \beta_1 - \beta\} + \&c.\,;$$

the coefficient of $2\,N N_1$, being reduced by [24] Int. This expression must be symmetrical in N, N_1, N_2, &c., hence from the term $2\,N N_1\,.\,\text{cos.}\,\{(g_1 - g)\,.\,t + \beta_1 - \beta\}$, it will evidently follow, that the general expression, corresponding to $N_n N_m$ is

$$2\,N_n N_m\,.\,\text{cos.}\,\{(g_m - g_n)\,.\,t + \beta_m - \beta_n\},$$

as in [1109]. If we suppose all the quantities N, N_1, N_2, &c., to be positive, the greatest possible value, of the second member of the equation [1109], will be when the cosines are all equal to unity, and then

$$e^2 = N^2 + N_1^2 + N_2^2 + \&c. + 2\,N N_1 + 2\,N N_2 + 2\,N_1 N_2 + \&c. = (N + N_1 + N_2 + \&c.)^2.$$

Now when g, g_1, g_2, &c., are all real, unequal, and incommensurable, these cosines cannot all become unity at the same instant; therefore the general value of e must be less than $(N + N_1 + N_2 + \&c.)$, as in [1109′]. This is the case with the solar system, as is observed in § 57. It may be remarked that in these values of e, e', &c., terms of the same order are neglected as in [1102b]. [1108a]

† (759) This value of tang. ϖ is deduced from that in [1108], by substituting h, l, [1102, 1102a], and if for brevity we put $g\,t + \beta = T$, $g_1\,t + \beta_1 = T_1$, &c., it becomes $\text{tang.}\,\varpi = \dfrac{N\,.\,\text{sin.}\,T + N_1\,.\,\text{sin.}\,T_1 + N_2\,.\,\text{sin.}\,T_2 + \&c.}{N.\text{cos.}\,T + N_1\,.\text{cos.}\,T_1 + N_2\,.\text{cos.}\,T_2 + \&c.}$ This gives

[1110] $$\text{tang.}\,\varpi = \frac{N.\,\text{sin.}\,(g\,t+\beta)+N_1.\,\text{sin.}\,(g_1\,t+\beta_1)+N_2.\,\text{sin.}\,(g_2\,t+\beta_2)+\&\text{c.}}{N.\,\text{cos.}\,(g\,t+\beta)+N_1.\,\text{cos.}\,(g_1\,t+\beta_1)+N_2.\,\text{cos.}\,(g_2\,t+\beta_2)+\&\text{c.}};$$

hence it is easy to deduce

[1111] $$\text{tang.}(\varpi-gt-\beta)=\frac{N_1.\,\text{sin.}\{(g_1-g).t+\beta_1-\beta\}+N_2.\,\text{sin.}\{(g_2-g).t+\beta_2-\beta\}+\&\text{c}}{N+N_1.\text{cos.}\{(g_1-g).t+\beta_1-\beta\}+N_2.\text{cos.}\{(g_2-g).t+\beta_2-\beta\}+\&\text{c.}}.$$

When the sum N_1+N_2+ &c. of the coefficients of the cosines of the [1111′] denominator, taken all positively, is less than N, $\text{tang.}(\varpi-gt-\beta)$ cannot

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$$\text{tang.}\,\varpi-\text{tang.}\,T=\frac{N.\,(\text{sin.}\,T-\text{cos}\,T.\,\text{tang.}\,T)+N_1.\,(\text{sin.}\,T_1-\text{cos.}\,T_1.\,\text{tang.}\,T)+\&\text{c.}}{N.\,\text{cos.}\,T+N_1.\,\text{cos.}\,T_1+\&\text{c.}};$$

also

$$1+\text{tang.}\,\varpi\,.\,\text{tang.}\,T=\frac{N.\,(\text{cos.}\,T+\text{sin.}\,T.\,\text{tang.}\,T)+N_1.\,(\text{cos.}\,T_1+\text{sin.}\,T_1.\,\text{tang.}\,T)+\&\text{c.}}{N.\,\text{cos.}\,T+N_1.\,\text{cos.}\,T_1+N_2.\,\text{cos.}\,T_2+\&\text{c.}}$$

Dividing the former by the latter, we get $\dfrac{\text{tang.}\,\varpi-\text{tang.}\,T}{1+\text{tang.}\,\varpi\,.\,\text{tang.}\,T}=\text{tang.}\,(\varpi-T)$, [30] Int., hence

$$\text{tang.}\,(\varpi-T)=\frac{N.\,(\text{sin.}\,T-\text{cos.}\,T.\,\text{tang.}\,T)+N_1.\,(\text{sin.}\,T_1-\text{cos.}\,T_1.\,\text{tang.}\,T)+\&\text{c.}}{N.\,(\text{cos.}\,T+\text{sin.}\,T.\,\text{tang.}\,T)+N_1.\,(\text{cos.}\,T_1+\text{sin.}\,T_1.\,\text{tang.}\,T)+\&\text{c.}}$$

Putting for tang. T its value $\dfrac{\text{sin.}\,T}{\text{cos.}\,T}$, and multiplying numerator and denominator by cos. T, it becomes

$$\frac{N.(\text{sin.}T.\text{cos.}T-\text{cos.}T.\text{sin.}T)+N_1.(\text{sin.}T_1.\text{cos.}T-\text{cos.}T_1.\text{sin.}T)+N_2.(\text{sin.}T_2.\text{cos.}T-\text{cos.}T_2.\,\text{sin.}\,T)+\&\text{c.}}{N.(\text{cos.}T.\text{cos.}T+\text{sin.}T.\text{sin.}T)+N_1.(\text{cos.}T_1.\text{cos.}T+\text{sin.}T_1.\text{sin.}T)+N_2.(\text{cos.}T_2.\text{cos.}T+\text{sin.}T_2.\,\text{sin.}\,T)+\&\text{c.}}$$

$$=\frac{N_1.\,\text{sin.}\,(T_1-T)+N_2.\,\text{sin.}\,(T_2-T)+\&\text{c.}}{N+N_1.\,\text{cos.}\,(T_1-T)+N_2.\,\text{cos.}\,(T_2-T)+\&\text{c.}},$$

by [22, 24] Int. Resubstituting the values of T, T_1, &c., it becomes as in [1111]. The terms N, N', &c., are of the order h, h', &c., [1102], and if the first term N of the denominator, [1111], be greater than the sum of all the following coefficients N_1+N_2+&c., considering them all as positive, the denominator will always be finite and positive; and as the numerator cannot exceed this sum of $N_1+N_2+N_3+$ &c., the expression $\text{tang.}\,(\varpi-T)$, cannot be infinite; therefore $\varpi-g\,t-\beta$, must be less than a right angle; now this cannot be the case unless the mean motion of ϖ be exactly equal to $g\,t$, β being constant. For if the difference, between the mean motion of ϖ and the angle $g\,t$, were even very small, it would, by increasing, in proportion to the time, finally become greater than a right angle, consequently the mean motion of the perihelion of m must in this case be $g\,t$. In all these computations, terms of the order mentioned in [1102*b*] are neglected.

become infinite; the angle $\varpi - gt - \beta$ cannot therefore become equal to a right angle; consequently the mean motion of the perihelion will be in [1111″] this case equal to gt.

57. It follows from what has been proved, that the excentricities of the orbits, and the positions of the transverse axes, are subject to considerable variations, which in the course of time change the form of these orbits, in periods depending on the roots g_1, g_2, &c.; and as it respects the planets, these periods include many centuries. We may therefore consider the [1111‴] excentricities as variable ellipticities, and the motions of the perihelia as not being uniform. These variations are very sensible in the satellites of [1111$^{\text{iv}}$] Jupiter; and we shall show hereafter, that they explain the singular inequalities which have been observed in the motion of the third satellite. [1111$^{\text{v}}$] But the question arises whether these variations of the excentricities are limited in extent, so that the orbits will always be nearly circular. This [1111$^{\text{vi}}$] is a subject which ought to be carefully examined. We have just shown [1109′] that if the roots of the equation in g be all real and unequal, the excentricity e of the orbit of m will be always less than the sum $N + N_1 + N_2 +$ &c. of the coefficients of the sines of the expression of h, taken positively; and as these coefficients are supposed to be very small, [1111$^{\text{vii}}$] the value of e will always be small. Therefore, if we notice only the secular variations, the orbits of the bodies m, m', m'', &c., may become more or less excentrical, but they will never vary much from a circular form, [1111$^{\text{viii}}$] though the positions of the transverse axes may suffer considerable variations. These axes will be invariably of the same magnitudes, and the mean motions, which depend on them will always be uniform, as we have seen in § 54 [1070$^{\text{x}}$]. [1111$^{\text{ix}}$] The preceding results, founded on the smallness of the excentricities of the orbits, will always take place, and may be extended to all past or future [1111$^{\text{x}}$] ages; so that we can affirm, that the orbits of the planets and satellites never were, at any former period of time, and never will be, hereafter, considerably excentrical, so far as it depends on their mutual attraction. But this would not be the case, if any of the roots g, g_1, g_2, &c., were equal [1111$^{\text{xi}}$] or imaginary: the sines and cosines of the expressions of h, l, h', l', &c., corresponding to these roots, would become arcs of a circle, or exponential

[1111$^{\text{xii}}$] quantities ; and as these quantities would increase indefinitely with the time, the orbits would at length become very excentrical.* The stability of the planetary system would in this case be destroyed, and the results we have found would cease to take place. It is therefore very interesting to ascertain [1111$^{\text{xiii}}$] whether the roots g, g_1, g_2, &c., are all real and unequal. This we shall prove to be the fact, in a very simple manner, in the case of nature, where [1111$^{\text{xiv}}$] the bodies m, m', m'', &c., of the system, revolve in the same direction.

We shall resume the equations (A) § 55 [1089]. If we multiply the first by $m\,.\sqrt{a}\,.\,h$, the second by $m\,.\sqrt{a}\,.\,l$, the third by $m'\,.\sqrt{a'}\,.\,h'$, the fourth by $m'\,.\sqrt{a'}\,.\,l'$, &c., and then add these products together, the [1111$^{\text{xv}}$] coefficients of hl, $h'l'$, $h''l''$, &c., in this sum, will be nothing ; the coefficient of $h'l-hl'$ will be $\boxed{0,1}\,m\,.\sqrt{a}-\boxed{1,0}\,.\,m'\,.\sqrt{a'}$, and this will become nothing by means of the equation $\boxed{0,1}\,.\,m\,.\sqrt{a}=\boxed{1,0}\,.\,m'\,.\sqrt{a'}$, found in § 55 [1094]. The coefficients of $h''l-hl''$, $h''l'-h'l''$, &c., will be nothing, [1111$^{\text{xvi}}$] for a similar reason ; therefore the sum of the equations (A) [1089], thus multiplied, will be reduced to the following equation,

$$\left(\frac{h\,dh+l\,dl}{dt}\right).\,m\,.\sqrt{a}+\left(\frac{h'\,dh'+l'\,dl'}{dt}\right).\,m'\,.\sqrt{a'}+\text{\&c.}=0\,; \qquad [1112]$$

which is equivalent to the following,†

$$0=e\,de\,.\,m\,.\sqrt{a}+e'\,de'\,.\,m'\,.\sqrt{a'}+\text{\&c.} \qquad [1113]$$

Taking the integral of this equation, and observing that by § 54 [1070$^{\text{xi}}$], the semi-transverse axes a, a', &c., are constant, we shall have

$$e^2.\,m.\sqrt{a}+e'^2.\,m'.\sqrt{a'}+\text{\&c.}=\text{constant.} \qquad (u) \qquad [1114]$$

Now the bodies m, m', m'', &c., being supposed to revolve in the same direction, the radicals $\sqrt{a}$, $\sqrt{a'}$, &c., ought to be positive in the preceding equation, as we have seen in § 55 [1094']; all the terms of the first member [1114'] of this equation are therefore positive, consequently each one of them must be less than the constant quantity, in the second member ; now if we

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* (760) This is shown more fully in [1114'''—1118$^{\text{iv}}$].

† (761) The differentials of the values of e^2, e'^2, &c., [1108] being substituted in [1112] give [1113].

suppose, at a given epoch, the excentricities to be very small, this constant quantity will be very small; each of the terms of the equation will therefore be small, and cannot increase indefinitely, so that the orbits will always be [1114′]
nearly circular.*

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* (762) If we substitute in the equation [1114] the values of the masses of the planets, given in Book vi, § 21, 22, the terms like $e^2 m . \sqrt{a}$, relative to Mercury, Venus, the Earth, Mars, Jupiter, Saturn, and Uranus, respectively, expressed in fractions of $\frac{1}{100000000000}$ of unity, will be nearly 50, 1, 9, 88, 258000, 276000, 95000, whose sum, 629148, is the value of the constant quantity of the second member of [1114], corresponding to the solar system, [1114a]
neglecting the terms depending on the comets and satellites, and on the very small planets, Vesta, Juno, Pallas, and Ceres, whose masses are unknown, but which are probably so very small that they could not sensibly affect the calculation. The constant term of the second member of [1114] being always equal to 629148, it will follow that the orbits of the three larger planets Jupiter, Saturn and Uranus, can never be very excentrical; but it does not follow, from the same equation, that the orbits of the smaller planets will always be nearly circular; since they might be very excentrical, or even parabolic, and the equation be satisfied. For if the orbits of Mercury, Venus, the Earth, and Mars, were parabolic, or their excentricities equal to unity, the preceding terms 50, 1, 9, 88, would become respectively, 1190, 16000, 30340, 10190, and the first sum would be increased by 57572, but this increment would be wholly balanced by decreasing the excentricity of Jupiter about one eighth part, which would decrease the term 258000 to $258000 . \frac{7^2}{8^2} = 197531$, the difference 60469 being greater than the preceding sum 57572; therefore the orbits of Mercury, Venus, the Earth, and the four lately discovered planets, might be supposed parabolic, and yet the equation [1114] would be satisfied. We cannot therefore conclude from that equation, *independent of other considerations of analogy*, that the orbits of all the planets will never vary much from a circular form. It may be observed that some of the [1114b]
terms of the order $m . \sqrt{a} . e^4$, &c., neglected in [1114, 1151″, &c.], exceed some of the terms of the order $m . \sqrt{a} . e^2$, which are retained; this is an additional reason why the equation [1114], should be restricted to the three greatest planets. This subject was mentioned by me in a paper presented to the American Academy of Arts and Sciences, and published in Vol. IV of their memoirs. The same defect was observed by La Grange in Vol. II, p. 148 of the second edition of his Mécanique Analytique. The equation [1114], is also affected by the small secular equation depending on the attraction of the fixed stars, treated of by the author, in Book vi, § 47; and M. Plana, in the paper mentioned in page 561, has shown that this attraction prevents the second member of the equation [1112] from becoming nothing, as will be shown in the notes on that book.

The case we have examined is that of the planets and satellites of the
[1114'''] solar system; since all these bodies revolve in the same direction, and at
the present epoch the excentricities of the orbits are very small. To leave
no doubt on this important result, we shall observe that if the equation, by
[1114iv] which g is determined, contains imaginary roots, some of the sines and cosines
of the expressions of h, l, h', l', &c., will become exponential quantities;*
therefore the expression of h will contain a finite number of terms of the form
[1114v] $P\,.\,c^{ft}$, c being the number whose hyperbolic logarithm is unity, and P a
real quantity, since h or $e\,.\,\sin.\varpi$ [1022] is a real quantity. Let $Q\,.\,c^{ft}$,
[1114vi] $P'.c^{ft}$, $Q'.c^{ft}$, $P''.c^{ft}$, &c., be the corresponding terms of l, h', l', h'', &c.;
Q, P', Q', P'', &c., being also real quantities; the expression of e^2 will
contain the term $(P^2+Q^2)\,.\,c^{2ft}$; the expression of e'^2 will contain the
[1114vii] term $(P'^2+Q'^2)\,.\,c^{2ft}$, and so on; the first member of the equation (u)
[1114] will therefore contain the term

[1115] $$\{(P^2+Q^2)\,.\,m\,.\sqrt{a}+(P'^2+Q'^2)\,.\,m'\,.\sqrt{a'}+(P''^2+Q''^2)\,.\,m''\,.\sqrt{a''}+\&c.\}\,.\,c^{2ft}.$$

If we suppose c^{ft} to be the greatest of the exponentials contained in h, l,
h', l', &c.; or, in other words, that one in which f is the greatest; c^{2ft} will
[1115'] be the greatest of the exponential quantities contained in the first member
of the equation [1114]; the preceding term [1115] cannot therefore be
balanced by any other term of the first member, and to render this first
member constant, it is necessary that the coefficient of c^{2ft} should be
nothing; hence we get

[1116] $$0=(P^2+Q^2)\,.\,m\,.\sqrt{a}+(P'^2+Q'^2)\,.\,m'\,.\sqrt{a'}+(P''^2+Q''^2)\,.\,m''\,.\sqrt{a''}+\&c.$$

When $\sqrt{a}$, $\sqrt{a'}$, $\sqrt{a''}$, &c., have the same sign; or, which is the same thing,
when the bodies m, m', m'', &c., revolve in the same direction [1094']; this
[1116'] equation will be impossible, except we suppose† $P=0$, $Q=0$, $P'=0$,

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* (763) As an example of the productions of real exponential quantities, we may observe that the expressions $\sin.g\,t$, $\cos.g\,t$, [11, 12] Int., depend on exponentials of the form $c^{\pm g\,t\,.\sqrt{-1}}$, and if g become imaginary, and equal to $-\gamma\,.\sqrt{-1}$, these exponentials will become real, and of the forms $c^{\mp\gamma\,t}$.

† (763a) In this case, the quantities $\sqrt{a}$, $\sqrt{a'}$, &c., being positive, the sum of all the terms [1116], cannot become nothing, except each term is separately equal to nothing; hence

[1115a] $$P^2+Q^2=0,\qquad P'^2+Q'^2=0,\quad \&c.$$

&c.; hence it follows that the quantities h, l, h', l', &c., contain no exponential quantities, consequently the equation in g contains no imaginary roots. [1116'']

If the equations [1114] have some equal roots, the expressions of h, l, h', l', &c., would contain arcs of a circle, as is well known; and we should have, in the expression of h, a finite number of terms of the form* $P\,.\,t^r$. [1116'''] Let $Q\,.\,t^r$, $P'\,.\,t^r$, $Q'\,.\,t^r$, &c., be the corresponding terms of l, h', l', &c.,

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Now P, Q, being supposed to be real quantities, [1114vi], P^2, Q^2, must be affirmative, their sum P^2+Q^2 cannot therefore become nothing, unless we have separately $P=0$, $Q=0$. In like manner $P'=0$, $Q'=0$, &c.

* (764) By [1102],

$$h=\mathcal{N}\,.\,\sin.(g\,t+\beta)+\mathcal{N}_1\,.\,\sin.(g_1\,t+\beta_1)+\&c.$$

$$=\mathcal{N}\,.\,(\sin.g\,t\,.\cos.\beta+\cos.g\,t\,.\sin.\beta)+\mathcal{N}_1\,.\,(\sin.g_1\,t\,.\cos.\beta_1+\cos.g_1\,t\,.\sin.\beta_1)+\&c.$$

[21] Int., and if any number of the roots g, g_1, &c., be supposed equal, the part of h depending on these roots will be

$$(\mathcal{N}\,.\cos.\beta+\mathcal{N}_1\,.\cos.\beta_1+\&c.)\,.\sin.g\,t+(\mathcal{N}\,.\sin.\beta+\mathcal{N}_1\,.\sin.\beta_1+\&c.)\,.\cos.g\,t\,;$$

which, by putting

$$\mathcal{N}\,.\cos.\beta+\mathcal{N}_1\,.\cos.\beta_1+\&c.=\nu\,.\cos.b,$$

$$\mathcal{N}\,.\sin.\beta+\mathcal{N}_1\,.\sin.\beta_1+\&c.=\nu\,.\sin.b,$$

will become $\nu\,.(\cos.b\,.\sin.g\,t+\sin.b\,.\cos.g\,t)=\nu\,.\sin.(g\,t+b)$; and whatever be the number of equal roots, the terms of h depending on them, may be thus reduced to one expression, containing only *two* arbitrary constant quantities, ν, b, instead of $\mathcal{N}$, $\mathcal{N}_1$, &c., β, β_1, &c. The expression of h will not therefore contain the requisite number of arbitrary quantities to render it the complete integral, but they may be obtained by putting $g_1=g+\alpha_1$, $g_2=g+\alpha_2$, &c., in [1102, 1102a], developing the quantities according to the powers of α_1, α_2, &c., changing the constant quantities so as to retain the requisite number, and afterwards putting $\alpha_1=0$, $\alpha_2=0$, &c. In this manner we shall get, by using [21] Int., and putting $\mathcal{N}_1\,.\cos.\beta_1=n_1$, $\mathcal{N}_1\,.\sin.\beta_1=m_1$,

$$\mathcal{N}_1\,.\sin.(g_1t+\beta_1)=\mathcal{N}_1\,.\,\{\sin.g_1\,t\,.\cos.\beta_1+\cos.g_1\,t\,.\sin.\beta_1\}=n_1\,.\sin.g_1\,t+m_1\,.\cos.g_1\,t$$

$$=n_1\,.\sin.(g\,t+\alpha_1\,t)+m_1\,.\cos.(g\,t+\alpha_1\,t),$$

and by [21, 23] Int.

$$=n_1\,.\,\{\sin.g\,t\,.\cos.\alpha_1\,t+\cos.g\,t\,.\sin.\alpha_1\,t\}+m_1\,.\,\{\cos.g\,t\,.\cos.\alpha_1\,t-\sin.g\,t\,.\sin.\alpha_1\,t)$$

$$=\{n_1\,.\cos.\alpha_1\,t-m_1\,.\sin.\alpha_1\,t\}\,.\sin.g\,t+\{m_1\,.\cos.\alpha_1\,t+n_1\,.\sin.\alpha_1\,t\}\,.\cos.g\,t,$$

[1116^{iv}] P, Q, P', Q', &c., being real quantities; the first member of the equation (u) [1114] would contain the term

[1117] $$\{(P^2+Q^2).m.\sqrt{a}+(P'^2+Q'^2).m'.\sqrt{a'}+(P''^2+Q''^2).m''.\sqrt{a''}+\&c.\}.t^{2r}.$$

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substituting the values of $\sin.\alpha_1 t$, $\cos.\alpha_1 t$, in series, like those at the bottom of page 487, we shall get

[1117a] $$\begin{aligned}N_1.\sin.(g_1 t+\beta_1)&=(n_1-m_1\alpha_1 t-\tfrac{1}{2}n_1\alpha_1^2 t^2+\&c.).\sin.g t\\&\quad+\{m_1+n_1\alpha_1 t-\tfrac{1}{2}m_1\alpha_1^2t^2-\&c.\}.\cos.g t.\end{aligned}$$

In like manner, by putting $N.\cos.\beta=n$, $N.\sin.\beta=m$, $N_2.\cos.\beta_2=n_2$, $N_2.\sin.\beta_2=m_2$, &c., we shall find

[1117b] $$\begin{aligned}&N.\sin.(g t+\beta)=n.\sin.g t+m.\cos.g t,\\&N_2.\sin.(g_2 t+\beta_2)=\{n_2-m_2\alpha_2 t-\tfrac{1}{2}n_2\alpha_2^2t^2+\&c.\}.\sin.g t\\&\qquad\qquad+\{m_2+n_2\alpha_2 t-\tfrac{1}{2}m_2\alpha_2^2t^2-\&c.\}.\cos.g t,\\&\&c.\end{aligned}$$

Taking the sum of the expressions [1117a, b], and putting $A_0=n+n_1+n_2+\&c.$,

$$\begin{aligned}&A_1=-m_1\alpha_1-m_2\alpha_2-\&c., &&A_2=-\tfrac{1}{2}n_1\alpha_1^2-\tfrac{1}{2}n_2\alpha_2^2-\&c.,\\&B=m+m_1+m_2+\&c., &&B_1=n_1\alpha_1+n_2\alpha_2+\&c.,\end{aligned}$$
$$B_2=-\tfrac{1}{2}m_1\alpha_1^2-\tfrac{1}{2}m_2\alpha_2^2-\&c.,$$

we shall get,

[1117c] $$\begin{aligned}&N.\sin.(g t+\beta)+N_1.\sin.(g_1 t+\beta_1)+N_2.\sin.(g_2 t+\beta_2)+\&c.\\&=(A_0+A_1 t+A_2t^2+\&c.).\sin.g t+(B_0+B_1 t+B_2t^2+\&c.).\cos.g t,\end{aligned}$$

in which the arbitrary constant quantities N, N_1, &c., β, β_1, &c., are replaced by the *same number* of arbitrary quantities, A_0, A_1, &c., B_0, B_1, &c. Thus if there were *three* equal roots g, g_1, g_2, there would be *six* arbitrary quantities N, N_1, N_2, β, β_1, β_2, depending on them, which would produce *six* arbitrary quantities n, n_1, n_2, m, m_1, m_2, whose places are supplied in the last expression [1117c], by the *six* arbitrary quantities, A_0, A_1, A_2, B_0, B_1, B_2, the terms A_3, A_4, &c., being supposed to vanish by putting $\alpha_1=0$, $\alpha_2=0$, &c. This expression [1117c], being substituted in h, [1102], instead of the first terms of its value, corresponding to the equal roots g, g_1, &c., will give the complete integral with the requisite number of arbitrary quantities, connected with angles and arcs of a circle which are not reducible to a more simple form, and which will be found to satisfy the original differential equations. Similar remarks may be made relative to the equal roots in the value of l, [1102a], and it is evident that if the term t^r occurs without the signs of sine and cosine in the value of h, it may also occur in those of l, h', l', &c., combined with the coefficients P, Q, &c., as above, which may therefore produce in the value of e^2, the terms mentioned in [1117].

If t^r be the highest power of t, which the values of h, l, h', l', &c., contain;
t^{2r} would be the highest power of t contained in the first member of the
equation (u) [1114]; to reduce this first member to a constant quantity, it [1117']
would therefore be necessary to put

$$0=(P^2+Q^2)\cdot m\cdot\sqrt{a}+(P'^2+Q'^2)\cdot m'\cdot\sqrt{a'}+\&c.;$$ [1118]

which would give* $P=0$, $Q=0$, $P'=0$, $Q'=0$, &c. The expressions [1118']
of h, l, h', l', &c., do not therefore contain exponential quantities, or arcs of
a circle; consequently all the roots of the equation in g are real and [1118'']
unequal.

Stability of the system.

The system of the orbits m, m', m'', &c., is therefore perfectly stable, as it
respects the excentricities. The ellipticities of the orbits oscillate about
their mean values, from which they vary but little, while the transverse axes [1118''']
remain invariable. These excentricities are always subject to this condition,
that the sum of their squares, multiplied respectively by the masses of the [1118iv]
bodies, and by the square roots of the transverse axes, is always constant.†

58. When e and ϖ have been found in the preceding manner, we must
substitute their values in all the terms of the expressions of r and $\frac{dv}{dt}$, given
in the preceding articles, neglecting the terms which contain the time t, [1118v]
without the signs of *sine* and *cosine*. The elliptical part of these expressions
will be the same as when the orbit is not troubled, excepting only that the
excentricity, and the position of the perihelion, will be variable; but the [1118vi]
periods of these variations being very long, on account of the smallness of
the masses m, m', m'', &c., in comparison with M; we may suppose these
variations to be proportional to the time, during a very great interval, which,
as it regards the planets, may be extended to several centuries before and
after the time selected for the epoch. It is useful, for astronomical purposes, [1118vii]
to have the secular variations of the excentricities and the perihelia of the
orbits expressed in this manner; we may easily obtain them from the

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* (764a) This follows from [1118], by reasoning as in [1115a].

† (765) This is to be understood to take place with the restrictions mentioned in note 762.

preceding formulas. For the equation $e^2=h^2+l^2$ [1108], gives
[1118viii] $e\,d\,e=h\,d\,h+l\,d\,l$; and noticing only the action of m', we shall have, by § 55 [1072],

[1119]
$$\frac{d\,h}{d\,t}=(0,1)\,.\,l-\boxed{0,1}\,.\,l'\,;$$
$$\frac{d\,l}{d\,t}=-(0,1).h+\boxed{0,1}.h'\,;$$

therefore*

[1120]
$$\frac{e\,d\,e}{d\,t}=\boxed{0,1}\,.\,\{h'\,l-h\,l'\}\,;$$

[1120'] but we have $h'\,l-h\,l'=e\,e'\,.\,\sin.(\varpi'-\varpi)$, therefore we shall get

[1121]
$$\frac{d\,e}{d\,t}=\boxed{0,1}\,.\,e'\,.\,\sin.(\varpi'-\varpi)\,;$$

hence by noticing the mutual action of the bodies m, m', m'', &c., we shall find

[1122]
$$\frac{d\,e}{d\,t}=\boxed{0,1}\,.\,e'\,.\,\sin.(\varpi'-\varpi)+\boxed{0,2}\,.\,e''\,.\,\sin.(\varpi''-\varpi)+\&c.\,;$$
$$\frac{d\,e'}{d\,t}=\boxed{1,0}\,.\,e\,.\,\sin.(\varpi-\varpi')+\boxed{1,2}\,.\,e''\,.\,\sin.(\varpi''-\varpi')+\&c.\,;$$
$$\frac{d\,e''}{d\,t}=\boxed{2,0}\,.\,e\,.\,\sin.(\varpi-\varpi'')+\boxed{2,1}\,.\,e'\,.\,\sin.(\varpi'-\varpi'')+\&c.\,;$$
&c.

The equation $\operatorname{tang.}\varpi=\frac{h}{l}$ [1108] gives, by taking its differential,†

[1123]
$$e^2\,.\,d\,\varpi=l\,.\,d\,h-h\,.\,d\,l.$$

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* (766) The differential of the first equation [1108] gives $e\,d\,e=h\,d\,h+l\,d\,l$. Dividing this by $d\,t$, and substituting the values of $\frac{d\,h}{d\,t}$, $\frac{d\,l}{d\,t}$, [1119], it becomes as in [1120]. The values of h, l, h', l', [1022] being substituted in $h'\,l-h\,l'$, it becomes

$$h'\,l-h\,l'=e\,e'\,.\,\{\sin.\varpi'\,.\,\cos.\varpi-\sin.\varpi\,.\,\cos.\varpi'\}=e\,e'\,.\sin.(\varpi'-\varpi),$$

[22] Int. Substituting this in [1120] and dividing by e we get [1121], from which the expressions [1122] are easily deduced by generalization. In these last equations terms of the order $m\,e^3$ are neglected.

† (767) The differential of the equation $\operatorname{tang.}\varpi=\frac{h}{l}$, [1108], is $\frac{d\varpi}{\cos.\varpi^2}=\frac{l\,d\,h-h\,d\,l}{l^2}$. Multiplying this by $e^2\,.\cos.\varpi^2=l^2$, [1022], we obtain $e^2\,.\,d\varpi=l\,d\,h-h\,d\,l$; [1123].

If we notice only the action of m', and substitute for dh and dl their values, we shall have

$$\frac{e^2 . d\varpi}{dt} = (0,1).(h^2+l^2) - \boxed{0,1}.(hh'+ll') \,; \qquad [1124]$$

which gives

$$\frac{d\varpi}{dt} = (0,1) - \boxed{0,1}.\frac{e'}{e}.\cos.(\varpi'-\varpi) \,; \qquad [1125]$$

therefore we shall have, by means of the mutual action of the bodies m, m', &c.,

$$\frac{d\varpi}{dt} = (0,1)+(0,2)+\&c. - \boxed{0,1}.\frac{e'}{e}.\cos.(\varpi'-\varpi) - \boxed{0,2}.\frac{e''}{e}.\cos.(\varpi''-\varpi) - \&c. \,;$$

$$\frac{d\varpi'}{dt} = (1,0)+(1,2)+\&c. - \boxed{1,0}.\frac{e}{e'}.\cos.(\varpi-\varpi') - \boxed{1,2}.\frac{e''}{e'}.\cos.(\varpi''-\varpi') - \&c. \,; \qquad [1126]$$

$$\frac{d\varpi''}{dt} = (2,0)+(2,1)+\&c. - \boxed{2,0}.\frac{e}{e''}.\cos.(\varpi-\varpi'') - \boxed{2,1}.\frac{e'}{e''}.\cos.(\varpi'-\varpi'') - \&c. \,;$$

&c.

If we multiply these values of $\frac{de}{dt}$, $\frac{de'}{dt}$, &c., $\frac{d\varpi}{dt}$, $\frac{d\varpi'}{dt}$, &c., by the time t, we shall have the differential expressions of the secular variations of the excentricities, and of the perihelia; and these expressions, which are [1126'] rigorously exact only when t is infinitely small, may however serve for a long interval as it respects the planets. If we compare these expressions [1126''] with accurate observations, made at distant intervals, we shall obtain in the most correct manner, the masses of the planets which have no satellites. We shall have, at any time t, the excentricity e equal to

$$e + t.\frac{de}{dt} + \frac{t^2}{1.2}.\frac{dde}{dt^2} + \&c. \,; \qquad [1126''']$$

e, $\frac{de}{dt}$, $\frac{dde}{dt^2}$, &c., being the values corresponding to the origin of the time t,

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Dividing this by dt, and substituting $\frac{dh}{dt}$, $\frac{dl}{dt}$, [1119], we get [1124]. Now from [1022] we have $hh+ll=ee$,

$$hh'+ll' = ee'.\{\cos.\varpi'.\cos.\varpi + \sin.\varpi'.\sin.\varpi\} = ee'.\cos.(\varpi'-\varpi),$$

[24] Int. Substituting these in [1124] we get [1125], and by generalization [1126].

or the epoch.* The differentials of the preceding value of $\frac{de}{dt}$, will give [1126$^{\rm iv}$] those of $\frac{dde}{dt^2}$, $\frac{d^3e}{dt^3}$, &c., observing that a, a', &c., are constant; we can therefore continue the preceding series as far as may be necessary, and by the same process we may obtain the series in ϖ; but with respect to [1126$^{\rm v}$] the planets, it will be sufficient, in the most ancient observations which have been transmitted to us, to notice the square of the time, in any of the expressions of e, e', &c., ϖ, ϖ', &c., in a series.

59. We shall now consider the equations, relative to the positions of the orbits; and shall resume, for this purpose, the equations (3) and (4) § 53 [1050],

[1127]
$$\frac{dp}{dt} = -\frac{m'n}{4} \cdot a^2a' \cdot B^{(1)} \cdot (q-q');$$
$$\frac{dq}{dt} = \frac{m'n}{4} \cdot a^2a' \cdot B^{(1)} \cdot (p-p').$$

Now by § 49 [1006], we have†

[1128]
$$a^2a' \cdot B^{(1)} = \alpha^2 \cdot b_{\frac{3}{2}}^{(1)};$$

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* (768) This value of the excentricity is easily deduced from the formula [617], by supposing the value of e, corresponding to the time t, to be $\varphi(t)$, and that corresponding to the time $t+\alpha$ to be in general, $e+\alpha \cdot \frac{de}{dt} + \frac{\alpha^2}{1.2} \cdot \frac{d^2e}{dt^2} + \&c.$, as in [1126$'''$], the time after the epoch being changed from α to t. In finding the differentials $\frac{de}{dt}$, $\frac{d^2e}{dt^2}$, &c., quantities a, a', &c., are constant, also $(0,1)$, $(0,2)$, &c., $\boxed{0,1}$, $\boxed{0,2}$, &c., which depend on a, a', &c., [1079, 1083]. In the equations [1122], terms multiplied by e^3, e'^3, &c., are neglected, or, as it may be generally expressed, terms of the order $m'e'^3$; and when we take the differentials to find $\frac{d^2e}{dt^2}$, &c., the neglected terms will be of the order $m'e'^2 \cdot \frac{de'}{dt}$. But $\frac{de'}{dt}$, [1122], is of the order $m'e'$, therefore the neglected terms will be of the order $m'^2e'^3$. Hence we perceive that in the coefficients of the terms t, t^2, &c., in the general value of the excentricity, there are terms neglected, which are to those retained, in the same coefficient, as the squares of the excentricities e, e', &c., to unity.

† (769) Putting $i=1$, in [1006], multiplying by $\frac{m'n}{4} \cdot a^2a'$, and substituting $\alpha^2 = \frac{a^2}{a'^2}$, [963$^{\rm iv}$], we shall get $\frac{m'n}{4} \cdot a^2a' \cdot B^{(1)} = \frac{m'n}{4} \cdot \alpha^2 b_{\frac{3}{2}}^{(1)}$, and by using $b_{\frac{3}{2}}^{(1)}$, [992],

and by the same article [992]

$$b_{\frac{3}{2}}^{(1)} = -\frac{3\,b_{-\frac{1}{2}}^{(1)}}{(1-\alpha^2)^2}\,;$$ [1129]

therefore we shall have

$$\frac{m'n}{4}\,.\,a^2\,a'\,.\,B^{(1)} = -\frac{3\,m'\,.\,n\,.\,\alpha^2\,.\,b_{-\frac{1}{2}}^{(1)}}{4\,.\,(1-\alpha^2)^2} = (0,1).$$ [1130]

The second member of this equation is what we have denoted by $(0,1)$ in § 55 [1076]; therefore we shall have

$$\frac{dp}{dt} = (0,1)\,.\,(q'-q)\,;$$
$$\frac{dq}{dt} = (0,1)\,.\,(p-p').$$ [1131]

Hence it follows, that the values of q, p, q', p', &c., will be determined by this system of differential equations,

Differential equations for finding the positions of the orbits.

$$\left.\begin{array}{l}
\frac{dq}{dt} = \{(0,1)+(0,2)+\&c.\}\,.\,p-(0,1)\,.\,p'-(0,2)\,.\,p''-\&c. \\
\frac{dp}{dt} = -\{(0,1)+(0,2)+\&c.\}\,.\,q+(0,1)\,.\,q'+(0,2)\,.\,q''+\&c. \\
\frac{dq'}{dt} = \{(1,0)+(1,2)+\&c.\}\,.\,p'-(1,0)\,.\,p-(1,2)\,.\,p''-\&c. \\
\frac{dp'}{dt} = -\{(1,0)+(1,2)+\&c.\}\,.\,q'+(1,0)\,.\,q+(1,2)\,.\,q''+\&c. \\
\frac{dq''}{dt} = \{(2,0)+(2,1)+\&c.\}\,.\,p''-(2,0)\,.\,p-(2,1)\,.\,p'-\&c. \\
\frac{dp''}{dt} = -\{(2,0)+(2,1)+\&c.\}\,.\,q''+(2,0)\,.\,q+(2,1)\,.\,q'+\&c. \\
\&c.
\end{array}\right\}\,. \qquad (C)$$ [1132]

This system of equations is similar to the system (A) § 55 [1089], and they would coincide wholly, if, in the equations (A), we should change h, l, h', l', &c., into q, p, q', p', &c.; and should also suppose $\boxed{0,1} = (0,1)$, $\boxed{1,0} = (1,0)$, &c.; so that the analysis, used in § 56, to integrate the equations [1089], may be applied to the equations [1132]. Therefore we shall suppose [1132']

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it will become as in [1130], being the same as the function $(0,1)$, [1076]. Substituting this in the equations [1127], they will become as in [1131], and, by generalization, we shall get the equation [1132].

Solution of the above system of equations of the first degree.

[1133]

$$q = N.\cos.(gt+\beta) + N_1.\cos.(g_1t+\beta_1) + N_2.\cos.(g_2t+\beta_2) + \&c.;$$
$$p = N.\sin.(gt+\beta) + N_1.\sin.(g_1t+\beta_1) + N_2.\sin.(g_2t+\beta_2) + \&c.;$$
$$q' = N'.\cos.(gt+\beta) + N_1'.\cos.(g_1t+\beta_1) + N_2'.\cos.(g_2t+\beta_2) + \&c.;$$
$$p' = N'.\sin.(gt+\beta) + N_1'.\sin.(g_1t+\beta_1) + N_2'.\sin.(g_2t+\beta_2) + \&c.;$$
$$\&c.;$$

[1133'] and we shall have, by § 56, an equation in g, of the degree i, whose roots
are g, g_1, g_2, &c.* It is easy to perceive that one of these roots is nothing;
for it is evident that we may satisfy the equations [1132], by supposing
[1133''] p, p', p'', &c., to be equal and constant, and also q, q', q'', &c.; this requires
that one of the roots of the equation in g should be nothing;† which reduces
it to the degree $i-1$. The arbitrary quantities N, N_1, N', &c., β, β_1, &c.,
[1133'''] may be determined by the method explained in § 56. Lastly, we shall find,
by the analysis of § 57,‡

[1134]
$$\text{constant} = (p^2+q^2).m.\sqrt{a} + (p'^2+q'^2).m'.\sqrt{a'} + \&c;$$

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* (770) These roots g, g_1, g_2, &c., as well as the values of N, N_1, N_2, &c., β, β', β'', &c., are different from those corresponding to the equations [1102, 1102a], but the form and manner of computation are the same.

† (771) If $p=p'=p''$, &c., $q=q'=q''$, &c., the second members of the equations [1132] will become nothing; and if all these quantities are constant, we shall have

$$0 = dp = dp' = dp'', \&c., \qquad \text{and} \qquad 0 = dq = dq' = dq'', \&c.,$$

which will render the first members of the same equations nothing; so that the equations [1132] will be satisfied, by the assumed values of p, p', &c., q, q', &c.,; and these values correspond to the supposition that one of the roots, as g, is equal to nothing, for then the parts of q, p, &c., [1133], depending on the angle $gt+\beta$, will become $N.\cos.\beta$, $N.\sin.\beta$, &c., which are constant.

‡ (772) Multiplying the first of the equations [1132] by $2m.\sqrt{a}.q$, the second by $2m.\sqrt{a}.p$, the third by $2m'.\sqrt{a'}.q'$, &c., adding the products, and making the reductions, similar to those used in computing [1112], we shall find that the sum becomes nothing, and its integral is as in [1134]. In the second member, terms of the order $m.\sqrt{a}.p^4$, &c., are neglected, as will also more evidently appear from another demonstration given in [1151—1155].

hence we may conclude, as in the article just mentioned [1114—1118''], that the expressions of p, q, p', q', &c., contain neither arcs of a circle, nor [1134'] exponential quantities, when the bodies revolve in the same direction; therefore the equation in g has all its roots real and unequal.

We may obtain two more integrals of the equations (C) [1132]. For if we multiply the first of these equations by $m.\sqrt{a}$, the third by $m'.\sqrt{a'}$, the [1134''] fifth by $m''.\sqrt{a''}$, &c.; we shall have, by means of the relations found in § 55,*

$$0=\frac{dq}{dt}\cdot m\cdot\sqrt{a}+\frac{dq'}{dt}\cdot m'\cdot\sqrt{a'}+\&c.; \qquad [1135]$$

which gives, by integration,

$$\text{constant}=q\cdot m\cdot\sqrt{a}+q'\ m'\cdot\sqrt{a'}+\&c.; \qquad (1) \qquad [1136]$$

and in like manner

$$\text{constant}=p\cdot m\ \sqrt{a}+p'\cdot m'\cdot\sqrt{a'}+\&c. \qquad (2) \qquad [1137]$$

If we put φ for the inclination of the orbit of m upon the fixed plane, and [1137'] θ for the longitude of the ascending node of this orbit, upon the same plane; the latitude of m will be nearly† $\text{tang.}\,\varphi\,.\,\text{sin.}\,(nt+\varepsilon-\theta)$. Comparing this [1137''] with the following, $q.\text{sin.}(nt+\varepsilon)-p.\text{cos.}(nt+\varepsilon)$, we shall get

$$p=\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta; \qquad q=\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta; \qquad [1138]$$

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* (773) The first member of the sum is evidently equal to the second member of [1135], and the second member of this sum must be equal to nothing. For the terms depending on (0, 1), and (1, 0), in this sum are $(p-p').\{m.\sqrt{a}.(0,1)-m'.\sqrt{a'}.(1,0)\}$, which by means of the equation [1093], become nothing. In the same manner the terms depending on (0, 2), (2, 0), &c., are nothing, hence it follows that the whole sum is equal to nothing. The integral of the equation [1135] is [1136]. Again, multiplying the second, fourth, &c., of the equations [1132], by $m.\sqrt{a}$, $m'.\sqrt{a'}$, &c., respectively, and taking the sum of all the products, it will be nothing, and its integral will be as in [1137]. [1136a] In both these equations, terms of the order $p^3 m.\sqrt{a}$ are neglected, as will evidently appear in [1158, &c.]

† (774) This expression is the same as that of tang. FG, [1030a], or FG, nearly, and if we develop it, with respect to θ, by [22] Int., it becomes

$$(\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta)\,.\,\text{sin.}\,(nt+\varepsilon)-(\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta)\,.\,\text{cos.}\,(nt+\varepsilon).$$

This expresses the part of the latitude depending on the angle $nt+\varepsilon$, which in the value

hence we deduce

[1139] $$\text{tang.}\,\varphi = \sqrt{p^2+q^2}\,; \qquad \text{tang.}\,\theta = \frac{p}{q}\,;$$

we shall therefore have the inclination of the orbit of m, and the position of its node, by means of the values of p and q. Marking the values of tang. φ,
[1139′] tang. θ, successively with one accent, two accents, &c., for the bodies m', m'', &c., we shall obtain the inclinations of the orbits of m', m'', &c., and the positions of their nodes, by means of the quantities p', q', p'', q'', &c.

The quantity $\sqrt{p^2+q^2}$ is less than the sum $N+N_1+N_2+$&c., of the
[1139″] coefficients of the sines of the expression of q;* and as these coefficients are very small, since the orbits are supposed to be but little inclined to the fixed plane, its inclination to this fixed plane will always be very small; hence it
[1139‴] follows, that the system of the orbits will also be permanent, relative to their inclinations,† as it is with regard to their excentricities. We may therefore consider the inclinations of the orbits as variable quantities, comprised between fixed limits, and the motions of the nodes as not being uniform. These
[1139iv] variations are very sensible in the satellites of Jupiter; and we shall see hereafter, [Book viii, § 30], that they account for the singular phenomena,
[1139v] observed in the inclination of the orbit of the fourth satellite.

From the preceding expressions of p and q, we obtain the following theorem:

Let there be a circle whose inclination to the fixed plane is N, and the longitude of its ascending node $gt+\beta$; upon this *first* circle, let there be

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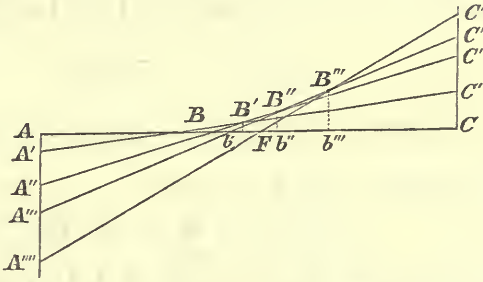
[1137a] of s, [1039], is put equal to $q\,.\,\text{sin.}\,(nt+\varepsilon)-p\,.\,\text{cos.}\,(nt+\varepsilon)$. Comparing these two expressions we get $p=\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta$, $q=\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta$, as in [1138]. The square root of the sum of the squares of p, q, is $\text{tang.}\,\varphi=\sqrt{p^2+q^2}$, and the value of p, divided by that of q, gives $\text{tang.}\,\theta=\frac{p}{q}$, as in [1139].

* (775) The sum of the squares of the values of p, q, [1133], gives for p^2+q^2, an expression precisely similar to that of e^2 or h^2+l^2 in [1109], hence we may prove, as in [1109′], that $\sqrt{p^2+q^2}$, is less than the sum $N+N_1+$&c., considering all the quantities N, N_1, &c., as positive.

† (776) This is liable to the same restrictions as in note 762.

placed a *second* circle, inclined to the first by the angle N_1, so that the [1139$^{\text{vi}}$] longitude of its intersection with the first circle may be $g_1t+\beta_1$; upon this second circle suppose a *third* to be placed, and inclined to the second by the angle N_2, the longitude of its intersection with the second circle being $g_2t+\beta_2$, and so on for others; the position of the last circle will be the orbit of m.*



* (777) Let ABC be the great circle in the heavens, corresponding to the fixed plane, A the point from which the angles $gt+\beta$, $g_1t+\beta_1$, &c., are counted, $A'B'C'$ the first circle, $A''B''C''$ the second circle, &c., B the intersection of the first circle and the fixed plane, B' the intersection of the first and second circles, B'' that of the second and third, B''' that of the third and fourth, &c.; and taking the arch AC equal to a right angle, or $\frac{1}{2}\pi$, we shall draw perpendicular to it, the arches AA'''', $B'b'$, $B''b''$, $B'''b'''$, &c., CC''''. Then by construction, $AB=gt+\beta$, $Ab'=g_1t+\beta_1$, $Ab''=g_2t+\beta_2$, &c., and the angles $ABA'=CBC'=N$, $A'B'A''=C'B'C''=N_1$, &c.; these angles being very small, we shall have nearly $A'B'=Ab'=g_1t+\beta_1$, $A''B''=Ab''=g_2t+\beta_2$, &c.; also $BC=\frac{1}{2}\pi-(gt+\beta)$, $B'C'=b'C=\frac{1}{2}\pi-(g_1t+\beta_1)$, &c. In the spherical triangle BAA', we shall have $\text{tang.}\,AA'=\text{tang.}\,ABA'\,.\,\text{sin.}\,AB$, which on account of the smallness of the arch AA', and the angle ABA', is nearly

$$AA'=ABA'\times\text{sin.}\,AB=N\,.\,\text{sin.}\,(gt+\beta).$$

In like manner, as the spherical triangle $B'A'A''$ is nearly right angled in A', we shall have $A'A''=A'B'A''\,.\,\text{sin.}\,A'B'=N_1\,.\,\text{sin.}\,(g_1t+\beta_1)$, and in the triangle $A''B''A'''$ we shall have $A''A'''=A''B''A'''\,.\,\text{sin.}\,A''B''=N_2\,.\,\text{sin.}\,(g_2t+\beta_2)$, &c. The sum of all these arches gives the value AA'''', corresponding to the last of the circles, which, in the present figure is $A''''FC''''$; and comparing this sum, with the value of p, [1133], we shall find $AA''''=p$. Again the spherical triangle CBC' gives nearly

$$CC'=CBC'\,.\,\text{sin.}\,BC=N\,.\,\text{cos.}\,(gt+\beta);$$

Applying the same construction to the expressions of h and l, § 56
[1139vii] [1102, 1102*a*], we shall find that the tangent of the inclination of the last circle to the fixed plane, will be equal to the excentricity of the orbit of m; and the longitude of the intersection of this circle with the same plane, will be equal to the longitude of the perihelion of the orbit of m.*

60. It is useful for astronomical purposes, to have the differential variations of the nodes and inclinations of the orbits. For this purpose, we shall resume the equations [1139] of the preceding article,

[1140] $$\text{tang.}\,\varphi=\sqrt{p^2+q^2}\,; \qquad \text{tang.}\,\theta=\frac{p}{q}.$$

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the triangle $C'\,B'\,C''$ gives

$$C'\,C''=C'\,B'\,C''\,.\,\text{sin.}\,B'\,C'=N_1\,.\,\text{cos.}\,(g_1\,t+\beta_1), \qquad \text{\&c.};$$

the sum of all these is the value of $C\,C''''$, which, being compared with q [1133], gives $C\,C''''=q$. This last arch, $A''''\,F\,C''''$, being supposed to intersect the fixed plane in F, we shall put $A\,F=\theta$, $F\,C=\frac{1}{2}\,\pi-\theta$, and the angle $A\,F\,A''''=\varphi=C\,F\,C''''$. The spherical angle $A\,F\,A''''$ gives nearly

$$A\,A''''=\text{tang.}\,A\,F\,A''''\,.\,\text{sin.}\,A\,F=\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta,$$

and the triangle $C\,F\,C''''$ gives

$$C\,C''''=\text{tang.}\,C\,F\,C''''\,.\,\text{sin.}\,F\,C=\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta\,;$$

substituting the preceding values of $A\,A''''$ $C\,C''''$, we shall get $p=\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta$, $q=\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta$, and as these are the same as the equations [1138], it follows that the angles φ, θ, or $A\,F\,A''''$ and $A\,F$, determined by this method, must be equal to those determined by the equation [1139], that is, they must be equal to the inclination, and the longitude of the node of the orbit of m.

* (778) In this supposition, the sum of the arches $A\,A'$, $A'\,A''$, &c., will be equal to h, [1102], and the sum of $C\,C'$, $C'\,C''$, $C''\,C'''$, &c., equal to l, [1102*a*], instead of p and q, [1133]. Hence, as in the last note, we shall get $h=\text{tang.}\,\varphi\,.\,\text{sin.}\,\theta$, $l=\text{tang.}\,\varphi\,.\,\text{cos.}\,\theta$, the sum of whose squares will give $\sqrt{h^2+l^2}=\text{tang.}\,\varphi$; and the first divided by the second is $\text{tang}\,\theta=\frac{h}{l}$, or by [1108], $e=\text{tang.}\,\varphi$, and $\text{tang.}\,\theta=\text{tang.}\,\varpi$, or $\theta=\varpi$, as in [1139^{vii}].

Taking the differentials, we shall have*

$$d\varphi = dp \,.\, \sin.\theta + dq \,.\, \cos.\theta\,;$$

$$d\theta = \frac{dp \,.\, \cos.\theta - dq \,.\, \sin.\theta}{\text{tang.}\,\varphi}. \qquad [1141]$$

If we substitute the values of dp, dq, given by the equations (C) [1132] of the preceding article, we shall have†

$$\frac{d\varphi}{dt} = (0,1)\,.\,\text{tang.}\,\varphi'\,.\,\sin.(\theta-\theta') + (0,2)\,.\,\text{tang.}\,\varphi''\,.\,\sin.(\theta-\theta'') + \&c.\,;$$

$$\frac{d\theta}{dt} = -\{(0,1)+(0,2)+\&c.\} + (0,1)\,.\,\frac{\text{tang.}\,\varphi'}{\text{tang.}\,\varphi}\,.\,\cos.(\theta-\theta') \qquad [1142]$$

$$+ (0,2)\,.\,\frac{\text{tang.}\,\varphi''}{\text{tang.}\,\varphi}\,.\,\cos.(\theta-\theta'') + \&c.$$

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* (779) The differential of $\text{tang.}\,\varphi = \sqrt{p^2+q^2}$ is $\frac{d\varphi}{\cos.^2\varphi} = \frac{p\,dp+q\,dq}{\sqrt{pp+qq}}$, and from $\text{tang.}\,\theta = \frac{p}{q}$ we get $\cos.\theta = \frac{q}{\sqrt{p^2+q^2}}$, and $\sin.\theta = \frac{p}{\sqrt{p^2+q^2}}$. Substituting these, we find $\frac{d\varphi}{\cos.^2\varphi} = dp\,.\,\sin.\theta + dq\,.\,\cos.\theta$, and neglecting terms of the order φ^2, we may put $\cos.^2\varphi = 1$, and we shall have $d\varphi = dp\,.\,\sin.\theta + dq\,.\,\cos.\theta$, as in [1141]. Again, the differential of $\text{tang.}\,\theta = \frac{p}{q}$, is $\frac{d\theta}{\cos.^2\theta} = \frac{q\,dp - p\,dq}{q^2}$, and the preceding values of $\cos.\theta$, $\sin.\theta$, give $q = \cos.\theta\,.\,\sqrt{p^2+q^2}$, $p = \sin.\theta\,.\,\sqrt{p^2+q^2}$; substituting [1141a] these in the preceding equation, and dividing the numerator and denominator of the second member by $\sqrt{p^2+q^2}$, we shall get $\frac{d\theta}{\cos.^2\theta} = \frac{dp\,.\cos.\theta - dq\,.\sin.\theta}{\cos.^2\theta\,.\,\sqrt{p^2+q^2}}$; substituting for $\sqrt{p^2+q^2}$, its value [1140] and multiplying by $\cos.^2\theta$, we shall get the value of $d\theta$, [1141].

† (780) Substituting in $\frac{d\varphi}{dt} = \frac{dp}{dt}\,.\,\sin.\theta + \frac{dq}{dt}\,.\,\cos.\theta$, [1141], the values $\frac{dp}{dt}$, $\frac{dq}{dt}$, [1132], we shall find

$$\frac{d\varphi}{dt} = \left[\{(0,1)+(0,2)+\&c.\}\,.\,p - (0,1)\,.\,p' - (0,2)\,.\,p'' - \&c.\right].\cos\theta$$

$$+ \left[-\{(0,1)+(0,2)+\&c.\}\,.\,q + (0,1)\,.\,q' + \&c.\right].\sin.\theta,$$

and the coefficient of $(0,1)$ is $p\,.\cos.\theta - p'\,.\cos.\theta - q\,.\sin.\theta + q'\,.\sin.\theta$, which, by using the values of p, q, [1141a], and the similar values $p' = \sin.\theta'\,.\,\sqrt{p'^2+q'^2}$, [1141b] $q' = \cos.\theta'\,.\,\sqrt{p'^2+q'^2}$, becomes

$$\sqrt{p^2+q^2}\,.\,\{\cos.\theta\,.\,\sin.\theta - \cos.\theta\,.\,\sin.\theta\} + \sqrt{p'^2+q'^2}\,.\,\{-\cos.\theta\,.\,\sin.\theta' + \cos.\theta'\,.\,\sin.\theta\},$$

We shall likewise have

$$\frac{d\varphi'}{dt} = (1,0)\,.\,\text{tang.}\,\varphi\,.\,\sin.\,(\theta'-\theta) + (1,2)\,.\,\text{tang.}\,\varphi''\,.\,\sin.\,(\theta'-\theta'') + \&c.\,;$$

[1143]
$$\frac{d\theta'}{dt} = -\{(1,0)+(1,2)+\&c.\} + (1,0)\,.\,\frac{\text{tang.}\,\varphi}{\text{tang.}\,\varphi'}\,.\,\cos.\,(\theta'-\theta)$$
$$+ (1,2)\,.\,\frac{\text{tang.}\,\varphi''}{\text{tang.}\,\varphi'}\,.\,\cos.\,(\theta'-\theta'') + \&c.$$

&c.

[1143'] Astronomers refer the motions of the heavenly bodies to the variable orbit of the earth; for it is in fact from the plane of this orbit that we make our

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or simply $\sqrt{p'^2+q'^2}\,.\,\{-\cos.\,\theta\,.\,\sin.\,\theta' + \cos.\,\theta'\,.\,\sin.\,\theta\} = \sqrt{p'^2+q'^2}\,.\,\sin.\,(\theta-\theta')$,
[22] Int. Now the formulas in p', q', φ', similar to those in [1139] give

$$\sqrt{p'^2+q'^2} = \text{tang.}\,\varphi'\,;$$

therefore the preceding term becomes $(0,1)\,.\,\text{tang.}\,\varphi'\,.\,\sin.\,(\theta-\theta')$. In like manner the term depending on $(0,2)$, is $(0,2)\,.\,\text{tang.}\,\varphi''\,.\,\sin.\,(\theta-\theta'')\,.\,\&c.$ The sum of all these terms is equal to the value of $\frac{d\varphi}{dt}$ in [1142]. Again, by substituting the values of dp, dq, [1132], in $\frac{d\theta}{dt}$, [1141], and multiplying it by tang. φ, we shall get

[1141c]
$$\frac{d\theta}{dt}\,.\,\text{tang.}\,\varphi = -\{(0,1)+(0,2)+\&c.\}\,.\,p\,.\,\sin.\,\theta + (0,1)\,.\,p'\,.\sin.\,\theta + (0,2)\,.\,p''\,.\sin.\,\theta + \&c.$$
$$-\{(0,1)+(0,2)+(0,3)+\&c.\}\,.\,q\,.\,\cos.\,\theta + (0,1)\,.\,q'\,.\,\cos.\,\theta + (0,2)\,.\,q''\,.\cos.\,\theta + \&c.$$

In which the coefficient of $(0,1)$, is $-p\,.\,\sin.\,\theta + p'\,.\,\sin.\,\theta - q\,.\,\cos.\,\theta + q'\,.\,\cos.\,\theta$, and by substituting the values of p, p', q, q', [1141*a*, *b*], it becomes

$$\sqrt{p^2+q^2}\,.\,\{-\sin.^2\theta - \cos.^2\theta\} + \sqrt{p'^2+q'^2}\,.\,\{\sin.\,\theta'\,.\,\sin.\,\theta + \cos.\,\theta\,.\,\cos.\,\theta'\}$$
$$= -\sqrt{p^2+q^2} + \sqrt{p'^2+q'^2}\,.\,\cos.\,(\theta-\theta'),$$

[24] Int. But by [1139] we have $\sqrt{p^2+q^2} = \text{tang.}\,\varphi$, $\sqrt{p'^2+q'^2} = \text{tang.}\,\varphi'$, therefore the term depending on $(0,1)$ will be $-(0,1)\,.\,\text{tang.}\,\varphi + (0,1)\,.\,\text{tang.}\,\varphi'\,.\,\cos.\,(\theta-\theta')$. In like manner the term depending on $(0,2)$ is

$$-(0,2)\,.\,\text{tang.}\,\varphi + (0,2)\,.\,\text{tang.}\,\varphi''\,.\,\cos.\,(\theta-\theta''), \quad \&c.$$

The sum of all these terms, representing the value of $\frac{d\theta}{dt}\,.\,\text{tang.}\,\varphi$, [1141*c*], is the same as the second member of the last of the equations [1142], multiplied by tang. φ. The equations [1143] are easily derived from [1142], by changing the accents, &c.

observations. It is therefore necessary to ascertain the variations of the nodes and of the inclinations of the orbits with respect to the ecliptic. Suppose now that it was required to determine the differential variations of the nodes and of the inclinations of the orbits, referred to the orbit of one of the bodies m, m', m'', &c., as, for example, the orbit of m. It is evident that

$$q\,.\,\sin.\,(n't+\varepsilon')-p\,.\,\cos.\,(n't+\varepsilon') \qquad [1143'']$$

would be the latitude of m' above the fixed plane,* if it moved in the orbit of m. Its real latitude above the same plane, is

$$q'\,.\,\sin.\,(n't+\varepsilon')-p'\,.\,\cos.\,(n't+\varepsilon')\,; \qquad [1143''']$$

and the difference of these two latitudes is nearly the latitude of m' above the orbit of m; putting therefore $\varphi_{\prime}'$ for the inclination, and $\theta_{\prime}'$ for the longitude of the node of the orbit of m', referred to that of m, we shall have, by what precedes, [1143^iv]

$$\text{tang.}\,\varphi_{\prime}'=\sqrt{(p'-p)^2+(q'-q)^2}\,; \qquad \text{tang.}\,\theta_{\prime}'=\frac{p'-p}{q'-q}. \qquad [1144]$$

If we take for the fixed plane that of the orbit of m, at a given epoch, we

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* (781) The general expression of the latitude of m above the fixed plane, is represented in [1137*a*] by $q\,.\,\sin.\,(nt+\varepsilon)-p\,.\,\cos.\,(nt+\varepsilon)$, and by accenting these quantities, we shall obtain the latitude of m', above the same plane, $q'\,.\,\sin.\,(n't+\varepsilon')-p'\,.\,\cos.\,(n't+\varepsilon')$. Now from the first equation it follows, that if a body move, on the plane of m, its latitude corresponding to the longitude $n't+\varepsilon'$, will be $q\,.\,\sin.\,(n't+\varepsilon')-p\,.\,\cos.\,(n't+\varepsilon')$, nearly. Subtracting this from the preceding expression, the remainder will represent, very nearly, the latitude of m', above the orbit of m,

$$(q'-q)\,.\,\sin.\,(n't+\varepsilon')-(p'-p)\,.\,\sin.\,(n't+\varepsilon')\,;$$

and this must be equal to $\text{tang.}\,\varphi_{\prime}'\,.\,\sin.\,(n't+\varepsilon'-\theta_{\prime}')$, which is similar to the expression used in [1137''], changing the accents. Now if we compare these expressions

$$(q'-q)\,.\,\sin.\,(n't+\varepsilon')-(p'-p)\,.\,\sin.\,(n't+\varepsilon') \qquad \text{and} \qquad \text{tang.}\,\varphi_{\prime}'\,.\,\sin.\,(n't+\varepsilon'-\theta_{\prime}')$$

with those in [1137''],

$$q\,.\,\sin.\,(nt+\varepsilon)-p\,.\,\cos.\,(nt+\varepsilon), \qquad \text{and} \qquad \text{tang.}\,\varphi\,.\,\sin.\,(nt+\varepsilon-\theta),$$

we shall find that the two former may be derived from the two latter, by changing p, q, n, ε, φ, θ, into $p'-p$, $q'-q$, n', ε', $\varphi_{\prime}'$, $\theta_{\prime}'$, respectively, and if we make the same changes in [1139], we shall get the equations [1144]. The same changes being made in [1141], we shall obtain the expressions [1145], observing that $\text{tang.}\,\varphi_{\prime}'$ becomes $\text{tang.}\,\varphi'$, if $p=0$, $q=0$.

[1144′] shall have, at this epoch,* $p=0$, $q=0$; but the differentials dp and dq will not vanish, and we shall have,

[1145]
$$d\varphi_{,}' = (dp'-dp)\,.\,\sin.\theta' + (dq'-dq)\,.\,\cos.\theta'\,;$$
$$d\theta_{,}' = \frac{(dp'-dp)\,.\,\cos.\theta' - (dq'-dq)\,.\,\sin.\theta'}{\text{tang.}\,\varphi'}.$$

Substituting for dp, dq, dp', dq', &c., their values given by the equations (C) [1132] of the preceding article, we shall have,†

$$\frac{d\varphi_{,}'}{dt} = \{(1,2)-(0,2)\}\,.\,\text{tang.}\,\varphi''\,.\,\sin.(\theta'-\theta'')$$
$$+\{(1,3)-(0,3)\}\,.\,\text{tang.}\,\varphi'''\,.\,\sin.(\theta'-\theta''') + \&c.\,;$$

[1146]
$$\frac{d\theta_{,}'}{dt} = -\{(1,0)+(1,2)+(1,3)+\&c.\}-(0,1)$$
$$+\{(1,2)-(0,2)\}\,.\,\frac{\text{tang.}\,\varphi''}{\text{tang.}\,\varphi'}\,.\,\cos.(\theta'-\theta'')$$
$$+\{(1,3)-(0,3)\}\,.\,\frac{\text{tang.}\,\varphi'''}{\text{tang.}\,\varphi'}\,.\,\cos.(\theta'-\theta''')+\&c.$$

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* (781*a*) The fixed plane being taken for the primitive orbit of m, we shall have, at the origin of the time, $\varphi=0$, [1137′], and in this case we shall have, by [1138], at that time, $p=0$, $q=0$.

† (782) Putting $p=0$, $q=0$, in the four first equations [1132], and deducing therefrom the values $\frac{dp'-dp}{dt}$, $\frac{dq'-dq}{dt}$, we get

[1146*a*]
$$\frac{dp'-dp}{dt} = -\{(0,1)+(1,0)+(1,2)+(1,3)+\&c.\}\,.\,q' + \{(1,2)-(0,2)\}\,.\,q''$$
$$+\{(1,3)-(0,3)\}\,.\,q''' + \{(1,4)-(0,4)\}\,.\,q'''' + \&c.$$
$$\frac{dq'-dq}{dt} = \{(0,1)+(1,0)+(1,2)+(1,3)+\&c.\}\,.\,p' - \{(1,2)-(0,2)\}\,.\,p''$$
$$-\{(1,3)-(0,3)\}\,.\,p''' - \{(1,4)-(0,4)\}\,.\,p'''' - \&c.$$

Multiplying the first by $\sin.\theta'$, the second by $\cos.\theta'$, and taking the sum of the products we shall get the value of $\frac{d\varphi_{,}'}{dt}$, [1145]. The terms depending on p', q', are

$$\{(0,1)+(1,0)+(1,2)+\&c.\}\,.\,(p'\,.\,\cos.\theta' - q'\,.\,\sin.\theta')\,;$$

[1146*b*] but in [1138, &c.], we have, $p'=\text{tang.}\,\varphi'\,.\,\sin.\theta'$, $q'=\text{tang.}\,\varphi'\,.\,\cos.\theta'$; hence $p'\,.\,\cos.\theta' - q'\,.\,\sin.\theta'$ becomes nothing. The terms depending on p'', q'', are

$$\{(1,2)-(0,2)\}\,.\,(q''\,.\,\sin.\theta' - p''\,.\,\cos.\theta'),$$

It is easy to deduce, from these expressions, the variations of the nodes and the inclinations of the orbits of the other bodies m'', m''', &c., to the variable orbit of m.

61. The integrals of the preceding differential equations, by which the elements of the orbit are determined, are merely approximate values, and the relations which they give, between all these elements, take place only upon [1146'] the supposition, that the excentricities of the orbits and their inclinations are very small. But the integrals [430, 431, 432, 442] which we have obtained in § 9, will give the same relations, whatever be the excentricities and inclinations. To prove this, we shall observe that $\frac{x\,dy-y\,dx}{dt}$ is double the area,* described by the planet m, during the time dt, by the projection [1146'']

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which, by substituting $p''=\text{tang.}\,\varphi''.\sin.\theta''$, $q''=\text{tang.}\,\varphi''.\cos.\theta''$, [1138, &c.], become [1146c]

$$\{(1,2)-(0,2)\}.\text{tang.}\,\varphi''.(\sin.\theta'.\cos.\theta''-\sin.\theta''.\cos.\theta')$$
$$=\{(1,2)-(0,2)\}.\text{tang.}\,\varphi''.\sin.(\theta'-\theta'').$$

In like manner the terms depending on p''', q''' become

$$\{(1,3)-(0,3)\}.\text{tang.}\,\varphi'''.\sin.(\theta'-\theta''');$$

and in like manner for the others, so that the whole expression becomes as in [1146]. Again, multiplying the values of $\frac{dp'-dp}{dt}$, $\frac{dq'-{}'dq}{dt}$, [1146a], by $\cos.\theta'$, and $-\sin.\theta'$, respectively, and adding the products, the sum will be, by the second equation [1145], equal to $d\,\theta'_{,}.\text{tang.}\,\varphi'$, which is therefore equal to

$$-\{(0,1)+(1,0)+(1,2)+\&c.\}.(q'.\cos.\theta'+p'.\sin.\theta')+\{(1,2)-(0,2)\}$$
$$\times(q''.\cos.\theta'+p''.\sin.\theta')+\{(1,3)-(0,3)\}.(q'''.\cos.\theta'+p'''.\sin.\theta')+\&c.$$

If we use the values of p', q', p'', &c., [1146b, c], we shall find,

$$q'.\cos.\theta'+p'.\sin.\theta'=\text{tang.}\,\varphi'.\{\cos.^2\theta'+\sin.^2\theta'\}=\text{tang.}\,\varphi';$$

$$q''.\cos.\theta'+p''.\sin.\theta'=\text{tang.}\,\varphi''.\{\cos.\theta''.\cos.\theta'+\sin.\theta''.\sin.\theta'\}=\text{tang.}\,\varphi''.\cos.(\theta'-\theta'')+\&c.$$

Substituting these in the preceding value of $d\,\varphi'_{,}.\text{tang.}\,\varphi'$, and dividing the whole by $\text{tang.}\,\varphi'$, we shall get $\frac{d\,\theta'_{,}}{dt}$, as in [1146].

* (783) This is proved in [167']. From the first of the equations [572] we have $c=\frac{x\,dy-y\,dx}{dt}$. Substituting the value of $c=\sqrt{\mu a.(1-e^2)}$, [596c], and putting $\mu=1$, which may be done, by neglecting the mass of the planet, in comparison with that of the sun, taken as unity [1013'], we shall obtain the formula [1147].

of its radius vector upon the plane of x, y. In the elliptical motion, if we neglect the mass of the planet in comparison with that of the sun, taken as unity; we shall have, by § 19, 20, relative to the plane of the orbit of m,

[1147] $$\frac{x\,dy-y\,dx}{dt}=\sqrt{a\,.\,(1-e^2)}.$$

To refer to the fixed plane the area described in the orbit, we must multiply it by the cosine of the inclination φ of the orbit to this plane;* therefore we shall have, as it respects this plane,

[1148] $$\frac{x\,dy-y\,dx}{dt}=\cos.\varphi\,.\sqrt{a\,.\,(1-e^2)}=\sqrt{\frac{a\,.(1-e^2)}{1+\operatorname{tang.}^2\varphi}};$$

we shall likewise have

[1149] $$\frac{x'\,dy'-y'\,dx'}{dt}=\sqrt{\frac{a'\,.(1-e'^2)}{1+\operatorname{tang.}^2\varphi'}};$$
$$\&c.$$

[1149′] These values of $x\,dy-y\,dx$, $x'\,dy'-y'\,dx'$, &c., may be used, when we neglect the perturbations of the motions of the planets, provided we suppose the elements e, e', &c., φ, φ', &c., to be variable, in consequence of their secular inequalities; the equation [430] will then become†

[1150] $$c=m\,.\sqrt{\frac{a\,.(1-e^2)}{1+\operatorname{tang.}^2\varphi}}+m'\,.\sqrt{\frac{a'\,.(1-e'^2)}{1+\operatorname{tang.}^2\varphi'}}+\&c.$$
$$+\Sigma\,.\,m\,m'\,.\left\{\frac{(x'-x)\,.\,(d\,y'-d\,y)-(y'-y)\,.\,(d\,x'-d\,x)}{dt}\right\}.$$

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* (784) This is evident from the principles of the orthographic projection, by which any area, in a given plane, being projected upon another plane, is reduced in the proportion of the cosine of the inclination φ of the two planes to the radius. This gives the two first expressions [1148], the third is found, by substituting for $\cos.\varphi$ its value $\frac{1}{\sqrt{1+\operatorname{tang.}^2\varphi}}$. The formula [1149] is deduced from [1148] by merely accenting the letters.

† (785) Putting $M=1$, and using the values of $\frac{x\,dy-y\,dx}{dt}$, $\frac{x'\,dy'-y'\,dx'}{dt}$, &c. [1148, 1149], we shall get from [430] the expression [1150], which is exact. The expression [1151] is exact in *all* terms of the order m', but not in those of the order m'^2. In the other expressions [1153, 1154, 1155], terms of the order $m\,.\sqrt{a}\,.\,e^4$, are neglected.

Neglecting this last term, which is of the order mm', we shall have [1150']

$$c = m \,.\, \sqrt{\frac{a.(1-e^2)}{1+\text{tang.}^2\varphi}} + m' \,.\, \sqrt{\frac{a'.(1-e'^2)}{1+\text{tang.}^2\varphi'}} + \&\text{c.} \qquad [1151]$$

Therefore whatever changes, in the course of time, may be made in the [1151'] values of e, e', &c., φ, φ', &c., by means of the secular variations, these values ought always to satisfy the preceding equation.

If we neglect the very small quantities of the order e^4, or $e^2\varphi^2$, this equation [1151''] will become

$$\begin{aligned} c = {} & m \,.\, \sqrt{a} + m' \,.\, \sqrt{a'} + \&\text{c.} \\ & - \tfrac{1}{2} m \,.\, \sqrt{a} \,.\, \{e^2 + \text{tang.}^2\varphi\} - \tfrac{1}{2} m' \,.\, \sqrt{a'} \,.\, \{e'^2 + \text{tang.}^2\varphi'\} - \&\text{c.}\,; \end{aligned} \qquad [1152]$$

therefore, if we neglect the squares of e, e', φ, &c., we shall have

$$\text{constant} = m \,.\, \sqrt{a} + m' \,.\, \sqrt{a'} + \&\text{c.} \qquad [1152']$$

We have already proved, [1070^{xi}], that if we notice only the first power of the disturbing force, each of the quantities a, a', &c., will be constant; therefore the preceding equation will give, by neglecting the very small [1152''] quantities of the order e^4, or $e^2\varphi^2$,

$$\text{constant} = m \,.\, \sqrt{a} \,.\, \{e^2 + \text{tang.}^2\varphi\} + m' \,.\, \sqrt{a'} \,.\, \{e'^2 + \text{tang.}^2\varphi'\} + \&\text{c.} \qquad [1153]$$

If we suppose the orbits to be nearly circular, and but very little inclined to each other, the secular variations of the excentricities of the orbits, will be determined, in § 55 [1089], by means of differential equations which are [1153'] independent of the inclinations, and therefore of the same form, as if the orbits were all in one plane; now, in this hypothesis, we shall have $\varphi = 0$, [1153''] $\varphi' = 0$, &c.; and the preceding equation will become

$$\text{constant} = e^2 \,.\, m \,.\, \sqrt{a} + e'^2 \,.\, m' \,.\, \sqrt{a'} + e''^2 \,.\, m'' \,.\, \sqrt{a''} + \&\text{c.}\,; \qquad [1154]$$

which we have already obtained in § 57 [1114].

Likewise the secular variations of the inclinations of the orbits, are, in § 59 [1132], determined by means of differential equations independent of [1154'] the excentricities, and which are therefore of the same form, as if the orbits were circular; now, in this hypothesis, we shall have $e = 0$, $e' = 0$, &c.; [1154''] therefore we shall get

[1155] $$\text{constant} = m \,.\, \sqrt{a} \,.\, \text{tang.}^2 \varphi + m' \,.\, \sqrt{a'} \,.\, \text{tang.}^2 \varphi' + m'' \,.\, \sqrt{a''} \,.\, \text{tang.}^2 \varphi'' + \&\text{c.};$$

which equation is the same as was found in § 59 [1134].*

If we suppose, as in the last article [1138],

[1156] $$p = \text{tang.}\, \varphi \,.\, \text{sin.}\, \theta\,; \qquad q = \text{tang.}\, \varphi \,.\, \text{cos.}\, \theta\,;$$

it is easy to prove, that if the inclination of the orbit of m to the plane of x, y, be φ, and the longitude of its ascending node, counted from the axis of [1156′] x, be θ; the cosine of the inclination of this orbit to the plane of x, z, will be†

[1157] $$\frac{q}{\sqrt{1 + \text{tang.}^2 \varphi}}.$$

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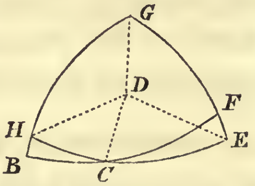
* (789) Substituting in [1134] the value $\sqrt{p^2+q^2} = \text{tang.}\, \varphi$, [1139], and the similar values of $\sqrt{p'^2+q'^2}$, &c., it becomes as in [1155].

† (790) In the adjoined figure, let DH, DE, DG, be the axes of x, y, z, respectively; $GHCE$, a spherical surface described about the origin D, with the radius 1, cutting the plane of the orbit of m in the great circle FCB, which plane intersects the plane of xy, in the point C; that of xz, in the point B, and that of zy, in the point F, and the inclinations of that orbit to those planes will be represented by the spherical angles FCE, CBH, CFE, respectively. To find these two last angles we have $FCE = \varphi$, $HC = \theta$, $CE = \frac{1}{2}\pi - \theta$, $\frac{1}{2}\pi$ being a right angle. Then in the right-angled spherical triangle CHB we have



$$\text{cos.}\, CBH = \text{cos.}\, CH \,.\, \text{sin.}\, HCB = \text{cos.}\, \theta \,.\, \text{sin.}\, \varphi,$$

and by the second equation [1138] or [1156], we have $\text{cos.}\, \theta = \frac{q}{\text{tang.}\, \varphi}$, hence

$$\text{cos.}\, CBH = \frac{q \,.\, \text{sin.}\, \varphi}{\text{tang.}\, \varphi} = q \,.\, \text{cos.}\, \varphi,$$

which is evidently equal to $\frac{q}{\sqrt{1+\text{tang.}^2 \varphi}}$, as in [1157]. Again, in the right-angled spherical triangle CEF we have

$$\text{cos.}\, CFE = \text{cos.}\, CE \,.\, \text{sin.}\, FCE = \text{sin.}\, \theta \,.\, \text{sin.}\, \varphi,$$

and by substituting the value $\text{sin.}\, \theta = \frac{p}{\text{tang.}\, \varphi}$, [1156], we get

$$\text{cos.}\, CFE = \frac{p \,.\, \text{sin.}\, \varphi}{\text{tang.}\, \varphi} = p \,.\, \text{cos.}\, \varphi = \frac{p}{\sqrt{1+\text{tang.}^2 \varphi}}.$$

Multiplying this quantity by $\frac{x\,dy-y\,dx}{dt}$, or by its value $\sqrt{a.(1-e^2)}$ [1147], we shall obtain the value of $\frac{x\,dz-z\,dx}{dt}$; the equation [431] will therefore [1157′] give, by neglecting quantities of the order m^2,

$$c'=m.q.\sqrt{\frac{a.(1-e^2)}{1+\text{tang.}^2\varphi}}+m'.q'.\sqrt{\frac{a'.(1-e'^2)}{1+\text{tang.}^2\varphi'}}+\&c. \qquad [1158]$$

We shall likewise find, from the equation [432],

$$c''=m.p.\sqrt{\frac{a.(1-e^2)}{1+\text{tang.}^2\varphi}}+m'.p'.\sqrt{\frac{a'.(1-e'^2)}{1+\text{tang.}^2\varphi'}}+\&c. \qquad [1159]$$

If in these two equations, we neglect quantities of the order e^3, or $e^2\varphi$, they [1159′] will become

$$\text{constant}=m\,q.\sqrt{a}+m'\,q'.\sqrt{a'}+\&c.\,;$$
$$\text{constant}=mp.\sqrt{a}+m'p'.\sqrt{a'}+\&c.\,; \qquad [1160]$$

which equations we have before found in § 59 [1136, 1137].

Lastly, the equation [442]* will give, by neglecting quantities of the [1160′]

The whole value of $\frac{x\,dy-y\,dx}{dt}$, corresponding to the supposition that the axes x, y, are situated in the plane of the orbit BCF is, by [1147], equal to $\sqrt{a.(1-e^2)}$. Multiplying this severally by the cosines of the angles FCE, CBH, CFE, that is, by $\cos.\varphi$, $\frac{q}{\sqrt{1+\text{tang.}^2\varphi}}$; $\frac{p}{\sqrt{1+\text{tang.}^2\varphi}}$; we shall, by the principles of the orthographic projection, obtain the values of the projections $\frac{x\,dy-y\,dx}{dt}$, $\frac{x\,dz-z\,dx}{dt}$, $\frac{y\,dz-z\,dy}{dt}$, respectively, the two last being substituted in [431, 432], putting $M=1$, neglecting terms of the order $m\,m'$, we get c', c'', as in [1158, 1159]. Developing these in series, and neglecting terms of the order e^3, $e^2\,\varphi$, $m\,m'$, we evidently obtain the equations [1160].

* (791) The equation [442] contains λ, which, by [397] is of the order $m\,m'$; now by neglecting such terms, and putting $M=1$, this equation becomes

$$h=\Sigma.m.\frac{(dx^2+dy^2+dz^2)}{dt}-2\Sigma.\frac{m}{r},$$

which, by the last of the equations [572], may be reduced to the form $h=-\Sigma.m.\frac{\mu}{a}$, and this, by means of [530$^{\text{iv}}$], will become $h=-\Sigma.m.\frac{1}{a}$, neglecting terms of the order m^2. This agrees with [1161].

order $m\,m'$, and observing, that by § 18 [572], $\frac{\mu}{a}=\frac{2\mu}{r}-\frac{(d\,x^2+d\,y^2+d\,z^2)}{d\,t^2}$,

[1161] $$\text{constant}=\frac{m}{a}+\frac{m'}{a'}+\frac{m''}{a''}+\&c.$$

All these equations take place in relation to the inequalities of a very long
[1161′] period, which might affect the elements of the orbit of m, m', &c. We have
observed, in § 54 [1070^{xii}], that if the ratio of the mean motions of these
bodies be nearly commensurable, it may introduce into the expressions of
the transverse axes of the orbits, considered as variable, some equations
whose arguments are proportional to the time, and which will increase very
[1161″] slowly; these equations having for divisors the coefficients of the time t
in these arguments, may therefore become sensible. Now it is evident,
that by noticing only terms which have such divisors, and considering the
orbits as ellipses, whose elements vary in consequence of these terms, the
[1161‴] integrals [430, 431, 432, 442], will always give the relations we have just
found between these elements; because the terms of the order mm', which
we have neglected in these integrals, in finding these relations, have not for
[1161^{iv}] divisors the very small coefficients we have mentioned; or, at least, they do
not contain them except they are multiplied by a power of the disturbing
forces, superior to that we have taken into consideration.

62. We have observed, in § 21, 22, [167^{iv}, 180, &c.] of the first book,
[1161^{v}] that in the motion of a system of bodies, there exists an invariable plane,
preserving always a parallel situation, which might at all times be found by
this principle, that the sum of the products, formed by multiplying each mass
Invariable plane. of the system, by the projection of the area described by its radius vector, in
[1161^{vi}] a given time, is a maximum. It is chiefly in the theory of the solar system,
that the investigation of this plane is important, on account of the proper
motions of the stars, and of the ecliptic, which makes it very difficult for
astronomers to determine with precision the motions of the heavenly bodies.
[1161^{vii}] If we put γ* for the inclination of this invariable plane to the plane of x, y;

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* (792) The equations [178, 179] give

$$\sin.\theta\,.\sin.\psi=\frac{c''}{\sqrt{c^2+c'^2+c''^2}},\qquad \sin.\theta\,.\cos.\psi=\frac{-c'}{\sqrt{c^2+c'^2+c''^2}};\qquad \cos.\theta=\frac{c}{\sqrt{c^2+c'^2+c''^2}}.$$

and Π for the longitude of its ascending node; it will follow, from what we have demonstrated in § 21, 22, of the first book, that we shall have,

$$\text{tang.}\,\gamma\,.\,\text{sin.}\,\Pi=\frac{c''}{c}; \qquad \text{tang.}\,\gamma\,.\,\text{cos.}\,\Pi=\frac{c'}{c}; \qquad [1162]$$

consequently

$$\text{tang.}\,\gamma\,.\,\text{sin.}\,\Pi=\frac{m\,.\sqrt{a\,.\,(1-e^2)}\,.\,\text{sin.}\,\varphi\,.\,\text{sin.}\,\theta+m'\,.\sqrt{a'\,.\,(1-e'^2)}\,.\,\text{sin.}\,\varphi'\,.\,\text{sin.}\,\theta'+\&\text{c.}}{m\,.\sqrt{a\,.\,(1-e^2)}\,.\,\text{cos.}\,\varphi+m'\,.\sqrt{a'\,.\,(1-e'^2)}\,.\,\text{cos.}\,\varphi'+\&\text{c.}}$$

$$\text{tang.}\,\gamma\,.\,\text{cos.}\,\Pi=\frac{m\,.\sqrt{a\,.\,(1-e^2)}\,.\,\text{sin.}\,\varphi\,.\,\text{cos.}\,\theta+m'\,.\sqrt{a'\,.\,(1-e'^2)}\,.\,\text{sin.}\,\varphi'\,.\,\text{cos.}\,\theta'+\&\text{c.}}{m\,.\sqrt{a\,.\,(1-e^2)}\,.\,\text{cos.}\,\varphi+m'\,.\sqrt{a'\,.\,(1-e'^2)}\,.\,\text{cos.}\,\varphi'+\&\text{c.}} \qquad [1162']$$

We may easily compute, by means of these values, the two angles γ and Π; and it is evident that to determine the invariable plane, we must know the [1162''] masses of the comets, and the elements of their orbits. Fortunately these masses seem to be very small, and it appears that we may, without sensible error, neglect their action on the planets; but time alone can give us the requisite information on this subject. We may also observe, that as it respects this invariable plane, the values of p, q, p', q', &c., do not contain

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Dividing the first and second by the third, we get $\text{tang.}\,\theta\,.\,\text{sin.}\,\psi=\frac{c''}{c}$, $\text{tang.}\,\theta\,.\,\text{cos.}\,\psi=-\frac{c'}{c}$, in which θ is the inclination of the fixed plane of $x_{\prime\prime\prime}$, $y_{\prime\prime\prime}$, to the plane of x, y, and by note 81, the longitude of the ascending node P, of the fixed plane, in the figure, page 112, is $\pi-\psi$, π being two right angles. To conform to the notation in [1161$^{\text{vii}}$], we must put $\theta=\gamma$, $\Pi=\pi-\psi$, or $\psi=\pi-\Pi$; substituting these in the two last equations, we shall get [1162]. The equation [1151], by putting $\text{cos.}\,\varphi$, $\text{cos.}\,\varphi'$, &c., for $\frac{1}{\sqrt{1+\text{tang.}^2\varphi}}$, &c., becomes

$$c=m\,.\sqrt{a\,.(1-e^2)}\,.\,\text{cos.}\,\varphi+m'\,.\sqrt{a'\,.(1-e'^2)}\,.\,\text{cos.}\,\varphi'+\&\text{c.} \qquad [1162a]$$

Substituting the values [1156], and the similar values of p', q', &c., in [1158, 1159], observing also that $\frac{\text{tang.}\,\varphi}{\sqrt{1+\text{tang.}^2\varphi}}=\text{sin.}\,\varphi$, $\frac{\text{tang.}\,\varphi'}{\sqrt{1+\text{tang.}^2\varphi'}}=\text{sin.}\,\varphi'$, &c., they will become $c'=m\,.\sqrt{a\,.(1-e^2)}\,.\,\text{sin.}\,\varphi\,.\,\text{cos.}\,\theta+m'\,.\sqrt{a'\,.(1-e'^2)}\,.\,\text{sin.}\,\varphi'\,.\,\text{cos.}\,\theta'+\&\text{c.}$, [1162b] $c''=m\,.\sqrt{a\,.(1-e^2)}\,.\,\text{sin.}\,\varphi\,.\,\text{sin.}\,\theta+\&\text{c.}$ Substituting these in [1162], we shall get the equations [1162'], which are exact in terms of the order m, neglecting m^2.

[1162‴] any constant terms;[*] for it is evident, by the equation (C) § 59 [1132], that these terms are the same for p, p', p'', &c.; and that they are also the same for q, q', q'', &c.; and as it respects the invariable plane, the constant terms [1162^{iv}] of the first members of the equations [1136, 1137] are nothing; therefore the constant terms disappear from the expressions of p, p', &c., q, q', &c., by means of these equations.

Considera-tion of two orbits. We shall now consider the motion of two orbits, supposing them inclined to each other by any angle; we shall have, by § 61,[†]

$$c' = \sin.\varphi . \cos.\theta . m . \sqrt{a.(1-e^2)} + \sin.\varphi' . \cos.\theta' . m' . \sqrt{a'.(1-e'^2)};$$

[1163]

$$c'' = \sin.\varphi . \sin.\theta . m . \sqrt{a.(1-e^2)} + \sin.\varphi' . \sin.\theta' . m' . \sqrt{a'.(1-e'^2)}.$$

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[*] (793) That is p, p', &c., q, q', &c., do not contain terms like $p=f$, $p'=f'$, &c., $q=l$, $q'=l'$, &c., in which f, f', &c., l, l', &c., are constant quantities, independent of the time. For the substitution of these, in [1132], would make the first members of those equations vanish, so that they would become

$$0 = \{(0,1)+(0,2)+\&c.\}.f-(0,1).f'-(0,2).f''+\&c.$$

$$0 = -\{(0,1)+(0,2)+\&c.\}.l+(0,1).l'+(0,2).l''+\&c.,$$

and we should have as many linear equations in f, f', &c., as there are different quantities f, f', &c., and from these we should obtain, by the usual rules of elimination of algebraic equations of the first degree, the values of the quantities f, f', &c. It is easy to perceive that these values may be obtained, by putting all the constant quantities f, f', f'', &c., equal to each other. In like manner, by putting all the constant quantities l, l', &c., equal to each other, we may satisfy the linear equations in l, l', &c., so that if we notice only the constant terms of the values of p, p', &c., q, q', &c., we shall have $p=f$, $p'=f$, $p''=f$, &c., [1163a] $q=l$, $q'=l$, $q''=l$, &c.; and the slightest inspection will show, that these values will satisfy the equations [1132]. Now, as it respects the invariable plane, we have $c'=0$, $c''=0$, [180′], therefore the first members of the equations [1158, 1159, 1160] must vanish, and if we substitute in [1160] the values [1163a], they will become

[1163b] $$0 = l.\{m.\sqrt{a}+m'.\sqrt{a'}+\&c.\}, \qquad 0 = f.\{m.\sqrt{a}+m'.\sqrt{a'}+\&c.\},$$

but the terms $m.\sqrt{a}$, $m'.\sqrt{a'}$, &c., [1114′], have all the same sign, therefore $m.\sqrt{a}+m'.\sqrt{a'}$, must be a finite quantity, in which case the equations [1163b] will give $l=0$, $f=0$, consequently, the constant terms must disappear from the values of p, p', &c., q, q', &c.

[†] (794) These values were computed from [1158, 1159], reduced, as in [1162b].

We shall suppose the fixed plane, to which we refer the motion of the orbits, to be the invariable plane just mentioned, with respect to which the constant quantities of the first members of these equations are nothing, as we have [1163'] seen in § 21, 22 [180'] of the first book. The angles φ and φ' being positive, the preceding equations will give* [1163'']

$$m.\sqrt{a.(1-e^2)}.\sin.\varphi = m'.\sqrt{a'.(1-e'^2)}.\sin.\varphi';$$
$$\sin.\theta = -\sin.\theta'; \qquad \cos.\theta = -\cos.\theta'; \qquad [1164]$$

hence we deduce $\theta' = \theta +$ the semi-circumference; therefore the nodes of [1164'] the orbits are upon the same line; but the ascending node of the one coincides with the descending node of the other; so that the mutual [1164''] inclination of the two orbits is $\varphi + \varphi'$.

We have, by § 61 [1162*a*],

$$c = m.\sqrt{a.(1-e^2)}.\cos.\varphi + m'.\sqrt{a'.(1-e'^2)}.\cos.\varphi'; \qquad [1165]$$

by combining this equation with the preceding between $\sin.\varphi$ and $\sin.\varphi'$, we shall find†

$$2mc.\cos.\varphi.\sqrt{a.(1-e^2)} = c^2 + m^2.a.(1-e^2) - m'^2.a'.(1-e'^2). \qquad [1166]$$

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* (795) Put $c'=0$, $c''=0$, in [1163], and we shall obtain

$$\sin.\varphi.\cos.\theta.m.\sqrt{a.(1-e^2)} = -\sin.\varphi'.\cos.\theta'.m'.\sqrt{a'.(1-e'^2)},$$
$$\sin.\varphi.\sin.\theta.m.\sqrt{a.(1-e^2)} = -\sin.\varphi'.\sin.\theta'.m'.\sqrt{a'.(1-e'^2)}. \qquad [1164a]$$

Dividing the second equation by the first, we shall get $\tang.\theta = \tang.\theta'$, which corresponds to $\theta' = \theta$, or $\theta' = \pi + \theta$. The first value cannot be used, for by substituting it in the first of the preceding equations, it would become divisible by $\cos.\theta$, and would give

$$\sin.\varphi.m.\sqrt{a.(1-e^2)} = -\sin.\varphi'.m'.\sqrt{a'.(1-e'^2)},$$

now by [1114'], the radicals $\sqrt{a}$, $\sqrt{a'}$, or $\sqrt{a.(1-e^2)}$, $\sqrt{a'.(1-e'^2)}$, must have the same sign, and as φ, φ', are both positive and acute, [1163''], their signs must be positive, the first member of the preceding equation will therefore be positive, the second negative, they cannot therefore be equal to each other, so that we cannot use the first value of θ', and must take the second $\theta' = \theta + \pi$, which gives, as in [1164], $\sin.\theta = -\sin.\theta'$, $\cos.\theta = -\cos.\theta'$; substituting these in the two equations [1164*a*], and dividing them respectively by $\cos.\theta$, $\sin.\theta$, we shall get $m.\sqrt{a.(1-e^2)}.\sin.\varphi = m'.\sqrt{a'.(1-e'^2)}.\sin.\varphi'$, as in [1164].

† (796) From [1165], we get $c - m.\sqrt{a.(1-e^2)}.\cos.\varphi = m'.\sqrt{a'.(1-e'^2)}.\cos.\varphi'$; squaring both sides, and substituting $\cos.^2\varphi = 1 - \sin.^2\varphi$, $\cos.^2\varphi' = 1 - \sin.^2\varphi'$, it becomes

$$c^2 - 2m.c.\sqrt{a.(1-e^2)}.\cos.\varphi + m^2a.(1-e^2).(1-\sin.^2\varphi) = m'^2a'.(1-e'^2).(1-\sin.^2\varphi');$$

If we suppose the orbits to be circular, or so little excentrical that we may neglect the squares of their excentricities, the preceding equation will make
[1166′] φ constant; and for the same reason, φ' will be constant; the inclinations of the planes of the orbits, to the fixed plane, and to each other, will therefore be constant; and these three planes will always have a common intersection.
[1166″] Hence it follows, that the mean momentary variation of this intersection is always the same, since it cannot be expressed but by a function of these inclinations. When they are very small, we shall easily find, from § 60, and by means of the preceding relation between $\sin.\varphi$ and $\sin.\varphi'$, that for the
[1166‴] time t, the motion of this intersection is* $-\{(0,1)+(1,0)\}.t$.

The position of the invariable plane to which we have just referred the motion of the orbits, is easily determined for any instant; it being only necessary to divide the angle of the mutual inclination of the two orbits, into
[1166iv] two angles, φ and φ', so that they may satisfy the preceding equation between

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adding to this the square of the first of the equations [1164],

$$m^2 a.(1-e^2).\sin.^2\varphi = m'^2 a'.(1-e'^2).\sin.^2\varphi',$$

we shall get $\quad c^2 - 2m.c.\sqrt{a.(1-e^2)}.\cos.\varphi + m^2 a.(1-e^2) = m'^2 a'.(1-e'^2),$
which, by transposition, gives [1166], and if e, e', are so small, that we may neglect their squares, this equation will give $\quad \cos.\varphi = \frac{c^2+m^2a-m'^2a'}{2mc.\sqrt{a}}, \quad$ in which each term of the second member is constant, consequently φ is constant, as in [1166′].

* (797) The second of the equations [1142], in this case, where there are only two bodies m, m', becomes $\quad \frac{d\theta}{dt} = -(0,1)+(0,1).\frac{\text{tang.}\varphi'}{\text{tang.}\varphi}.\cos.(\theta-\theta')$. Now by [1164′], $\cos.(\theta-\theta') = \cos.(-\pi) = -1$, and the first of the equations [1164], neglecting terms of the order $e^2\varphi$, becomes $\quad m.\sqrt{a}.\sin.\varphi = m'.\sqrt{a'}.\sin.\varphi'$; or, by neglecting terms of the order φ^3, $\quad m.\sqrt{a}.\text{tang.}\varphi = m'.\sqrt{a'}.\text{tang.}\varphi' \quad$ hence $\quad \frac{\text{tang.}\varphi'}{\text{tang.}\varphi} = \frac{m.\sqrt{a}}{m'.\sqrt{a'}}$, consequently $\quad \frac{d\theta}{dt} = -(0,1)-(0,1).\frac{m.\sqrt{a}}{m'.\sqrt{a'}}$; but from [1093] we have

$$(0,1).\frac{m.\sqrt{a}}{m'.\sqrt{a'}} = (1,0), \qquad \text{hence} \qquad \frac{d\theta}{dt} = -\{(0,1)+(1,0)\};$$

Multiplying this by dt, and integrating, we get $\quad \theta = -\{(0,1)+(0,1)\}.t$, as in [1166′]″.

$\sin.\varphi$ and $\sin.\varphi'$. Denoting, therefore, this mutual inclination by ϖ, we shall have,*

$$\tang.\varphi=\frac{m'.\sqrt{a'.(1-e'^2)}.\sin.\varpi}{m.\sqrt{a.(1-e^2)}+m'.\sqrt{a'.(1-e'^2)}.\cos.\varpi}. \qquad [1167]$$

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* (798) Put $\varphi+\varphi'=\varpi$ or $\varphi'=\varpi-\varphi$, hence

$$\sin.\varphi'=\sin.\varpi.\cos.\varphi-\cos.\varpi.\sin.\varphi, \qquad [22]\ \text{Int.}$$

Substituting this in the first equation [1164], we get

$$m.\sqrt{a.(1-e^2)}.\sin.\varphi=m'.\sqrt{a'.(1-e'^2)}.\{\sin.\varpi.\cos.\varphi-\cos.\varpi.\sin.\varphi\},$$

which, being divided by $\cos.\varphi$, becomes

$$m.\sqrt{a.(1-e^2)}.\tang.\varphi=m'.\sqrt{a'.(1-e'^2)}.\{\sin.\varpi-\cos.\varpi.\tang.\varphi\};$$

transposing the last term, and dividing by the coefficient of $\tang.\varphi$, we get [1167].

CHAPTER VIII.

SECOND METHOD OF APPROXIMATION OF THE MOTIONS OF THE HEAVENLY BODIES.

63. We have seen, in Chapter II, that the co-ordinates of the heavenly bodies, referred to the foci of the principal forces which act on them, are determined by differential equations of the second order. These equations have been integrated in Chapter III, noticing only the principal forces, and it has been shown, in this case, that the orbits are conic sections, whose elements are the arbitrary constant quantities introduced by the integrations; and as the disturbing forces produce but small variations in the motions, it is very natural to endeavor to reduce the disturbed motions of the heavenly bodies to the laws of the elliptical motion. If we apply to the differential equations of the elliptical motion, increased by small terms arising from the disturbing forces, the method of approximation explained in § 45, we may suppose the motions which are performed in oval or returning curves, to be elliptical; but then the elements of these motions will be variable, and we
[1167′] may find the variations by that method. These differential equations being of the second order, *their finite integrals, and also their integrals of the first*
[1167″] *order, will be the same as if the ellipses were invariable*;* *so that we may*
Important principles of this method. *take the differential of the finite equations of the elliptical motions, supposing the elements of these motions to be constant.* It follows also from the same method, that in the equations of this motion, which are differentials of the first order, we may again take the differentials, *considering as variable only the elements of the orbits, and the first differentials of the co-ordinates; provided*

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* (798) This is conformable to the remarks [898′], the term i being here equal to 2,
[1167a] consequently, by what is there said, the functions of the *first* order will be the same in both ellipses.

*that instead of the second differentials of these co-ordinates, we substitute the part of their values, depending upon the disturbing forces.** These results may also be deduced from the consideration of the elliptical motion. [1167‴]

For this purpose, suppose an ellipsis to pass through a planet, and through the infinitely small arch which it describes; the centre of the sun being in its focus. This ellipsis is that which the planet would invariably describe, if the disturbing forces should cease to act upon it. Its elements are constant during the time dt, but they will vary from one instant to another. Therefore [1167ⁱᵛ] let $V = 0$ be a finite equation of the invariable ellipsis, V being a function of the rectangular co-ordinates x, y, z, and the parameters c, c', &c., which are functions of the elements of the elliptical motion. This equation will also take place in the variable ellipsis; but the parameters c, c', &c., will no [1167ᵛ] longer be constant. However, since this ellipsis appertains to the infinitely small part of the curve described by the planet during the instant dt; the equation $V = 0$ will take place for the first and last point of this infinitely [1167ᵛⁱ] small arch, supposing c, c', &c., to be constant quantities. We may therefore take the first differential of this equation, supposing only x, y, z, to be variable, and we shall get,

$$0 = \left(\frac{dV}{dx}\right) . dx + \left(\frac{dV}{dy}\right) . dy + \left(\frac{dV}{dz}\right) . dz. \qquad (i) \quad [1168]$$

Hence it is evident, that if we have a finite equation of the *invariable* ellipsis, we may take its first differential, supposing the parameters to be *constant*, and it will nevertheless correspond to the *variable* ellipsis. In like [1168′] manner, every differential equation of the first order, in the invariable ellipsis, will take place also in the variable ellipsis. For let $V' = 0$ be an equation of this order; V' being a function of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and of the [1168″] parameters c, c', &c. It is evident, that all these quantities are the same, in the variable ellipsis, as in the invariable ellipsis, which coincide with each [1168‴] other, during the instant dt.

Now, if we consider the planet, at the end of the instant dt, or at the commencement of the following instant, the function V will not vary, from

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* (799) This method of differentiation is proved in § 45, in the equation [903] and in the remarks immediately following it.

[1168iv] the ellipsis relative to the instant dt, to the consecutive ellipsis, except by the variation of the parameters, since the co-ordinates x, y, z, corresponding to the end of the first instant, are the same for both ellipses ; the function V being nothing, we shall have

[1169] $$0=\left(\frac{dV}{dc}\right).dc+\left(\frac{dV}{dc'}\right).dc'+\&c. \qquad (i')$$

This equation may also be deduced from the equation $V=0$, supposing all the quantities x, y, z, c, c', &c., to be variable ; for if we subtract the equation (i) [1168], from this equation, we shall obtain the equation (i') [1169].

Taking the differential of the equation (i) [1168], we shall obtain another [1169′] equation in dc, dc', &c., which, with the equation (i') [1169], will serve to determine the parameters c, c', &c. It was by this method, that the mathematicians who first attempted the computation of the theory of the perturbations of the heavenly bodies, determined the variations of the nodes and of the inclinations of the orbits : but we may simplify this method in the following manner.

We shall consider generally the differential equation of the first order [1169″] $V'=0$; this, as we have just seen, corresponds both to the variable and the invariable ellipsis, which during the instant dt coincide with each other. In the following instant the same equation corresponds to both ellipses, but with this difference, that c, c', &c., remain the same in the invariable ellipsis, but [1169‴] vary in the variable ellipsis. Let V'' be what V' becomes, when the ellipsis is invariable ; $V'_{\prime}$ what the same function becomes, when the ellipsis is [1169iv] variable. It is evident that to obtain V'', we must change in V', the co-ordinates x, y, z, corresponding to the beginning of the first instant dt, into those, corresponding to the beginning of the second instant ; we must [1169v] then increase the first differentials dx, dy, dz, respectively by the quantities ddx, ddy, ddz, corresponding to the invariable ellipsis, the element of the time dt being supposed constant.

Moreover, to obtain $V'_{\prime}$, we must change in V', the co-ordinates x, y, z, into those corresponding to the beginning of the second instant, which are also [1169vi] the same in the two ellipses ; we must then increase dx, dy, dz, by the quantities $d\,dx, d\,dy, d\,dz$, respectively ; lastly we must change the parameters c, c', &c., into $c+dc$, $c'+dc'$, &c.

The values of ddx, ddy, ddz, are not the same in both ellipses; they
are increased in the case of the variable ellipsis, by quantities arising from [1169^{vii}]
the disturbing forces. We see therefore that the two functions V'' and $V'_{\prime}$
differ only in this respect, that in the second expression, the parameters c, c',
&c., increase by dc, dc', &c.; and the values of ddx, ddy, ddz,
corresponding to the invariable ellipsis, increase by quantities arising from [1169^{viii}]
the disturbing forces. We may therefore compute $V'_{\prime} - V''$, by taking the
differential of V', supposing x, y, z, to be constant, and dx, dy, dz, c, c', &c.,
to be variable, provided we substitute in this differential, for ddx, ddy, [1169^{ix}]
ddz, &c., the parts of their values arising only from the disturbing forces.

Now if in the function $V'' - V'$, we substitute for ddx, ddy, ddz,
their values corresponding to the elliptical motion, we shall have, for the
function $V'' - V'$, an expression in terms of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, c, [1169^{x}]
c', &c., which, in the case of the invariable ellipsis, will be nothing; this
function will therefore be nothing, in case the ellipsis is variable.* We
evidently have, in this last case, $V'_{\prime} - V' = 0$; since this equation is the [1169^{xi}]
differential of the equation $V' = 0$; subtracting from it the equation
$V'' - V' = 0$, we shall get $V'_{\prime} - V'' = 0$. *Therefore we may, in this case, take
the differential of the equation* $V' = 0$, *supposing only* dx, dy, dz, c, c', &c.,
to be variable, and substituting for ddx, ddy, ddz, *the parts of their values* [1169^{xii}]
corresponding to the disturbing forces. These results are exactly the same
as those we have obtained in § 45, by a pure analytical method; but on
account of the importance of the subject, we have thought it proper to

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* (800) This function $V'' - V'$, after the substitution of the elliptical values of ddx, ddy, ddz, becomes a differential function of the *first* order, which must therefore, by using the method explained in [1167''], be the same for the *variable* as for the *invariable* ellipsis. On the contrary, $V'_{\prime}$ contains dc, dc', &c., and the values of ddx, ddy, ddz, corresponding to the variable ellipsis. These last values of ddx, ddy, ddz, may be considered as consisting of two parts, namely, the elliptical values, and the parts arising from the perturbations; and as V'' contains the elliptical values, $V'_{\prime} - V'' = 0$, must contain only the parts of ddx, ddy, ddz, arising from the perturbations. Hence we see the reason of the method of differentiation [1169^{xii}].

deduce them also from the consideration of the elliptical motion. This being supposed,

64. We shall resume the equations (*P*) § 46 [915],

[1170]

$$\left.\begin{aligned} 0 &= \frac{ddx}{dt^2} + \frac{\mu x}{r^3} + \left(\frac{dR}{dx}\right) \\ 0 &= \frac{ddy}{dt^2} + \frac{\mu y}{r^3} + \left(\frac{dR}{dy}\right) \\ 0 &= \frac{ddz}{dt^2} + \frac{\mu z}{r^3} + \left(\frac{dR}{dz}\right) \end{aligned}\right\} ; \qquad (P)$$

If we suppose $R = 0$, we shall obtain the equations of the elliptical motion, [1170'] which we have integrated in Chapter III [545]. We have obtained, in § 18 [572], the seven following integrals,

[1171]

$$\left.\begin{aligned} & c = \frac{x\,dy - y\,dx}{dt} ; \qquad c' = \frac{x\,dz - z\,dx}{dt} ; \qquad c'' = \frac{y\,dz - z\,dy}{dt} ; \\ & 0 = f + x \,.\left\{\frac{\mu}{r} - \left(\frac{dy^2 + dz^2}{dt^2}\right)\right\} + \frac{y\,dy\,.dx}{dt^2} + \frac{z\,dz\,.dx}{dt^2} ; \\ & 0 = f' + y \,.\left\{\frac{\mu}{r} - \left(\frac{dx^2 + dz^2}{dt^2}\right)\right\} + \frac{x\,dx\,.dy}{dt^2} + \frac{z\,dz\,.dy}{dt^2} ; \\ & 0 = f'' + z \,.\left\{\frac{\mu}{r} - \left(\frac{dx^2 + dy^2}{dt^2}\right)\right\} + \frac{x\,dx\,.dz}{dt^2} + \frac{y\,dy\,.dz}{dt^2} ; \\ & 0 = \frac{\mu}{a} - \frac{2\mu}{r} + \left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) ; \end{aligned}\right\} . \qquad (p)$$

As these integrals express the arbitrary quantities in functions of the co-ordinates and their first differentials, they are under a very convenient form, for computing the variations of the arbitrary quantities. The three first integrals give, by differentiation, supposing only the parameters c, c', c'', and the first differentials of the co-ordinates to be variable, as in the preceding article [1167'''],*

[1172]

$$dc = \frac{x\,ddy - y\,ddx}{dt} ; \qquad dc' = \frac{x\,ddz - z\,ddx}{dt} ; \qquad dc'' = \frac{y\,ddz - z\,ddy}{dt} .$$

Substituting, for ddx, ddy, ddz, the parts of their values, arising from the disturbing forces, which are easily deduced from the equations [1170], namely, $-dt^2.\left(\frac{dR}{dx}\right)$, $-dt^2.\left(\frac{dR}{dy}\right)$, $-dt^2.\left(\frac{dR}{dz}\right)$; we shall [1172''] find,

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* (801) These differentiations are made, and the equations [1173] deduced, upon the principles mentioned in [1167'''] or [1169^{xii}].

$$dc=dt.\left\{y.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dy}\right)\right\};$$
$$dc'=dt.\left\{z.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dz}\right)\right\};\qquad [1173]$$
$$dc''=dt.\left\{z.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dz}\right)\right\};$$

We have seen, in § 18, 19, [591, 599, 575'], that from the parameters c, c', c'', we may determine three elements of the elliptical orbit; namely, the [1173'] inclination φ of the orbit to the plane of x, y, and the longitude θ of its node, by means of the equations* [591]

$$\text{tang.}\,\varphi=\frac{\sqrt{c'^2+c''^2}}{c};\qquad \text{tang.}\,\theta=\frac{c''}{c'};\qquad [1174]$$

also the semi-parameter of the ellipsis $a.(1-e^2)$ [378s], by means of the [1174'] equation

$$\mu a.(1-e^2)=c^2+c'^2+c''^2.\qquad [1175]$$

These equations take place also in the variable ellipsis, provided we determine c, c', c'', by means of the preceding differential equations. We shall thus have the parameter of the variable ellipsis, its inclination to the fixed [1175'] plane of x and y, and the position of its node.

From the three first of the equations (P) [572], we have deduced, in § 19 [579], the finite integral $0=c''x-c'y+cz$; this equation takes [1175''] place in case the ellipsis is disturbed [1167''], and its first differential, $0=c''.dx-c'dy+c.dz$, found upon the supposition that c, c', c'', are constant, also takes place.

If we take the differentials of the fourth, fifth and sixth of the integrals (p) [1171], supposing only the parameters f, f', f'', and the differentials

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* (802) The equations [1174] are the same as [591]. The equation [1175] is deduced from $h^2=c^2+c'^2+c''^2$, [575'], substituting $h^2=\mu a.(1-e^2)$, [599].

dx, dy, dz, to be variable; and then substitute for ddx, ddy, ddz, [1175'''] the values [1172'] $-dt^2.\left(\frac{dR}{dx}\right)$, $-dt^2.\left(\frac{dR}{dy}\right)$, $-dt^2.\left(\frac{dR}{dz}\right)$, we shall find*

$$df=dy.\left\{y.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dy}\right)\right\}+dz.\left\{z.\left(\frac{dR}{dx}\right)-x.\left(\frac{dR}{dz}\right)\right\}$$
$$+(y\,dx-x\,dy).\left(\frac{dR}{dy}\right)+(z\,dx-x\,dz).\left(\frac{dR}{dz}\right);$$

[1176]
$$df'=dx.\left\{x.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dx}\right)\right\}+dz.\left\{z.\left(\frac{dR}{dy}\right)-y.\left(\frac{dR}{dz}\right)\right\}$$
$$+(x\,dy-y\,dx).\left(\frac{dR}{dx}\right)+(z\,dy-y\,dz).\left(\frac{dR}{dz}\right);$$

$$df''=dx.\left\{x.\left(\frac{dR}{dz}\right)-z.\left(\frac{dR}{dx}\right)\right\}+dy.\left\{y.\left(\frac{dR}{dz}\right)-z.\left(\frac{dR}{dy}\right)\right\}$$
$$+(x\,dz-z\,dx).\left(\frac{dR}{dx}\right)+(y\,dz-z\,dy).\left(\frac{dR}{dy}\right).$$

[1176'] Lastly, the differential of the seventh of the integrals (p) [1171], taken in the same manner, will give the variation of the semi-transverse axis a, by means of the equation†

[1177]
$$d.\frac{\mu}{a}=2.\mathrm{d}R;$$

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* (803) The differential of the fourth of the equations [1171], taken as above directed, and observing that $r=\sqrt{x^2+y^2+z^2}$, is

$$0=df-2x.\frac{(dy.ddy+dz.ddz)}{dt^2}+y.\frac{(ddy.dx+dy.ddx)}{dt^2}+z.\frac{(ddz.dx+dz.ddx)}{dt^2},$$

or, as it may be written,

$$df=dy.\left(\frac{-y\,ddx+x\,ddy}{dt^2}\right)+dz.\left(\frac{-z\,ddx+x\,ddz}{dt^2}\right)-(y\,dx-x\,dy).\frac{ddy}{dt^2}-(z\,dx-x\,dz).\frac{ddz}{dt^2};$$

substituting, for ddx, ddy, ddz, their values [1175'''], we shall obtain df, [1176]. The fifth of the equations [1171], may be deduced from the fourth, by changing f into f', x into y, and y into x. The sixth may be deduced from the fifth, by changing f' into f'', y into z, and z into y. The same changes being successively made in df, [1176], we shall get df'. df'', [1176].

* (804) Taking the differential of the last of the equations [1171], in the abovementioned manner, we shall get $0=d.\frac{\mu}{a}+2.\frac{dx.ddx+dy.ddy+dz.ddz}{dt^2}$, substituting ddx,

the differential $\mathrm{d}R$ refers only to the co-ordinates x, y, z, of the body m.

The longitude of the projection of the perihelion of the orbit, upon the
fixed plane, and the ratio of the excentricity to the semi-transverse axis, are
determined by means of the values of f, f', f''. For I being the longitude [1177′]
of this projection, we shall have, by § 19 [594′]

$$\text{tang.}\, I = \frac{f'}{f}; \qquad [1178]$$

and e being the ratio of the excentricity to the semi-transverse axis, we shall [1178′]
have, by the same article,*

$$\mu e = \sqrt{f^2 + f'^2 + f''^2}. \qquad [1179]$$

This ratio may also be determined, if we divide the semi-parameter
$a\,.\,(1 - e^2)$, by the semi-transverse axis a: the quotient subtracted from [1179′]
unity, will give e^2.

The integrals (p) [1171] have given by elimination, in § 19 [582], the
finite integral $0 = \mu r - h^2 + fx + f'y + f''z$; this equation takes place [1179″]
also in the disturbed ellipsis [1167″], and it determines, at each instant, the
nature of the variable ellipsis. We may take its differential, supposing f,
f', f'', to be constant, and we shall get

$$0 = \mu\, dr + f\, dx + f'\, dy + f''\, dz. \qquad [1180]$$

The semi-transverse axis a gives the mean motion of m, or more accurately,
that which, in the disturbed orbit, corresponds to the mean motion in the
undisturbed orbit; for we have, by § 20 [605′] $n = a^{-\frac{3}{2}}.\sqrt{\mu}$; moreover, [1180′]
if we put ζ equal to the mean motion of m, we shall have, in the invariable
ellipsis $d\zeta = n\,dt$ [1044′, &c.]; this equation generally takes place in the [1180″]

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ddy, ddz, [1172′], we shall find

$$d\,.\,\frac{\mu}{a} = 2\,.\left\{ \left(\frac{dR}{dx}\right).\,dx + \left(\frac{dR}{dy}\right).\,dy + \left(\frac{dR}{dz}\right).\,dz \right\},$$

the second member of which is evidently equal to $2\,\mathrm{d}R$, the characteristic d being supposed to affect only the co-ordinates of the body m.

* (805) We have $f^2 + f'^2 + f''^2 = l^2$, [574″], and $l = \mu e$, [597′], hence $\mu e = \sqrt{f^2 + f'^2 + f''^2}$, as in [1179]. The value of e may also be determined, as in [1179′], from that of $\mu a\,.\,(1 - e^2)$, found in [1175], and the semi-transverse axis a, deduced from [1177].

variable ellipsis, since it is a differential of the first order. Taking its
[1180‴] differential, we shall have $d\,d\zeta = dn\,.\,dt$; now we have*

[1181] $$dn = \frac{3an}{2\mu}\,.\,d\,.\,\frac{\mu}{a} = \frac{3an\,.\,\mathrm{d}R}{\mu},$$

therefore

[1182] $$d\,d\zeta = \frac{3an\,.\,dt\,.\,\mathrm{d}R}{\mu};$$

and by integration,

[1183] $$\zeta = \frac{3}{\mu}\,.\int\!\!\int an\,dt\,.\,\mathrm{d}R.$$

Lastly, we have seen in § 18, that the integrals (p) [1171] are equivalent only to five distinct integrals, and that they give, between the seven
[1183′] parameters, c, c', c'', f, f', f'', a, the two following equations of condition,†

[1184] $$0 = fc'' - f'c' + f''c;$$
$$0 = \frac{\mu}{a} + \frac{f^2 + f'^2 + f''^2 - \mu^2}{c^2 + c'^2 + c''^2};$$

these equations take place also in the variable ellipsis, provided the parameters are determined in the preceding manner. Which may also easily be proved *a posteriori*.

We have thus determined five elements of the disturbed orbit, namely, the inclination; the position of the nodes; the semi-transverse axis, which gives
[1184′] the mean motion; the excentricity, and the position of the perihelion. It now remains to find the sixth element of the elliptical motion, being that

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* (806) The differential of the logarithm of $n = a^{-\frac{3}{2}}.\sqrt{\mu}$, [1180′], is

$$\frac{dn}{n} = -\tfrac{3}{2}\,.\,\frac{da}{a} = \frac{3a}{2}\,.\,d\,.\,\frac{1}{a} = \frac{3a}{2\mu}\,.\,d\,.\,\frac{\mu}{a};$$

multiplying by n, we get $dn = \frac{3an}{2\mu}\,.\,d\,.\,\frac{\mu}{a}$; substituting [1177], we find the second value of dn, [1181].

† (807) The first of these equations is given in the same form in [574′], the second is deduced from [578], substituting for l^2, h^2, their values [574″, 575′].

which, in the undisturbed ellipsis, corresponds to the position of m, at a given epoch. For this purpose, we shall resume the expression of dt § 16,*

$$\frac{dt.\sqrt{\mu}}{a^{\frac{3}{2}}}=\frac{dv.(1-e^2)^{\frac{3}{2}}}{\{1+e.\cos.(v-\varpi)\}^2}.$$ [1185]

This equation being developed in a series, as in that article, becomes

$$n\,dt=dv.\{1+E^{(1)}.\cos.(v-\varpi)+E^{(2)}.\cos.2.(v-\varpi)+\&c.\}$$ [1186]

Integrating this equation, supposing e and ϖ to be constant, we shall get

$$\textstyle\int n\,dt+\varepsilon=v+E^{(1)}.\sin.(v-\varpi)+\frac{E^{(2)}}{2}.\sin.2.(v-\varpi)+\&c.\;;$$ [1187]

ε being an arbitrary constant quantity. This integral corresponds to the invariable ellipsis : to extend it to the disturbed ellipsis, we must make its differential agree with the preceding, when all the terms, including even the arbitrary quantities ε, e, ϖ, are supposed to be variable ; hence we get† [1187′]

$$d\varepsilon=de.\left\{\left(\frac{dE^{(1)}}{de}\right).\sin.(v-\varpi)+\tfrac{1}{2}.\left(\frac{dE^{(2)}}{de}\right).\sin.2.(v-\varpi)+\&c.\right\}$$
$$-d\varpi.\{E^{(1)}.\cos.(v-\varpi)+E^{(2)}.\cos.2.(v-\varpi)+\&c.\}$$ [1188]

[1188′] $v-\varpi$ is the true anomaly of m counted upon the orbit, and ϖ is the longitude of the perihelion, counted also upon the orbit. We have already found the longitude I [1178] of the projection of the perihelion upon the fixed plane ; now we shall have, by § 22 [676′], changing v into ϖ, and $v_{,}$ into I, in the expression $v-\beta$ of that article,‡

$$\varpi-\beta=I-\theta+\text{tang.}^2\tfrac{1}{2}\varphi.\sin.2.(I-\theta)+\&c.$$ [1189]

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* (808) This is the equation [535], multiplied by $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$, and it is developed [542], in the form [1186], whose integral is [1187].

† (809) Take the differential of [1187], supposing all the quantities n, ε, v, e, ϖ, to be variable and $E^{(1)}$, $E^{(2)}$, &c., to be functions of e; subtract from it the equation [1186], the difference will be [1188].

‡ (810) The longitude I is given by the formula [1178]. With this value of I we may find that of ϖ, by changing in [676′], v into ϖ, $v_{,}$ into I, from which we get [1189]. Putting $v=0$, $v_{,}=0$, in [676′], it becomes $-\beta=-\theta+\text{tang.}^2\frac{1}{2}\varphi.\sin.(-2\theta)+\&c.$, or $\beta=\theta+\text{tang.}^2\frac{1}{2}\varphi.\sin.2\theta+\&c.$, as in [1190].

Supposing v, $v_{\prime}$, to be nothing, in the same expression [676'], we shall find

[1190] $$\beta = \theta + \text{tang.}^2 \tfrac{1}{2} \varphi \,.\, \text{sin.}\, 2\theta + \&c.\,;$$

therefore*

[1191] $$\varpi = I + \text{tang.}^2 \tfrac{1}{2} \varphi \,.\, \{\text{sin.}\, 2\theta + \text{sin.}\, 2\,.\,(I - \theta)\} + \&c.\,;$$

hence we get

[1192] $$\begin{aligned} d\varpi = {} & dI\,.\,\{1 + 2\,\text{tang.}^2 \tfrac{1}{2}\varphi \,.\, \text{cos.}\, 2\,.\,(I - \theta) + \&c.\} \\ & + 2\,d\theta \,.\, \text{tang.}^2 \tfrac{1}{2}\varphi \,.\, \{\text{cos.}\, 2\theta - \text{cos.}\, 2\,.\,(I - \theta) + \&c.\} \\ & + \frac{d\varphi \,.\, \text{tang.}\, \frac{1}{2}\varphi}{\text{cos.}^2 \frac{1}{2}\varphi} \,.\, \{\text{sin.}\, 2\theta + \text{sin.}\, 2\,.\,(I - \theta) + \&c.\} \end{aligned}$$

The values of dI, $d\theta$, and $d\varphi$, having been determined, by what precedes, we shall, from [1192], get the value of $d\varpi$, and then, from [1188], the value of $d\varepsilon$.

Hence it follows, that the expressions in series, of the radius vector, and its projection upon the fixed plane, the longitude of the body in its orbit, or referred to the fixed plane, and the latitude, which we have given in § 22,
[1192'] for the case of the invariable ellipsis, take place also in the variable ellipsis; provided we change nt into $\int ndt$, and determine the elements of the variable ellipsis, by the preceding formulas. For the finite equations between r, v, s, x, y, z, and $\int n\,dt$, are the same in both cases; and the
[1192''] series of § 22, result from these equations by analytical operations, wholly independent of the constancy or variableness of the elements; therefore it is evident, that these expressions also take place when the elements are variable.

When the ellipses are very excentrical, like the orbits of comets, we must alter a little the preceding analysis. The inclination of the orbit to the fixed plane φ, the longitude of its ascending node θ, the semi-transverse axis
[1192'''] a, the semi-parameter $a\,.\,(1 - e^2)$, the excentricity e, and the longitude I of the perihelion, upon the fixed plane, may be found as before. But the

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* (811) The sum of the two expressions [1189, 1190] gives ϖ, [1191], and its differential is [1192]. The values of dI, $d\theta$, $d\varphi$, are found from the differentials of the equations [1178, 1174], substituting the values of the differentials of [1173, 1176]. This value of $d\varpi$ substituted in [1188], and also the value of de, deduced from [1179, 1176], will give the value of $d\varepsilon$.

values of ϖ and $d\varpi$, being given in series arranged according to the powers [1192iv] of tang. $\frac{1}{2}\varphi$, we must, in order to render them converging, make choice of the fixed plane so that tang. $\frac{1}{2}\varphi$ may be very small; and the most simple method of doing this, is to take for the fixed plane, the orbit of m at a given epoch.

The preceding value of $d\varepsilon$ [1188], is expressed by a series, which converges only when the excentricity of the orbit is small; it cannot [1192v] therefore be used in the present case. To find a substitute for it, we shall resume the equation [1185],

$$\frac{dt\,.\sqrt{\mu}}{a^{\frac{3}{2}}}=\frac{dv\,.\,(1-e^2)^{\frac{3}{2}}}{\{1+e\,.\cos.(v-\varpi)\}^2}. \qquad [1193]$$

If we put $1-e=\alpha$, we shall find, by the analysis of § 23, in the invariable [1193′] ellipsis,*

$$t+T=\frac{2a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{\mu}}.\,\mathrm{tang.}\tfrac{1}{2}.(v-\varpi).\left\{1+\frac{(\frac{2}{3}-\alpha)}{2-\alpha}.\,\mathrm{tang.}^2\tfrac{1}{2}.(v-\varpi)-\&\mathrm{c.}\right\} \qquad [1194]$$

T being an arbitrary constant quantity. To apply this equation to the variable ellipsis, we must take its differential, supposing T, the semi-parameter $a.(1-e^2)$, α, and ϖ, to be the only variable quantities. We shall thus have a differential equation, which will determine T; and then the finite equations, [1194′] which take place in the invariable ellipsis, will take place also in the variable ellipsis.

65. We shall now consider particularly the variations of the elements of the orbit of m, when the excentricities of the orbits, and their inclinations to

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* (812) In the equation [690] the angles t, v, are supposed to commence together, but if we suppose t to be equal to $-T$, when $v=\varpi$, the first member of the equation will become $t+T$, and the angle v, in the second member, will become $v-\varpi$. Substituting $D=a\alpha$, [681″], in the factor $\frac{2D^{\frac{3}{2}}}{\sqrt{(2-\alpha).\mu}}$, [690], it becomes

$$\frac{2a^{\frac{3}{2}}.\alpha^{\frac{3}{2}}}{(2-\alpha)^{\frac{1}{2}}.\sqrt{\mu}}=\frac{2a^{\frac{3}{2}}.\alpha^{\frac{3}{2}}.(2-\alpha)^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{\mu}}=\frac{2a^{\frac{3}{2}}.\{\alpha.(2-\alpha)\}^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{\mu}}=\frac{2a^{\frac{3}{2}}.\{(1-e).(1+e)\}^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{\mu}}=\frac{2a^{\frac{3}{2}}.(1-e^2)^{\frac{3}{2}}}{(2-\alpha)^2.\sqrt{\mu}},$$

These substitutions being made in [690], it becomes as in [1194]. We may observe that [1194a] no terms are neglected in § 64, so that the equations of that article are accurate.

each other are small. We have given in § 48 [957] the method of developing the value of R, in this case, by a series of sines and cosines of the form
[1194''] $m'k.\cos.(i'n't - int + A)$, k and A being functions of the excentricities, inclinations of the orbits, positions of the nodes and perihelia, longitudes of the bodies at a given epoch, and transverse axes. When the ellipses are variable, all these quantities must be supposed to vary, in the manner already explained; we must also change, in the preceding quantity, the angle
[1194'''] $i'n't - int$, into $i'\int n'dt - i\int n\,dt$,* or which is the same thing, into $i'\zeta' - i\zeta$.

Now we have, by the preceding article [1177, 1183],

[1195]
$$\frac{\mu}{a} = 2\int \mathrm{d}R\,;$$
$$\zeta = \int n\,dt = \frac{3}{\mu}.\int\int a\,n\,dt.\mathrm{d}R.$$

The differential $\mathrm{d}R$ being taken, supposing only the co-ordinates x, y, z, of
[1195'] the body m to be variable, we must, in the term $m'k.\cos.(i'\zeta' - i\zeta + A)$ of the expression of R, developed in a series, consider as variable, only those quantities which depend on the motion of this body; moreover, R being a finite function† of x, y, z, x', y', z', we may, by § 63, [1167''], suppose the elements of the orbit to be constant, in the differential $\mathrm{d}R$; therefore it will be sufficient to vary ζ, in the preceding term, and as the differential of ζ is
[1195''] ndt [1180''], we shall have $im'.k.n\,dt.\sin.(i'\zeta' - i\zeta + A)$, for the part of $\mathrm{d}R$, corresponding to the preceding term of R; and if we notice only this term, we shall have [1195]

[1196]
$$\frac{1}{a} = \frac{2\,im'}{\mu}.\int k.n\,dt.\sin.(i'\zeta' - i\zeta + A)\,;$$
$$\zeta = \frac{3\,im'}{\mu}.\int\int ak.n^2dt^2.\sin.(i'\zeta' - i\zeta + A).$$

[1195a] * (813) This change of the angle $i'n't - int$, appears evident, by comparing the value of R, [951], with that of [957]; it being easy to perceive that the last value would more accurately conform to the first, and to the principles above explained, by making this substitution.

† (814) This follows from [913, 914]. The differential of R, being of the first order, we may, in finding it, suppose the arbitrary quantities a, e, &c., to be constant, conformably to [1167''].

If we neglect the squares and products of the disturbing masses, we may, in the integration of these terms, suppose the elements of the elliptical motion [1196'] to be constant, which will change ζ into nt, and ζ' into $n't$; and we shall obtain

$$\frac{1}{a} = -\frac{2im'.n.k}{\mu.(i'n'-in)}.\cos.(i'n't-int+A);$$

$$\zeta = -\frac{3im'.an^2.k}{\mu.(i'n'-in)^2}.\sin.(i'n't-int+A).$$ [1197]

Great inequalities of the transverse axis and mean motion.

Hence we see, that if $i'n'-in$ does not vanish, the quantities a and ζ will contain only periodical inequalities, provided we notice only the first power of the disturbing force;† now i' and i being integral numbers, the equation [1197'] $i'n'-in=0$, cannot take place if the mean motions of m and m' be incommensurable, which is the case with the planets, and may be admitted generally, since n and n' are arbitrary constant quantities, susceptible of all [1197''] possible values, and the supposition that this ratio can be exactly defined in whole numbers, is in the highest degree improbable.

We are therefore led to this remarkable result, that the transverse axes of the orbits of the planets, and their mean motions, are subjected only to periodical inequalities, depending on their mutual configuration, and by [1197'''] neglecting such quantities, these axes will be constant, and the mean motions will be uniform; this result is conformable to that we have found, by another method, in § 54 [1070^{xi}].

If the mean motions nt, $n't$, without being exactly commensurable, approach very nearly to the ratio of i' to i, the divisor $i'n'-in$, will be [1197^{iv}] very small, and there may result in ζ and ζ' some inequalities, which vary so slowly, that observers may be induced to suppose the mean motions of the two bodies m and m' not to be uniform. We shall see in the theory of Jupiter and Saturn, that this has happened relative to these two planets: their mean motions are such, that twice that of Jupiter is nearly equal to five times that of Saturn; so that $5n'-2n$ is but the seventy-fourth part [1197^{v}] of n [3818d]. The smallness of this divisor renders the term of ζ, depending upon the angle $5n't-2nt$, very sensible, although it is of the order $i'-i$,

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† (814a) This is true, even if we include the second power, and some terms of the third power, of the disturbing masses, as has been observed in [1070a].

[1197vi] or of the third order relative to the excentricities and inclinations of the orbits, as we have seen in § 48 [957viii]. The preceding analysis gives the most important part of these inequalities. For the variation of the mean longitude depends on two integrations, whilst the variations of the other elements of the elliptical motion depend but on one integration; [1197vii] therefore the terms of the mean longitude only can have the square $(i'n'-in)^2$ for a divisor; so that if we notice only those terms which, on account of the smallness of this divisor, ought to be the most considerable, [1197viii] it will suffice, in the expressions of the radius vector, the longitude and the latitude, to increase the mean longitude by these terms.*

When we have found the inequalities of this kind, which the action of m' [1197ix] produces in the mean motion of m, it will be easy to deduce the corresponding inequalities produced by the action of m in the mean motion of m'. For, if we notice only the mutual action of the three bodies M, m, and m', the formula [442] will give,†

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* (815) This also follows from the equations [1066, 1070], as well as from the method here used.

† (816) Dividing the equation [442] by $M+\Sigma.m$, or $M+m+m'$, we get

$$\frac{h}{M+\Sigma.m}=\frac{M}{M+\Sigma.m}.\Sigma.m.\frac{dx^2+dy^2+dz^2}{dt^2}$$
$$+\frac{\Sigma.mm'.\left\{\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2}\right\}}{M+\Sigma.m}-2M.\Sigma.\frac{m}{r}-2\lambda,$$

and as $\frac{M}{M+\Sigma.m}=1-\frac{\Sigma.m}{M+\Sigma.m}=1-\frac{(m+m')}{M+\Sigma.m}$, it becomes

$$\frac{h}{M+\Sigma.m}=\Sigma.m.\frac{dx^2+dy^2+dz^2}{dt^2}$$
$$+\frac{\Sigma.mm'.\{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2\}-(m+m').\Sigma.m.(dx^2+dy^2+dz^2)}{(M+m+m').dt^2}$$
$$-2M.\Sigma.\frac{m}{r}-2\lambda.$$

The term of the second member, having the denominator $(M+m+m').dt^2$, may be put under a more simple form, since the terms of the numerator, depending on x, x', are $mm'.(dx'-dx)^2-(m+m').(m.dx^2+m'.dx'^2)$, which by reduction become

$$\text{constant} = m \,.\, \frac{(d x^2 + d y^2 + d z^2)}{d t^2} + \frac{m' . (d x'^2 + d y'^2 + d z'^2)}{d t^2}$$
$$- \frac{\{(m\, d x + m'\, d x')^2 + (m\, d y + m'\, d y')^2 + (m\, d z + m'\, d z')^2\}}{(M + m + m') \,.\, d t^2} \qquad (a)$$ [1198]
$$- \frac{2 M m}{\sqrt{x^2+y^2+z^2}} - \frac{2 M m'}{\sqrt{x'^2+y'^2+z'^2}} - \frac{2 m m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} .$$

The last of the integrals (p) [1171] of the preceding article, substituting for $\frac{\mu}{a}$ the integral $2 \int \mathrm{d} R$ [1195], gives*

$$\frac{d x^2 + d y^2 + d z^2}{d t^2} = \frac{2 . (M + m)}{\sqrt{x^2+y^2+z^2}} - 2 \int \mathrm{d} R .$$ [1199]

If we put R' for what R becomes, when we consider the action of m upon m', we shall have [1199']

$$R' = \frac{m . (x x' + y y' + z z')}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{m}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}} ;$$ [1200]
$$\frac{d x'^2 + d y'^2 + d z'^2}{d t^2} = \frac{2 . (M + m')}{\sqrt{x'^2+y'^2+z'^2}} - 2 \int \mathrm{d}' R' ;$$

the differential characteristic d' affects only the co-ordinates x', y', z', of the body m'. Substituting these values of $\frac{d x^2 + d y^2 + d z^2}{d t^2}$, $\frac{d x'^2 + d y'^2 + d z'^2}{d t^2}$, in the equation (a) [1198], we shall have† [1200'] Symbol d'

$-(m . d x + m' . d x')^2$, and as the numerator is symmetrical in x, x', y, y', z, z', the whole numerator will be $-(m . d x + m' . d x')^2 - (m . d y + m' . d y')^2 - (m . d z + m' . d z')^2$; substituting this, and putting $\lambda = \frac{m m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}$, [397], it becomes as in [1198].

* (817) Substituting $\frac{\mu}{a} = 2 \int \mathrm{d} R$, and $\mu = M + m$, [530$^{\text{iv}}$], in the last of the equations [1171], we shall get [1199]. The value of R, [949], changing the terms relative to m', into those relative to m, and the contrary, becomes the same as R', [1200]. Similar changes being made in [1199], we shall get the second equation [1200].

† (818) After substituting the values of $\frac{d x^2 + d y^2 + d z^2}{d t^2}$, $\frac{d x'^2 + d y'^2 + d z'^2}{d t^2}$, transposing the terms $2 m . \int \mathrm{d} R$, $2 m' . \int \mathrm{d}' R'$, dividing by 2 and reducing, the expression becomes as in [1201]; the last term, depending on m, m', was accidentally omitted in the original work.

[1201]
$$m.\int \mathrm{d}R+m'.\int \mathrm{d}'R'=\text{constant}-\frac{(m\,dx+m'dx')^2+(m\,dy+m'dy')^2+(m\,dz+m'dz')^2}{2.(M+m+m').dt^2}$$
$$+\frac{m^2}{\sqrt{x^2+y^2+z^2}}+\frac{m'^2}{\sqrt{x'^2+y'^2+z'^2}}-\frac{m\,m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}.$$

It is evident that the second member of this equation contains no terms of the order of the squares and products of the masses m and m', which [1201'] has $i'n'-in$ for a divisor;* and by noticing only these terms, we shall have

[1202]
$$m.\int \mathrm{d}R+m'.\int \mathrm{d}'R'=0\,;$$

therefore, by considering only the terms, which have $(i'n'-in)^2$ for a divisor, we shall get†

[1203]
$$\frac{3\iint a'n'dt.\mathrm{d}'R'}{M+m'}=-\frac{m.(M+m).a'n'}{m'.(M+m').an}\cdot\frac{3\iint a\,n\,dt.\mathrm{d}R}{M+m}\,;$$

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* (819) The terms having $i'n'-in$ for a divisor [1066, 1070, &c.], must be those arising from the disturbing force, and they will therefore be of the order m or m'; that is, the parts of x, y, z, dx, dy, dz, x', &c., depending on such angles, must be of the order m or m'; these parts being substituted in the second member of the equation [1201], will produce terms of the third order, as it respects the powers and products of the masses m, m'; and by noticing only terms of the second power, we may put that second member equal to nothing, as in [1202].

† (820) R, [949], is of the order m', hence $m.\iint \mathrm{d}R.dt$, is of the order $m\,m'$, and the like is to be said of $m'.\iint \mathrm{d}'R.dt$, and if we neglect terms of a higher order, we may, from [1196'], write $\frac{m.\iint an.dt.\mathrm{d}R}{an}$, for $m.\iint dt.\mathrm{d}R$, and $\frac{m'.\iint a'n'.dt.\mathrm{d}'R'}{a'n'}$, for $m'.\iint dt.\mathrm{d}'R'$. Multiplying the expression [1202] by $3\,dt$, integrating, and making the preceding substitutions we shall get,

$$\frac{3m.\iint an.dt.\mathrm{d}R}{an}+\frac{3m'.\iint a'n'.dt.\mathrm{d}'R'}{a'\,n'}=0.$$

The constant quantity of the second member is put equal to nothing, because no terms, except those depending on the angle, $i'n't-int$ are here noticed. Multiplying the numerator and denominator of the first of these terms by $M+m$, and those of the second by $M+m'$, we shall obtain, by reduction, the formula [1203]. The equations [1204] are deduced from the second equation [1195], putting successively, $\mu'=M+m$, $\mu=M+m'$. Substituting [1204] in [1203], we get $\zeta'=-\frac{m.(M+m).a'n'}{m'.(M+m').an}.\zeta$, from which we easily now we have

$$\zeta = \frac{3\int\int a\,n\,dt\,.\,\mathrm{d}R}{M+m}\,; \qquad \zeta' = \frac{3\int\int a'n'dt\,.\,\mathrm{d}'R'}{M+m'}\,; \qquad [1204]$$

therefore we shall obtain

$$m'\,.\,(M+m')\,.\,an\,.\,\zeta' + m\,.\,(M+m)\,.\,a'n'\,.\,\zeta = 0. \qquad [1205]$$

We then have [605′]

$$n = \frac{\sqrt{M+m}}{a^{\frac{3}{2}}}\,; \qquad n' = \frac{\sqrt{M+m'}}{a'^{\frac{3}{2}}}\,; \qquad [1206]$$

neglecting therefore m and m', in comparison with M, we shall find

$$m\,.\,\sqrt{a}\,.\,\zeta + m'\,.\,\sqrt{a'}\,.\,\zeta' = 0\,; \qquad [1207]$$

or

$$\zeta' = -\,\frac{m\,.\,\sqrt{a}}{m'.\,\sqrt{a'}}.\zeta. \qquad [1208]$$

Therefore the inequalities of ζ, which have $(i'n' - in)^2$ for a divisor, will give those of ζ' which have the same divisor. These inequalities will [1208′] evidently be affected with contrary signs, if n and n' have the same sign; or, which is the same thing, if both bodies m, m', revolve in the same direction;* they are, moreover, in a constant ratio to each other: hence it follows, that if these inequalities appear to accelerate the mean motion of m, they will appear to retard that of m', according to the same law; and the apparent acceleration of m, will be to the apparent retardation of m', as [1208″]

LaPlace's Theorem on the inequalities of a long period, like those of Jupiter and Saturn.

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obtain [1205]; now $a\,n = \frac{\sqrt{M+m}}{a^{\frac{1}{2}}}$, $a'\,n' = \frac{\sqrt{M+m'}}{a'^{\frac{1}{2}}}$, [1206], these being substituted in [1205], it becomes $m'\,.\,(M+m')\,.\,\frac{\sqrt{M+m}}{a^{\frac{1}{2}}}\,.\,\zeta' + m\,.\,(M+m)\,.\,\frac{\sqrt{M+m'}}{a'^{\frac{1}{2}}}\,.\,\zeta = 0$, or by reduction $m'\,.\,\sqrt{M+m'}\,.\,\sqrt{a'}\,.\,\zeta' + m\,.\,\sqrt{M+m}\,.\,\sqrt{a}\,.\,\zeta = 0$, and if we neglect m and m' in comparison with M, we may divide it by $\sqrt{M+m}$, or $\sqrt{M+m'}$, and we shall get [1207], from which [1208] is easily deduced. This beautiful theorem is frequently used by the author, particularly in the third volume, where it is applied, without restriction, to all terms of the order of the square of the disturbing forces; which has been objected to by M. Plana, in a paper published in Vol. II, of the Memoirs of the Astronomical Society of London, as will be more particularly stated in the notes upon that part of the third volume of this work.

* (821) Conformably to note 746.

$m'.\sqrt{a'}$ to $m.\sqrt{a}$. The acceleration of the mean motion of Jupiter, and
the retardation of that of Saturn, discovered by Halley, in comparing ancient
and modern observations, are very nearly in this ratio; and I inferred, from
[1208'''] the preceding theorem, that they were produced by the mutual action of
those planets upon each other; and since it is evident, that this action cannot
produce any alteration in the mean motions, independent of the configuration
of the planets, I did not hesitate in the belief, that there must exist, in the
theory of Jupiter and Saturn, an important equation of a very long period.
[1208^{iv}] Taking also into consideration, that five times the mean motion of Saturn,
minus twice that of Jupiter, is nearly equal to nothing [3818*d*]; it appeared
to me, to be highly probable, that the phenomenon observed by Halley, was
produced by an equation, depending on this argument. The calculation of
this equation proved the conjecture to be true.

[1208^{v}] The period of the argument $i'n't-int$ being supposed very long, the
elements of the orbits of m and m' vary so much during this interval, that it
is necessary to notice this circumstance in the double integral

$$\int\int ak.n^2dt^2.\sin.(i'n't-int+A).$$

For this purpose, we shall put the function $k.\sin.(i'n't-int+A)$ under
[1208^{vi}] the form* $Q.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)+Q'.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)$;

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* (822) In [1194''] the general term of R is put equal to $m'k.\cos.(i'n't-int+A)$. Comparing this with the expressions [958, 961], we shall get

$$A=i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta-g'''\theta',$$

and k equal to a function of a, a', e, e', φ, φ'. If for brevity we put

$$N=i'n't-int+i'\varepsilon'-i\varepsilon, \qquad N'=g\varpi+g'\varpi'+g''\theta+g'''\theta',$$

the corresponding term of $k.\sin.(i'n't-int+A)$, will be

$$k.\sin.(N-N')=k.\{\sin.N.\cos.N'-\cos.N.\sin.N'\},$$

and if we substitute $Q=k.\cos.N'$, $Q'=-k.\sin.N'$, it will become

$$Q.\sin.N+Q'.\cos.N,$$

as in [1208^{vi}], Q, Q', being functions of a, a', e, e', φ, φ', &c. During the period of the
argument $i'n't-int$, the quantities e, e', φ, φ', &c., will vary by reason of the secular
[1209*a*] inequalities; therefore, Q, Q', must vary for the same cause; a, a', n, n', not being liable
to such variations [1044', &c.].

Q and Q' being functions of the elements of the orbits; we shall then have

$$\int\!\!\int a\,k\,.\,n^2\,d\,t^2\,.\,\sin.\,(i'n't-i\,n\,t+A)=$$

$$-\frac{n^2a.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)}{(i'n'-in)^2}.\left\{Q-\frac{2\,d\,Q'}{(i'n'-in).dt}-\frac{3\,d\,d\,Q}{(i'n'-in)^2.dt^2}+\frac{4\,d^3Q'}{(i'n'-in)^3.dt^3}+\&c.\right\} \qquad [1209]$$

$$-\frac{n^2a.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)}{(i'n'-in)^2}.\left\{Q'+\frac{2\,d\,Q}{(i'n'-in).dt}-\frac{3\,d\,d\,Q'}{(i'n'-in)^2.dt^2}-\frac{4\,d^3Q}{(i'n'-in)^3.dt^3}+\&c.\right\}$$

If we substitute the value of $k\,.\,\sin.\,(i'\,n'\,t-i\,n\,t+\mathcal{A})$, in the first member of [1209] we shall get

$$\int\!\!\int a\,k\,n^2\,.\,d\,t^2\,.\,\sin.\,(i'\,n'\,t-i\,n\,t+\mathcal{A})=\int\!\!\int a\,n^2\,.\,d\,t^2\,.\,\{Q\,.\,\sin.\,\mathcal{N}+Q'\,.\,\cos.\,\mathcal{N}\}.$$

And it easy to prove, by integrating by parts, that if $\mathcal{A}$, B, are any functions of t, we shall have

$$\int\!\!\int \mathcal{A}\,B\,.\,d\,t^2=\mathcal{A}\int^2 B\,.\,d\,t^2-2\,.\,\frac{d\,\mathcal{A}}{dt}\,.\int^3 B\,.\,dt^3+3\,.\,\frac{d^2\,\mathcal{A}}{d\,t^2}\,.\int^4 B.dt^4-4\,.\,\frac{d^3\,\mathcal{A}}{d\,t^3}\,.\int^5 B\,.\,d\,t^5+\&c. \qquad [1209b]$$

For if we take the differential of this equation, and connect the similar terms of the second member we shall get

$$\int \mathcal{A}\,B\,.\,d\,t^2=\mathcal{A}\int B\,.\,d\,t^2-\frac{d\,\mathcal{A}}{dt}\,.\int^2 B\,.\,d\,t^3+\frac{d^2\,\mathcal{A}}{d\,t^2}\,.\int^3 B\,.\,d\,t^4-\&c., \qquad [1209c]$$

$d\,t$ being constant. Again, taking the differential, all the terms of the second member will be destroyed, except the first term, $\mathcal{A}\,B\,.\,d\,t^2$, which is the same as in the first member.

Putting in this formula $\mathcal{A}=Q$, $B=a\,n^2\,.\,\sin.\,\mathcal{N}$, it becomes

$$\int\!\!\int a\,n^2\,.\,d\,t^2\,.\,Q\,.\,\sin.\,\mathcal{N}=Q\,.\int^2 a\,n^2\,.\,d\,t^2\,.\,\sin.\,\mathcal{N}$$
$$-\frac{2\,d\,Q}{dt}\,.\int^3 a\,n^2\,.\,d\,t^3\,.\,\sin.\,\mathcal{N}+\frac{3\,d^2\,Q}{d\,t^2}\,.\int^4 a\,n^2\,.\,d\,t^4\,.\,\sin.\,\mathcal{N}-\&c.$$

Taking the integrals of the second member, it becomes equal to

$$-Q\,.\,\frac{a\,n^2.\sin.\mathcal{N}}{(i'n'-in)^2}-\frac{2\,d\,Q}{dt}\,.\,\frac{a\,n^2.\cos.\mathcal{N}}{(i'n'-in)^3}+\frac{3\,d^2\,Q}{d\,t^2}\,.\,\frac{a\,n^2.\sin.\mathcal{N}}{(i'n'-in)^4}+\frac{4\,d^3\,Q}{d\,t^3}\,.\,\frac{a\,n^2.\cos.\mathcal{N}}{(i'n'-i\,n)^5}+\&c.$$

In like manner, putting $\mathcal{A}=Q'$, $B=a\,n^2\,.\,\cos.\,\mathcal{N}$, we shall obtain

$$\int\!\!\int a\,n^2.dt^2.Q'.\cos.\,\mathcal{N}=Q'.\int^2 a\,n^2\,.\,d\,t^2.\cos.\,\mathcal{N}-\frac{2\,d\,Q'}{dt}\,.\int^3 a\,n^2\,.\,d\,t^3\,.\,\cos.\,\mathcal{N}+\&c.$$
$$=-Q'\,.\,\frac{a\,n^2\,.\cos.\mathcal{N}}{(i'n'-in)^2}+\frac{2\,d\,Q'}{dt}\,.\,\frac{a\,n^2.\sin.\mathcal{N}}{(i'n'-in)^3}+\frac{3\,d^2\,Q'}{d\,t^2}\,.\,\frac{a\,n^2.\cos.\mathcal{N}}{(i'n'-i\,n)^4}-\&c.;$$

adding these expressions, and connecting the terms depending on sin. $\mathcal{N}$, also those depending on cos. $\mathcal{N}$, we shall have the value of $\int\!\!\int a\,k\,n^2\,.\,d\,t^2\,.\,\sin.\,(i'\,n'\,t-i\,n\,t+\mathcal{A})$, as in the second member of [1209].

As the terms of these two series decrease very rapidly, on account of the slowness of the secular variations of the elliptical elements, we need only retain the two first terms of each series. Then substituting the values of the elements, arranged according to the powers of the time, and retaining
[1209'] only the first power; the preceding double integral may be transformed into one single term of the form*

[1210] $$(F+E\,.\,t)\,.\,\sin.\,(i'n't-int+A+H\,.\,t).$$

[1210'] With respect to Jupiter and Saturn, this expression will serve for several centuries before and after the time selected for the epoch.

The great inequalities we have just mentioned, produce similar ones
[1210''] among the terms depending upon the second power of the disturbing masses. For, if in the formula [1196],

[1211] $$\zeta=\frac{3\,i\,m'}{\mu}\,.\int\!\!\int a\,k\,.\,n^2\,dt^2\,.\,\sin.\,(i'\zeta'-i\zeta+A),$$

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* (823) The terms Q, $\frac{dQ}{dt}$, &c., vary very slowly, and their values may be arranged in a series, proceeding according to the powers of the time, $D+D't+D''t^2+\&c.$ If we retain only the first power of t, the coefficients of the sine and cosine of

$$(i'n't-int+i'\varepsilon'-i\varepsilon),$$

in the second member of [1209], may be put under the forms $E'+E''t$, $F'+F''t$, E'' and F'' being very small in comparison with E', F', so that

$$\int\!\!\int a\,k\,n^2\,.\,\sin.\,(i'n't-int+\mathcal{A})=(E'+E''t)\,.\,\sin.\,\mathcal{N}+(F'+F''t)\,.\,\cos.\,\mathcal{N}.$$

If we now put $F'=F\,.\,\sin.\,\mathcal{A}'$, $E'=F\,.\,\cos.\,\mathcal{A}'$, $E''=E\,.\,\cos.\,\mathcal{A}'-FH\,.\,\sin.\,\mathcal{A}'$, $F''=E\,.\,\sin.\,\mathcal{A}'+FH\,.\,\cos.\,\mathcal{A}'$, the preceding expression will become

$$\{F.\cos.\mathcal{A}'+Et.\cos.\mathcal{A}'-FHt.\sin.\mathcal{A}'\}.\sin.\mathcal{N}+\{F.\sin.\mathcal{A}'+Et.\sin.\mathcal{A}'+FHt.\cos.\mathcal{A}'\}.\cos.\mathcal{N},$$

or, as it may be written,

$$(F+Et)\,.\,\{\cos.\mathcal{A}'.\sin.\mathcal{N}+\sin.\mathcal{A}'.\cos.\mathcal{N}\}+FHt\,.\,\{-\sin.\mathcal{A}'.\sin.\mathcal{N}+\cos.\mathcal{A}'.\cos.\mathcal{N}\}$$
$$=(F+Et)\,.\,\sin.\,(\mathcal{N}+\mathcal{A}')+FHt\,.\,\cos.\,(\mathcal{N}+\mathcal{A}'),$$

[21, 23] Int. In this last term we may write $F+Et$, for F, neglecting terms of the order t^2, and then it will become

$$(F+Et)\,.\,\{\sin.\,(\mathcal{N}+\mathcal{A}')+Ht\,.\,\cos.\,(\mathcal{N}+\mathcal{A}')\}=(F+Et)\,.\,\sin.\,(\mathcal{N}+\mathcal{A}'+Ht),$$

[60] Int. If we neglect E, H, this ought to agree with ζ, [1197], which would give

$$\mathcal{N}+\mathcal{A}'=i'n't-int+\mathcal{A},$$

and the preceding expression would become as in [1210].

we substitute for ζ and ζ' their values,*

$$n\,t - \frac{3i\,.\,m'an^2\,.\,k}{\mu\,.\,(i'n' - in)^2}\,.\,\sin.\,(i'n't - int + A)\,;$$

$$n'\,t + \frac{3i\,.\,m\,a\,n^2\,.\,k}{\mu\,.\,(i'n' - in)^2}\,.\,\frac{\sqrt{a}}{\sqrt{a'}}\,.\,\sin.\,(i'n't - int + A)\,; \qquad [1212]$$

there will arise, among the terms of the order m^2, the following :†

$$-\frac{9i^2\,.\,m'^2\,.\,a^2n^4\,.\,k^2}{8\,\mu^2\,.\,(i'n' - in)^4}\,.\,\frac{\{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}\}}{m'\,.\sqrt{a'}}\,.\,\sin.\,2\,.\,(i'n't - int + A). \qquad [1213]$$

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* (824) It follows from [1197], that the value of ζ, which is $n\,t$, when the elliptical elements are constant, becomes as in the first formula, [1212], when they vary by terms depending on the angle $(i'n't - int + \mathcal{A})$. The corresponding variation of ζ', or $n'\,t$, [1212], is found by means of [1208], multiplying the value of the decrement of ζ, [1197] by $-\frac{m\,.\sqrt{a}}{m'\,.\sqrt{a'}}$.

† (825) Putting for brevity $i'n't - int + \mathcal{A} = \mathcal{A}'$, $\frac{3i\,.\,m'an^2\,.\,k}{\mu\,.\,(i'n' - in)^2} = b$, the expressions [1212] will become $\zeta = nt - b\,.\,\sin.\,\mathcal{A}'$; $\zeta' = n't + b\,.\,\frac{m\,.\sqrt{a}}{m'\,.\sqrt{a'}}\,.\,\sin.\,\mathcal{A}'$. These values being substituted in $i'\zeta' - i\zeta + \mathcal{A}$, it becomes

$$i'\zeta' - i\zeta + \mathcal{A} = i'n't - int + \mathcal{A} + \frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{m'\,.\sqrt{a'}}\,.\,b\,.\,\sin.\,\mathcal{A}'$$
$$= \mathcal{A}' + \frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{m'\,.\sqrt{a'}}\,.\,b\,.\,\sin.\,\mathcal{A}',$$

and as the part depending on b is very small, we shall get, by [60] Int.,

$$\sin.\,(i'\zeta' - i\zeta + \mathcal{A}) = \sin.\,\mathcal{A}' + \left\{\frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{m'\,.\sqrt{a'}}\,.\,b\,.\,\sin.\,\mathcal{A}'\right\}.\,\cos.\,\mathcal{A}'$$
$$= \sin.\,\mathcal{A}' + \frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{2m'\,.\sqrt{a'}}\,.\,b\,.\,\sin.\,2\,\mathcal{A}',$$

substituting this in [1211], we shall find

$$\zeta = \frac{3im'}{\mu}\,.\iint a\,k\,n^2\,.\,dt^2\,.\left\{\sin.\,\mathcal{A}' + \frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{2m'\,.\sqrt{a'}}\,.\,b\,.\,\sin.\,2\mathcal{A}\right\}.$$

In the part depending on b, we must resubstitute the values of $\mathcal{A}$, b, and it will become

$$\frac{9\,i^2\,.\,m'^2\,.\,a^2\,.\,n^4\,.\,k^2}{\mu^2\,.\,(i'n' - in)^2}\,.\,\frac{i\,.\,m'\,.\sqrt{a'} + i'\,.\,m\,.\sqrt{a}}{2m'\,.\sqrt{a'}}\,.\iint dt^2\,.\,\sin.\,2\,.\,(i'n't - int + \mathcal{A}),$$

and the integrals being taken we shall get [1213]. The corresponding term of ζ' is found, as in [1208], from multiplying the preceding expression by $-\frac{m\,.\sqrt{a}}{m'\,.\sqrt{a'}}$, which gives [1214].

The value of ζ' contains a corresponding term, which is to the preceding, in
[1213'] the ratio of $m.\sqrt{a}$ to $-m'.\sqrt{a'}$,

[1214] $$\frac{9i^2.m'^2.a^2n^4.k^2}{8\mu^2.(i'n'-in)^4}.\{i.m'.\sqrt{a'}+i'.m.\sqrt{a}\}.\frac{m.\sqrt{a}}{m'^2.a'}.\sin.2.(i'n't-int+A).$$

66. It may happen, that the most remarkable equations of the mean
motions will occur only in terms of the order of the square of the disturbing
[1214'] forces. If we take into consideration three bodies, m, m', m'', revolving
about M; the expression of dR, as it respects terms of that order, will
contain equations of the form $k.\sin.(int-i'n't+i''n''t+A)$; now if
[1214''] we suppose the mean motions nt, $n't$, $n''t$, to be such, that $in-i'n'+i''n''$
is an extremely small fraction of n, there will result a very sensible equation
[1214'''] in the value of ζ. This equation may even make the quantity $in-i'n'+i''n''$
rigorously vanish, which will establish an equation of condition between the
mean motions, and the mean longitudes, of the three bodies m, m', m''. As
[1214iv] this very singular case occurs, in the system of the satellites of Jupiter, we
shall here investigate the analytical expression of this equation.

[1214v] If we put M for the unity of mass, and neglect m, m', m'', in comparison
with M, we shall have [1206]

[1215] $$n^2=\frac{1}{a^3}; \qquad n'^2=\frac{1}{a'^3}; \qquad n''^2=\frac{1}{a''^3}.$$

We shall also have [1195'']

[1216] $$d\zeta=n\,dt; \qquad d\zeta'=n'\,dt; \qquad d\zeta''=n''\,dt;$$

therefore*

[1217] $$\frac{dd\zeta}{dt}=-\tfrac{3}{2}.n^{\frac{1}{3}}.\frac{da}{a^2}; \qquad \frac{dd\zeta'}{dt}=-\tfrac{3}{2}.n'^{\frac{1}{3}}.\frac{da'}{a'^2}; \qquad \frac{dd\zeta''}{dt}=-\tfrac{3}{2}.n''^{\frac{1}{3}}.\frac{da''}{a''^2}.$$

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* (826) The differential of $d\zeta=n\,dt$, [1216], is $dd\zeta=dn.dt$; and the differential of $n=a^{-\frac{3}{2}}$, [1215], is

$$dn=-\tfrac{3}{2}a^{-\frac{5}{2}}.da=-\tfrac{3}{2}a^{-\frac{1}{2}}.\frac{da}{a^2}=-\tfrac{3}{2}n^{\frac{1}{3}}.\frac{da}{a^2};$$

hence $dd\zeta=-\frac{3}{2}n^{\frac{1}{3}}.\frac{da}{a^2}$, as in [1217]; the values of $dd\zeta'$, $dd\zeta''$, are found in the same manner, by merely accenting the letters.

We have seen in § 61 [1161, 1161′], that if we notice only equations of a long period, we shall have [1217′]

$$\text{constant} = \frac{m}{a} + \frac{m'}{a'} + \frac{m''}{a''}; \qquad [1218]$$

which gives, by taking the differential,

$$0 = m \,.\, \frac{d\,a}{a^2} + m' \,.\, \frac{d\,a'}{a'^2} + m'' \,.\, \frac{d\,a''}{a''^2}. \qquad [1219]$$

We have seen in the same article, that if we neglect the squares of the excentricities, and of the inclinations of the orbits, we shall have [1152′]

$$\text{constant} = m \,.\, \sqrt{a} + m' \,.\, \sqrt{a'} + m'' \,.\, \sqrt{a''}; \qquad [1220]$$

hence, taking the differential,

$$0 = m \,.\, \frac{d\,a}{\sqrt{a}} + m' \,.\, \frac{d\,a'}{\sqrt{a'}} + m'' \,.\, \frac{d\,a''}{\sqrt{a''}}. \qquad [1221]$$

From these equations it is easy to deduce*

$$\begin{aligned} \frac{d\,d\,\zeta}{d\,t} &= -\tfrac{3}{2} \,.\, n^{\frac{1}{3}} \,.\, \frac{d\,a}{a^2}; \\ \frac{d\,d\,\zeta'}{d\,t} &= \tfrac{3}{2} \,.\, \frac{m \,.\, n'^{\frac{4}{3}}}{m' \,.\, n} \,.\, \frac{(n - n'')}{(n' - n'')} \,.\, \frac{d\,a}{a^2}; \\ \frac{d\,d\,\zeta''}{d\,t} &= -\tfrac{3}{2} \,.\, \frac{m \,.\, n''^{\frac{4}{3}}}{m'' \,.\, n} \,.\, \frac{(n - n')}{(n' - n'')} \,.\, \frac{d\,a}{a^2}. \end{aligned} \qquad [1222]$$

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* (828) The first of the equations [1222] is the same as the first of [1217]. Multiply the formula [1221] by $-\frac{1}{a''^{\frac{3}{2}}}$, and add it to [1219], we shall get

$$0 = m \,.\, \frac{d\,a}{a^2} \,.\, \left(1 - \frac{a^{\frac{3}{2}}}{a''^{\frac{3}{2}}}\right) + m' \,.\, \frac{d\,a'}{a'^2} \,.\, \left(1 - \frac{a'^{\frac{3}{2}}}{a''^{\frac{3}{2}}}\right); \qquad [1222a]$$

now $a^{\frac{3}{2}} = \frac{1}{n}$, $a'^{\frac{3}{2}} = \frac{1}{n'}$, $a''^{\frac{3}{2}} = \frac{1}{n''}$, [1215]; substituting these we get

$$0 = m \,.\, \frac{d\,a}{a^2} \,.\, \left(1 - \frac{n''}{n}\right) + m' \,.\, \frac{d\,a'}{a'^2} \,.\, \left(1 - \frac{n''}{n'}\right),$$

hence $\frac{d\,a'}{a'^2} = -\frac{m}{m'} \,.\, \frac{n'}{n} \,.\, \frac{n - n''}{n' - n''} \,.\, \frac{d\,a}{a^2}$; and if we substitute this in the second of the equations [1217], we shall get the second of [1222]. The equations [1217, 1219, 1221], being symmetrical, as it respects the elements of the orbits of m', m'', we may change, in the equation just found, m', n', into m'', n'', and the contrary, to get the last equation [1222].

Lastly, the equation $\frac{\mu}{a} = 2 \int \mathrm{d} R$, § 64 [1177], gives, by taking its differential,*

[1223] $$-\frac{d\,a}{a^2} = 2\,\mathrm{d}\,R.$$

It now remains to determine $\mathrm{d}R$.

[1223′] We have in § 46, neglecting the squares and products of the inclinations of the orbits,†

[1224] $$R = \frac{m'.r}{r'^2}.\cos.(v'-v) - m'.\{r^2 - 2\,r\,r'.\cos.(v'-v) + r'^2\}^{-\frac{1}{2}}$$
$$+\frac{m''.r}{r''^2}.\cos.(v''-v) - m''.\{r^2 - 2\,r\,r''.\cos.(v''-v) + r''^2\}^{-\frac{1}{2}}.$$

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[*] (829) The equations [1217—1222], are defective in terms of the order of the square of the disturbing forces, but [1216, 1223] are correct.

[†] (830) Neglecting the squares and products of z, z', z'', in [913, 914], we shall get

[1224a] $$R = \frac{m'.(x x' + y y')}{(x'^2 + y'^2)^{\frac{3}{2}}} + \frac{m''.(x x'' + y y'')}{(x''^2 + y''^2)^{\frac{3}{2}}} - \frac{m'}{\{(x'-x)^2 + (y'-y)^2\}^{\frac{1}{2}}}$$
$$-\frac{m''}{\{(x''-x)^2 + (y''-y)^2\}^{\frac{1}{2}}} - \frac{m'\,m''}{m.\{(x''-x')^2 + (y''-y')^2\}^{\frac{1}{2}}}.$$

Substituting the values of x, y, x', y', [950], and the similar values $x'' = r''.\cos.v''$, $y'' = r''.\sin.v''$, which give

[1224b] $$x\,x' + y\,y' = r\,r'.\cos.(v'-v); \qquad x\,x'' + y\,y'' = r\,r''.\cos.(v''-v);$$
$$(x'-x)^2 + (y'-y)^2 = r\,r - 2\,r\,r'.\cos.(v'-v) + r'^2; \qquad [950a],$$
$$(x''-x)^2 + (y''-y)^2 = r\,r - 2\,r\,r''.\cos.(v''-v) + r''^2;$$
$$x'^2 + y'^2 = r'^2; \qquad x''^2 + y''^2 = r''^2.$$

This value of R will become like that in [1224], with the addition of the term arising from $-\frac{m'\,m''}{m.\{(x''-x')^2 + (y''-y')^2\}^{\frac{1}{2}}}$, which term may however be neglected, because this value of R is only used for finding $\mathrm{d}\,R$, and the characteristic d does not affect x', x'', y', y'', so that the result of this term in [1226] will be nothing. Now if we use a notation, similar to that in [1077], putting

[1225a] $$\frac{r}{r'^2}.\cos.(v'-v) - \{r^2 - 2\,r\,r'.\cos.(v'-v) + r'^2\}^{-\frac{1}{2}}$$
$$= \tfrac{1}{2}(r,r')^{(0)} + (r,r')^{(1)}.\cos.(v'-v) + (r,r')^{(2)}.\cos.2.(v'-v) + \&c.;$$

using also a precisely similar expression in r'', v'', found by changing, in [1225a], r' into r'', and v' into v'', we shall obtain the value [1225], whose differential relative to d is in [1226].

If we develop this function in a series arranged according to the cosines of the angles $(v'-v)$, $(v''-v)$, and their multiples, we shall have an expression of this form,

$$\begin{aligned} R = \tfrac{m'}{2}\cdot(r,r')^{(0)} + m'\cdot(r,r')^{(1)}\cdot\cos.(v'-v) + m'\cdot(r,r')^{(2)}\cdot\cos.2.(v'-v) \\ + m'\cdot(r,r')^{(3)}\cdot\cos.3.(v'-v) + \&c. \\ + \tfrac{m''}{2}\cdot(r,r'')^{(0)} + m''.(r,r'')^{(1)}\cdot\cos.(v''-v) + m''.(r,r'')^{(2)}\cdot\cos.2.(v''-v) \\ + m''.(r,r'')^{(3)}\cdot\cos.3.(v''-v) + \&c.\,; \end{aligned} \qquad [1225]$$

hence we deduce

$$\mathrm{d}R = \left\{ \begin{array}{l} dr.\left\{ \begin{array}{r} \frac{m'}{2}\cdot\left(\frac{d.(r,r')^{(0)}}{dr}\right) + m'\cdot\left(\frac{d.(r,r')^{(1)}}{dr}\right)\cdot\cos.(v'-v) \\ + m'\cdot\left(\frac{d.(r,r')^{(2)}}{dr}\right)\cdot\cos.2.(v'-v) + \&c. \\ + \frac{m''}{2}\cdot\left(\frac{d.(r,r'')^{(0)}}{dr}\right) + m''.\left(\frac{d.(r,r'')^{(1)}}{dr}\right)\cdot\cos.(v''-v) \\ + m''.\left(\frac{d.(r,r'')^{(2)}}{dr}\right)\cdot\cos.2.(v''-v) + \&c. \end{array} \right\} \\ + dv.\left\{ \begin{array}{l} m'.(r,r')^{(1)}.\sin.(v'-v) + 2m'.(r,r')^{(2)}.\sin.2.(v'-v) + \&c. \\ + m''.(r,r'')^{(1)}.\sin.(v''-v) + 2m''.(r,r'')^{(2)}.\sin.2.(v''-v) + \&c. \end{array} \right\} \end{array} \right\} \qquad [1226]$$

Suppose now, in conformity to what appears to be the case in the system of the three first satellites of Jupiter, that $n-2n'$ and $n'-2n''$ are very small fractions of n, and that their difference $(n-2n')-(n'-2n'')$, or $n-3n'+2n''$, is incomparably smaller than either of them.* It will follow from the expressions of $\frac{\delta r}{a}$ and δv § 50 [1020, 1021], that the action [1226′]

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* (831) It is shown in Book VIII, § 20, [6782, &c.], that $n=n'''.9{,}433419$, $n'=n'''.4{,}699569$, $n''=n'''.2{,}332643$, hence $n-2n'=n'''.0.034281$, [1226a] $n'-2n''=n'''.0{,}034283$, so that either of these quantities is much smaller than n, n', n''. Also $(n-2n')-(n'-2n'')=-n'''.0{,}000002$, which is also [1226b] incomparably smaller than either of the preceding quantities.

[1226″] of m' will produce in the radius vector and in the longitude of m, a very sensible equation depending on the argument* $2.(n't-nt+\varepsilon'-\varepsilon)$. The terms, corresponding to this equation, have for a divisor $4.(n'-n)^2-n^2$,

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* (832) *First*, This term of $\frac{\delta r}{a}$ arises from the part of the expression [1020], depending on $\cos.i.(n't-nt+\varepsilon'-\varepsilon)$, putting $i=\pm 2$; the same angle, in [1021], gives the corresponding term of δv. *Second*, If in these expressions of $\frac{\delta r}{a}$, δv, we change what relates to m into m', and the contrary, we shall have the parts of $\frac{\delta r'}{a'}$, $\delta v'$, arising from the action of m upon m'; and by putting $i=\pm 1$, in the terms depending on $\overset{\cos.}{\sin.}\, i.(nt-n't+\varepsilon-\varepsilon')$, they will have the divisor $(n'-n)^2-n'^2$, as above. *Third*, Changing in [1020, 1021], the terms relative to m, m', into those of m', m'', respectively, we shall have the values of $\frac{\delta r'}{a'}$, $\delta v'$, arising from the action of m'' upon m'; and the terms depending on $\overset{\cos.}{\sin.}\, i.(n''t-n't+\varepsilon''-\varepsilon')$, will, by putting $i=\pm 2$, furnish terms having the divisor $4.(n''-n')^2-n'^2$, or $(n'-2n'').(3n'-2n'')$. *Fourth*, Changing in [1020, 1021], the terms relative to m into those relative to m'', the terms depending on $\overset{\cos.}{\sin.}\, i.(n''t-n't+\varepsilon''-\varepsilon')$, for the case of $i=\pm 1$, will have the divisor $(n''-n')^2-n''^2$, or $n'.(n'-2n'')$. All the preceding terms have the small divisors of the order $n-2n'$, or $n'-2n''$, and give for δr, δv, &c., expressions of the same forms as in [1227]. The object of the present calculation is merely to find the greatest terms of δr, δv, &c., of forms similar to [1227], so that their substitution might produce terms depending on the angle [1227′], therefore it is not necessary to introduce any terms multiplied by the excentricities, which are very small, and would generally produce angles different from those which are here noticed, as is observed in [1228*c*]. It is easy to prove that there are no other terms similar to those above retained, which have those small divisors $n-2n'$, or $n'-2n''$. For n, n', n'', are nearly to each other, as 4, 2, 1, [1226*a*], respectively, and if we put $n=4$, $n'=2$, $n''=1$, the proposed divisors $n-2n'$, $n'-2n''$, will become nothing. Those *integral* values of i only, ought therefore to be retained, which make the divisors become nothing, by putting
[1226*c*] $n=4$, $n'=2$, $n''=1$. Now in the action of m' upon m, the angle $i.(n't-nt+\varepsilon'-\varepsilon)$, produces the divisor $i^2.(n'-n)^2-n^2$, or $4i^2-16$, which, being put equal to nothing, gives $i=\pm 2$; in the action of m'' upon m, we have the angle $i.(n''t-nt+\varepsilon''-\varepsilon)$, and the divisor $i^2.(n''-n)^2-n^2=9i^2-16$, which, being put nothing, will not give an integer for i, and it must therefore be neglected; this corresponds with δr, δv, [1227].

or $(n-2n').(3n-2n')$; and this divisor is very small, on account of $[1226''']$
the smallness of the factor $n-2n'$. We perceive also, in examining the same expressions, that the action of m produces, in the radius vector, and in
the longitude of m', an inequality depending on the argument $(n't-nt+\varepsilon'-\varepsilon)$, $[1226^{iv}]$
which, having $(n'-n)^2-n'^2$ or $n.(n-2n')$ for a divisor, is very sensible. We also find that the action of m'' upon m' produces, in the same quantities, a considerable inequality, depending on the argument $2.(n''t-n't+\varepsilon''-\varepsilon')$.
Lastly, we find that the action of m' produces, in the radius vector, and in $[1226^{v}]$
the longitude of m'', a considerable inequality, depending on the argument $n''t-n't+\varepsilon''-\varepsilon'$. These inequalities were first discovered by observations;
we shall fully develop them, in the theory of the satellites of Jupiter: their $[1226^{vi}]$
magnitudes, in comparison with the other inequalities, permit us to neglect these in the present case. We shall therefore suppose

$$\begin{aligned}
\delta r &= m'.E'.\cos.2.(n't-nt+\varepsilon'-\varepsilon);\\
\delta v &= m'.F'.\sin.2.(n't-nt+\varepsilon'-\varepsilon);\\
\delta r' &= m''.E''.\cos.2.(n''t-n't+\varepsilon''-\varepsilon')+m.G.\cos.(n't-nt+\varepsilon'-\varepsilon);\\
\delta v' &= m''.F''.\sin.2.(n''t-n't+\varepsilon''-\varepsilon')+m.H.\sin.(n't-nt+\varepsilon'-\varepsilon);\\
\delta r'' &= m'.G'.\cos.(n''t-n't+\varepsilon''-\varepsilon');\\
\delta v'' &= m'.H'.\sin.(n''t-n't+\varepsilon''-\varepsilon').
\end{aligned}$$
[1227]

We must now substitute, in the preceding expression of $\mathrm{d}R$, for r, v, r', v', r'', v'', the values of $a+\delta r$, $nt+\varepsilon+\delta v$, $a'+\delta r'$, $n't+\varepsilon'+\delta v'$, $a''+\delta r''$,

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Again, the action of m upon m', depending on the angle $i.(n't-nt+\varepsilon'-\varepsilon)$, produces the divisor $i^2.(n'-n)^2-n'^2=4i^2-4$, which becomes nothing, by putting $i=\pm 1$; the action of m'' upon m', depending on the angle $i.(n''t-n't+\varepsilon''-\varepsilon')$, has the divisor $i^2.(n''-n')^2-n'^2=i^2-4$, which becomes nothing, by putting $i=\pm 2$; these furnish the two terms of $\delta r'$, $\delta v'$, [1227]. Lastly, the action of m upon m'', depending upon the angle $i.(n''t-nt+\varepsilon''-\varepsilon)$, has the divisor $i^2.(n''-n)^2-n''^2=9i^2-1$, which gives a fractional value of i, and must therefore be neglected; the action of m' upon m'', depending upon the angle $i.(n''t-n't+\varepsilon''-\varepsilon')$, has the divisor

$$i^2.(n''-n')^2-n''^2=i^2-1,$$

which gives $i=\pm 1$, and furnishes the terms $\delta r''$, $\delta v''$, [1227]. Therefore the terms above noticed, are the only ones necessary to be retained.

$n''t+\varepsilon''+\delta v''$,* and retain only the terms depending on the argument [1227'] $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''$; and it is easy to perceive that the substitution of the values of δr, δv, $\delta r''$, $\delta v''$, can produce no similar terms; but this is not the case with the substitution of the values of $\delta r'$, $\delta v'$; since the term $m'.(r,r')^{(1)}.dv.\sin.(v'-v)$ of the expression of $\mathrm{d}R$, produces the following quantity,

[1228] $$-\frac{m'.m''.ndt}{2}.\left\{E''.\left(\frac{d.(a,a')^{(1)}}{da'}\right)-F''.(a,a')^{(1)}\right\}.\sin.(nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon'');$$

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* (833) In the value of $\mathrm{d}R$, [1226], we may neglect terms of the order m^3. Now $r=a+\delta r$, gives $dr=d\delta r$, which, by the first of the equations [1227], is of the order m', also $dv=ndt+d\delta v$, and the last term $d\delta v$, by reason of the second of the equations [1227], is also of the order m'. If we now put, for brevity,

[1226d] $$\tfrac{1}{2}.\left(\frac{d.(r,r')^{(0)}}{dr}\right)+\left(\frac{d.(r,r')^{(1)}}{dr}\right).\cos.(v'-v)+\&c.=\tfrac{1}{2}\Sigma.\left(\frac{d.(r,r')^{(i)}}{dr}\right).\cos.i.(v'-v);$$

$$(r,r')^{(1)}.\sin.(v'-v)+2.(r,r')^{(2)}.\sin.2.(v'-v)+\&c.=\tfrac{1}{2}\Sigma.i.(r,r')^{(i)}.\sin.i.(v'-v);$$

and use also similar expressions for the terms in r'', &c., we shall have from [1226],

[1227a] $$\begin{aligned}2\,\mathrm{d}R=d\delta r.&\left\{m'.\Sigma.\left(\frac{d.(r,r')^{(i)}}{dr}\right).\cos.i.(v'-v)+m''.\Sigma.\left(\frac{d.(r,r'')^{(i)}}{dr}\right).\cos.i.(v''-v)\right\}\\&+ndt.\{m'.\Sigma.i.(r,r')^{(i)}.\sin.i.(v'-v)+m''.\Sigma.i.(r,r'')^{(i)}.\sin.i.(v''-v)\}\\&+d\delta v.\{m'.\Sigma.i.(r,r')^{(i)}.\sin.i.(v'-v)+m''.\Sigma.i.(r,r'')^{(i)}.\sin.i.(v''-v)\}.\end{aligned}$$

Substituting the values of r, r', r'', v, v', v'', in the preceding equation, and rejecting terms of the order m'^3, we may, in the terms multiplied by $d\delta r$, $d\delta v$, put a, a', a'', $nt+\varepsilon-\varpi$, &c., for r, r', r'', v, &c., respectively; because these terms are multiplied by m' or m'', and the terms $d\delta r$, $d\delta v$, are of the order m' or m'', neglecting the excentricities and inclinations, as we shall hereafter find, may be done [1228c]; but in the term multiplied by ndt, we must develop $(r,r')^{(i)}$, $\sin.i.(v'-v)$, &c., to terms of the order m. Thus the development of $(r,r')^{(i)}$, made by means of the formula [610], becomes

$$(r,r')^{(i)}=(a,a')^{(i)}+\left(\frac{d.(a,a')^{(i)}}{da}\right).\delta r+\left(\frac{d.(a,a')^{(i)}}{da'}\right).\delta r',$$

and

$$(r,r'')^{(i)}=(a,a'')^{(i)}+\left(\frac{d.(a,a'')^{(i)}}{da}\right).\delta r+\left(\frac{d.(a,a'')^{(i)}}{da''}\right).\delta r'',$$

Again, as in the formula [678], we have

$$\sin.i.(v'-v)=\sin.i.(n't-nt+\varepsilon'-\varepsilon)+i.(\delta v'-\delta v).\cos.i.(n't-nt+\varepsilon'-\varepsilon);$$

$$\sin.i.(v''-v)=\sin.i.(n''t-nt+\varepsilon''-\varepsilon)+i.(\delta v''-\delta v).\cos.i.(n''t-nt+\varepsilon''-\varepsilon);$$

it is the only quantity of this kind which the expression of $\mathrm{d}R$ contains. The expressions of $\frac{\delta r}{a}$, δv, [1020, 1021], applied to the action of m'' upon m', [1228']

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hence

$$2\,\mathrm{d}R = d\delta r . \left\{ \begin{array}{l} m' . \Sigma . \left(\frac{d . (a, a')^{(i)}}{da}\right) . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \\ + m'' . \Sigma . \left(\frac{d . (a, a'')^{(i)}}{da}\right) . \cos . i . (n''t - nt + \varepsilon'' - \varepsilon) \end{array} \right\}$$

$$+ m' . n dt . \Sigma . i . \left\{ \begin{array}{r} (a,a')^{(i)} . \sin . i . (n't - nt + \varepsilon' - \varepsilon) + \left(\frac{d . (a, a')^{(i)}}{da}\right) . \delta r . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\ + \left(\frac{d . (a, a')^{(i)}}{da'}\right) . \delta r' . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\ - (a,a')^{(i)} . i\delta v . \cos . i . (n't - nt + \varepsilon' - \varepsilon) + (a,a')^{(i)} . i\delta v' . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \end{array} \right\} \qquad [1227b]$$

$$+ m'' . n dt . \Sigma . i . \left\{ \begin{array}{r} (a,a'')^{(i)} . \sin . i . (n''t - nt + \varepsilon'' - \varepsilon) + \left(\frac{d . (a,a'')^{(i)}}{da}\right) . \delta r . \sin . i . (n''t - nt + \varepsilon'' - \varepsilon) \\ + \left(\frac{d . (a,a'')^{(i)}}{da''}\right) . \delta r'' . \sin . i . (n''t - nt + \varepsilon'' - \varepsilon) \\ - (a,a'')^{(i)} . i \delta v . \cos . i . (n''t - nt + \varepsilon'' - \varepsilon) + (a,a'')^{(i)} . i \delta v'' . \cos . i . (n''t - nt + \varepsilon'' - \varepsilon) \end{array} \right\}$$

$$+ m' . d \delta v . \Sigma . i . (a, a')^{(i)} . \sin . i . (n't - nt + \varepsilon' - \varepsilon) + m'' . d \delta v . \Sigma . i . (a, a'') . \sin . i . (n''t - nt + \varepsilon'' - \varepsilon).$$

Now if we substitute the values of δr, δv, [1227], and their differentials, in the preceding expression, and reduce it, by the formulas [954*b*, 955*b*, *c*], the angles resulting will all be of the form $(i+2) . (n't - nt + \varepsilon' - \varepsilon)$, or $i . (n''t - nt + \varepsilon'' - \varepsilon) + 2 . (n't - nt + \varepsilon' - \varepsilon)$; in the first of these forms the coefficients of n, n', are the same, therefore the angle cannot be of the proposed form $nt - 3n't + 2n''t + \varepsilon - 3\varepsilon' + 2\varepsilon''$, [1227']; and in the second form the coefficient of $n't$ is 2, which cannot be of the proposed form, so that we may neglect δr, δv, and their differentials. In like manner, if we substitute the values of $\delta r''$, $\delta v''$, [1227], in $2\,\mathrm{d}R$, and reduce the angles, by the formulas [954*b*, 955*b*, *c*], they will become of the form $i . (n''t - nt + \varepsilon'' - \varepsilon) + n''t - n't + \varepsilon'' - \varepsilon'$, in which the coefficient of $n't$ is -1, which cannot agree with the proposed form, [1227'], so that we may neglect $\delta r''$, $\delta v''$, and as the terms of $2\,\mathrm{d}R$ [1227*b*], independent of δr, $\delta r'$, $\delta r''$, $\delta v'$, &c., cannot produce the proposed angle, there will only remain the terms depending on $\delta r'$, $\delta v'$, namely

$$2\,\mathrm{d}R = m' . n dt . \Sigma . i . \left\{ \begin{array}{l} \left(\frac{d . (a, a')^{(i)}}{da'}\right) . \delta r' . \sin . i . (n't - nt + \varepsilon' - \varepsilon) \\ + (a,a')^{(i)} . i . \delta v' . \cos . i . (n't - nt + \varepsilon' - \varepsilon) \end{array} \right\} . \qquad [1228a]$$

The parts of $\delta r'$, $\delta v'$, [1227], depending on *G*, *H*, produce the angle

$$(i+1) . (n't - nt + \varepsilon' - \varepsilon),$$

give, by retaining only the terms that have $n'-2n''$ for a divisor, and observing that n'' is very nearly equal to $\frac{1}{2}n'$,*

[1229]
$$\frac{E''}{a'}=n'^2\,.\,\frac{\left\{a'^2\,.\left(\frac{d\,.\,(a',a'')^{(2)}}{d\,a'}\right)+\frac{2\,n'}{n'-n''}\,.\,a'\,.\,(a',a'')^{(2)}\right\}}{(n'-2\,n'')\,.\,(3\,n'-2\,n'')}\,;$$

$$F''=\frac{2\,E''}{a'}\,;$$

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which does not contain n'', consequently it cannot be of the proposed form, [1227′]. The part depending on E'', F'', will produce the angle

$$i\,.\,(n'\,t-n\,t+\varepsilon'-\varepsilon)+2\,.\,(n''\,t-n'\,t+\varepsilon''-\varepsilon'),$$

which, by putting $i=-1$, becomes of the proposed form. Substituting therefore

[1228b] $\delta\,r'=m''\,.\,E''\,.\cos.2\,.\,(n''t-n'\,t+\varepsilon''-\varepsilon')$; $\delta\,v'=m''\,.\,F''\,.\sin.2\,.\,(n''\,t-n'\,t+\varepsilon''-\varepsilon')$;

we shall have

$$2\,\mathrm{d}R=m'.n\,dt.\Sigma.\left\{\begin{array}{l} i\,.\left(\frac{d\,.\,(a,a')^{(i)}}{d\,a'}\right).m''.E''.\cos.2\,.(n''t-n't+\varepsilon''-\varepsilon').\sin i.(n't-n\,t+\varepsilon'-\varepsilon) \\ +(a,a')^{(i)}.i^2.m''.F''.\sin.2\,.(n''t-n't+\varepsilon''-\varepsilon').\cos.i\,.(n't-n\,t+\varepsilon'-\varepsilon) \end{array}\right\},$$

and if we reduce the angles by the formulas [955b, e], it becomes

$$2\,\mathrm{d}\,R=m'\,.\,n\,d\,t\,.\,\Sigma\,.\left\{i\,.\left(\frac{d\,.\,(a,a')^{(i)}}{d\,a'}\right).\,m''\,.\,E''\,.+i^2\,.\,(a,a')^{(i)}\,.\,m''\,.\,F''\right\}$$
$$\times\sin.\,\{i\,.\,(n'\,t-n\,t+\varepsilon'-\varepsilon)+2\,.\,(n''\,t-n'\,t+\varepsilon''-\varepsilon')\}.$$

Now putting $i=-1$, and $(a,a')^{(-1)}=(a,a')^{(1)}$, which is similar to [954″], we shall get $2\,d\,R$, corresponding to $d\,R$, [1228].

As it respects the terms multiplied by the excentricities and inclinations, mentioned in the last note, it may be observed, that the terms of $\delta\,r$, $\delta\,r'$, &c., depending on the pure elliptical motion, will in general be of the form

$$\delta\,r=\Sigma\,.\,E^{(i)}\,.\cos.\,(k\,n\,t+A'),\qquad \delta\,v=\Sigma\,.\,F^{(i)}\,.\sin.\,(k\,n\,t+A'),\quad \&c.,$$

k being an integral number, as appears from [669, 675, 676′, &c.] These values being substituted in $2\,\mathrm{d}\,R$, [1227a], would produce terms depending on the angles

[1228c] $\{i\,.\,(n'-n)+k\,n\}\,.\,t+i\,.\,(\varepsilon'-\varepsilon)+A'$; $\{i\,.\,(n''-n)+k\,n\}\,.t+i\,.\,(\varepsilon''-\varepsilon)+A'$, &c.,

and it is evident that no integral value of i will reduce this to the form [1227′], particularly as this term, would generally introduce the longitudes of the perigee and node into the argument, so that on every account these terms are to be neglected.

* (834) Comparing the expression [954] with [1224, 1225], we easily perceive that $A^{(2)}=(a,a')^{(2)}$, and if we put another accent upon a, a', we shall get the value of $A^{(2)}$, corresponding to the action of the planet m'' upon m', namely, $A^{(2)}=(a',a'')^{(2)}$. Now the

therefore we shall have

$$\mathrm{d}R = \frac{m'.m''.n\,dt}{2} . E'' . \left\{ \frac{2.(a,a')^{(1)}}{a'} - \left(\frac{d.(a,a')^{(1)}}{da'}\right) \right\} \times \sin.(nt - 3n't + 2n''t + \varepsilon - 3\varepsilon' + 2\varepsilon'') = -\tfrac{1}{2} . \frac{da}{a^2} . \qquad [1230]$$

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values of $m''.E''$, $m''.F''$, which occur in $\delta r'$, $\delta v'$, [1227], are easily deduced from the terms of $\frac{\delta r}{a}$, δv, [1020, 1021], depending on the angle $i.(n't - nt + \varepsilon' - \varepsilon)$, putting another accent on n, n', a, a', m', &c., by which means we obtain the parts of $\frac{\delta r'}{a}$, $\delta v'$, arising from the action of m'' upon m', and these become

$$\frac{\delta r'}{a'} = \frac{m''.n'^2}{2} . \Sigma . \frac{\left\{ a'^2 . \left(\frac{dA^{(i)}}{da'}\right) + \frac{2n'}{n'-n''} . a' . A^{(i)} \right\}}{i^2.(n'-n'')^2 - n'^2} . \cos. i.(n''t - n't + \varepsilon'' - \varepsilon'),$$

$$\delta v' = \frac{m''}{2} . \Sigma . \frac{2n'^3 . \left\{ a'^2 . \left(\frac{dA^{(i)}}{da'}\right) + \frac{2n'}{n'-n''} . a' . A^{(i)} \right\}}{i.(n'-n'') . \left\{ i^2.(n'-n'')^2 - n'^2 \right\}} . \sin. i.(n''t - n't + \varepsilon'' - \varepsilon'), \qquad [1228d]$$

using $A^{(2)} = A^{(-2)} = (a', a'')^{(2)}$, and retaining only the values $i = 2$, $i = -2$, corresponding to the angle $2.(n''t - n't + \varepsilon'' - \varepsilon')$, we shall get

$$\frac{\delta r'}{a'} = m''.n'^2 . \frac{\left\{ a'^2 . \left(\frac{d.(a',a'')^{(2)}}{da'}\right) + \frac{2n'}{n'-n''} . a' . (a',a'')^{(2)} \right\}}{4.(n'-n'')^2 - n'^2} . \cos. 2.(n''t - n't + \varepsilon'' - \varepsilon'),$$

$$\delta v' = m'' . \frac{n'^3 . \left\{ a'^2 . \left(\frac{d.(a',a'')^{(2)}}{da'}\right) + \frac{2n'}{n'-n''} . a' . (a',a'')^{(2)} \right\}}{(n'-n'') . \left\{ 4.(n'-n'')^2 - n'^2 \right\}} . \sin. 2.(n''t - n't + \varepsilon'' - \varepsilon'). \qquad [1228e]$$

Substituting in these, for $4.(n'-n'')^2 - n'^2$, its value $(n' - 2n'').(3n' - 2n'')$; then comparing the value of $\frac{\delta r'}{a'}$, with that in [1228b], $\frac{m''E''}{a'}.\cos.2.(n''t - n't + \varepsilon'' - \varepsilon')$, we shall obtain the expression of $\frac{E''}{a'}$, [1229]. Again the values [1226a], give nearly $n' - n'' = \tfrac{1}{2}n'$, and if we substitute this in $\delta v'$, [1228e], the coefficient of

$$\sin. 2.(n''t - n't + \varepsilon'' - \varepsilon'),$$

will become equal to twice the preceding coefficient of $\frac{\delta r'}{a'}$, or $m''.\frac{2E''}{a'}$, as in [1229]. Now substituting the value of F'', [1229] in [1228], we shall get $\mathrm{d}R$, [1230], which is equal to $-\tfrac{1}{2}.\frac{da}{a^2}$, [1223].

Substituting this value of $\frac{da}{a^2}$ in the values of $\frac{dd\zeta}{dt}$, $\frac{dd\zeta'}{dt}$, $\frac{dd\zeta''}{dt}$, and putting for brevity*

[1231] $$\beta=\tfrac{3}{2}.E''.\left\{2.(a,a')^{(1)}-a'.\left(\frac{d.(a,a')}{da'}\right)\right\}.\left\{\frac{a}{a'}.m'.m''+\frac{9}{4}.m.m''+\frac{a''}{4a'}.m.m'\right\};$$

we shall have, because n is very nearly equal to $2n'$, and n' very nearly equal to $2n''$,

[1232] $$\frac{dd\zeta}{dt^2}-3.\frac{dd\zeta'}{dt^2}+2.\frac{dd\zeta''}{dt^2}=\beta.n^2.\sin.(nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon'');$$

or more accurately,†

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* (835) Substituting the value of $\frac{-da}{a^2}$, deduced from [1230] in [1222], and putting for brevity,

[1230a] $$G=E''.\left\{\frac{2.(a,a')^{(1)}}{a'}-\left(\frac{d.(a,a')^{(1)}}{da'}\right)\right\}.\sin.(nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''),$$

we shall get $\frac{dd\zeta}{dt^2}=\tfrac{3}{2}.m'm''.n^{\frac{4}{3}}.G$; $\frac{dd\zeta'}{dt^2}=-\tfrac{3}{2}.mm''.n'^{\frac{4}{3}}.\frac{n-n''}{n'-n''}.G$; $\frac{dd\zeta''}{dt^2}=\tfrac{3}{2}.mm'.n''^{\frac{4}{3}}.\frac{n-n'}{n'-n''}.G$; but the values n, n', n'', [1226a], give nearly $\frac{n-n''}{n'-n''}=3$, $\frac{n-n'}{n'-n''}=2$, therefore

$$\frac{dd\zeta'}{dt^2}=-\tfrac{9}{2}.mm''.n'^{\frac{4}{3}}.G;\qquad\frac{dd\zeta''}{dt^2}=3mm'.n''^{\frac{4}{3}}.G;$$

hence we get

[1230b] $$\frac{dd\zeta}{dt^2}-3.\frac{dd\zeta'}{dt^2}+2.\frac{dd\zeta''}{dt^2}=\tfrac{3}{2}G.\left\{m'm''.n^{\frac{4}{3}}+9mm''.n'^{\frac{4}{3}}+4mm'.n''^{\frac{4}{3}}\right\};$$

but from [1215] we obtain $n^{\frac{4}{3}}=n^2.n^{-\frac{2}{3}}=n^2a$, also $n'^{\frac{4}{3}}=n'^2a'$, $n''^{\frac{4}{3}}=n''^2a''$, and from [1226a], we have nearly $n'=\tfrac{1}{2}n$, $n''=\tfrac{1}{4}n$, therefore we shall have $n'^{\frac{4}{3}}=\tfrac{1}{4}n^2a'$; $n''^{\frac{4}{3}}=\tfrac{1}{16}n^2a''$; substituting these in [1230b], we shall get

$$\frac{dd\zeta}{dt^2}-3.\frac{dd\zeta'}{dt^2}+2.\frac{dd\zeta''}{dt^2}=\tfrac{3}{2}G.n^2.\{m'm''.a+\tfrac{9}{4}.mm''.a'+\tfrac{1}{4}mm'.a''\};$$

resubstituting the value of G, [1230a], and using β, [1231], we shall obtain [1232].

† (835a) It is evident that the value of $\frac{-da}{2a^2}$, [1230], depending on the configuration of the planets, will be more accurately expressed, if we change, as in [1194'''], nt into $\int ndt$, or ζ; also $n't$ into ζ', and $n''t$ into ζ''; since this part of R will be rendered more accurate by these changes, as has been observed in [1195a]. [1232a]

$$\frac{dd\zeta}{dt^2}-3\,.\frac{dd\zeta'}{dt^2}+2\,.\frac{dd\zeta''}{dt^2}=\beta\,.\,n^2\,.\,\sin.\,(\zeta-3\,\zeta'+2\,\zeta''+\varepsilon-3\,\varepsilon'+2\,\varepsilon'')\,;$$ [1233]

and if we put

$$V=\zeta-3\,\zeta'+2\,\zeta''+\varepsilon-3\,\varepsilon'+2\,\varepsilon''\,;$$ [1234]

we shall find by substitution in [1233],

$$\frac{dd\,V}{dt^2}=\beta\,.\,n^2\,.\,\sin.\,V.$$ [1235]

The mean distances a, a', a'', vary but little, and the same may be observed
relative to the quantity n; we may therefore suppose $\beta\,.\,n^2$ to be constant, [1235']
in this equation, and then, by integration, we shall find*

$$dt=\frac{\pm\,d\,V}{\sqrt{c-2\,\beta\,.\,n^2\,.\,\cos.\,V}}\,,$$ [1236]

c being an arbitrary constant quantity. The different values, which might be given to this constant quantity, furnish the three following cases.

If c be *positive* and greater than $\pm\,2\,\beta\,.\,n^2$, the angle V will always [1236']
increase,† and this must happen, if at the commencement of the motion
$(n-3\,n'+2\,n'')^2$ exceed $\pm\,2\,\beta\,.\,n^2\,.\,(1\mp\cos.\,V)$, the *upper* signs taking [1236'']

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* (836) Multiplying [1235] by $2\,d\,V$, we get $\frac{2\,d\,V.\,dd\,V}{dt^2}=2\,\beta\,.\,n^2\,d\,V\,.\,\sin.\,V$, whose integral, supposing β, n, to be constant, is

$$\frac{d\,V^2}{dt^2}=c-2\,\beta\,.\,n^2\,.\,\cos.\,V,$$ [1236a]

hence $dt=\frac{\pm\,d\,V}{\sqrt{c-2\,\beta\,.\,n^2.\cos.\,V}}$, as in [1236]. The supposition that β, n, are constant is allowable in this integration, because it has been proved in § 54, that the values of a, a', a'', are constant, if we neglect terms of the order m^2, and in the appendix to the third volume it will be shown, that the same is true, if we neglect terms of the order m^3, and higher powers.

† (836a) Because the denominator of the value of dt, [1236], will always be real, oscillating between $\sqrt{c-2\beta\,.\,n^2}$, and $\sqrt{c+2\beta\,.\,n^2}$, so that $dt>\frac{\pm\,d\,V}{\sqrt{c+2\,\beta\,.\,n^2}}$, and its integral will give $t>\frac{\pm\,V}{\sqrt{c+\beta\,.\,n^2}}$, the angles t, V, being supposed to commence together.

place if β be *positive*, the *lower* signs if β be *negative*.* It is easy to prove, and we shall particularly show, in the theory of the satellites of Jupiter, that the value of β, corresponding to the three first satellites of Jupiter, is a
[1236'''] positive quantity ;† supposing therefore $\mp \varpi = \pi - V$, π being the semi-circumference, we shall get [1236]

[1237] $$dt = \frac{d\varpi}{\sqrt{c + 2\beta \,.\, n^2 \,.\, \cos.\varpi}}.$$

In the interval from $\varpi = 0$ to $\varpi = \frac{\pi}{2}$, the radical $\sqrt{c + 2\beta \,.\, n^2 \,.\, \cos.\varpi}$
[1237'] would exceed $\sqrt{2\beta \,.\, n^2}$, provided c should be equal or greater than $\sqrt{2\beta \,.\, n^2}$; in which case we should have, during that interval,‡ $\varpi > nt \,.\, \sqrt{2\beta}$; and the

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* (837) Taking the differential of the equation [1234], and substituting the values of $d\zeta$, $d\zeta'$, $d\zeta''$, [1216], it becomes $dV = dt \,.\, (n - 3n' + 2n'')$, hence the equation [1236*a*] will give

[1236*b*] $$(n - 3n' + 2n'')^2 = c - 2\beta \,.\, n^2 \,.\, \cos.V;$$

subtracting $c - 2\beta \,.\, n^2$, we get

$$(n - 3n' + 2n'')^2 - (c - 2\beta \,.\, n^2) = 2\beta \,.\, n^2 \,.\, (1 - \cos.V),$$

in which the terms $(n - 3n' + 2n'')^2$, n^2, $1 - \cos.V$, are evidently positive ; also by hypothesis, c is positive, and greater than $2\beta \,.\, n^2$, therefore $c - 2\beta \,.\, n^2$, is positive ; and, if β be positive, the preceding equation will give

$$(n - 3n' + 2n'')^2 > 2\beta \,.\, n^2 \,.\, (1 - \cos.V).$$

Again, if we subtract $c + 2\beta \,.\, n^2$ from the equation [1236*b*], we shall obtain

$$(n - 3n' + 2n'')^2 - (c + 2\beta \,.\, n^2) = -2\beta \,.\, n^2 \,.\, (1 + \cos.V),$$

and if β be negative, the three terms

$$(n - 3n' + 2n'')^2, \qquad (c + 2\beta \,.\, n^2), \qquad -2\beta \,.\, n^2 \,.\, (1 + \cos.V),$$

must be positive, hence $(n - 3n' + 2n'')^2 > -2\beta \,.\, n^2 \,.\, (1 + \cos.V)$. Both these cases are included in the form [1236''], $(n - 3n' + 2n'')^2 > \pm 2\beta \,.\, n^2 \,.\, (1 \mp \cos.V)$.

† (838) Comparing [1235] with Book VIII, § 15, [6611], we find $\beta = k$, and by
[1236*c*] Book VIII, § 29, [7272], we have $k = 0{,}000000607302 = \beta$, which is positive.

‡ (839) When $\varpi = 0$, $\sqrt{c + 2\beta \,.\, n^2 \,.\, \cos.\varpi}$, becomes $\sqrt{c + 2\beta \,.\, n^2}$, and when $\varpi = \frac{1}{2}\pi$, it becomes $\sqrt{c}$, hence if c be equal to, or greater than, $\sqrt{2\beta \,.\, n^2}$, the quantity $\sqrt{c + 2\beta \,.\, n^2 \,.\, \cos.\varpi}$, will exceed $\sqrt{2\beta \,.\, n^2}$, or $n \,.\, \sqrt{2\beta}$, whilst ϖ varies from 0 to $\frac{1}{2}\pi$, and within these limits the equation [1237] will give $dt < \frac{d\varpi}{n \,.\, \sqrt{2\beta}}$, hence

$$d\varpi > n\,dt\,\sqrt{2\beta};$$

time t required for the angle ϖ to increase from 0 to a right angle, would be less than $\frac{\pi}{2n.\sqrt{2\beta}}$. The value of β depends on the masses m, m', m''. [1237″]
The inequalities observed in the motions of the three first satellites of Jupiter, of which we have spoken above, give the ratios between their masses and the mass of Jupiter; from which it follows, that $\frac{\pi}{2n.\sqrt{2\beta}}$ is [1237‴]
less than two years, as we shall see in the theory of the satellites [7274]; therefore the angle ϖ would, on this supposition, require less than two years to increase from nothing to a right angle; now from all the observations of the satellites of Jupiter, since the time of their discovery, ϖ has been found [1237ⁱᵛ]
to be nothing, or insensible: the case we are now examining does not therefore correspond with that of the three first satellites of Jupiter.

If the constant quantity c be less than $\pm 2\beta.n^2$, the angle V [1236] will merely oscillate; it will never attain to two right angles, if β be negative, because then the radical $\sqrt{c-2\beta.n^2.\cos.V}$ would become [1237ᵛ]
imaginary. On the contrary, V will never vanish, if β be positive. In the first case, its value will be alternately greater and less than nothing; in the [1237ᵛⁱ]
second case, it will be alternately greater and less than two right angles. [1237ᵛⁱⁱ]
All the observations of the three first satellites of Jupiter prove that this second case is that which corresponds to these bodies; therefore the value of [1237ᵛⁱⁱⁱ]
β must be positive, as it respects them; *and as the theory of gravity gives β positive, we may consider this phenomenon as a new confirmation of that* [1237ⁱˣ]
theory.

We shall now resume the equation [1237],

$$dt=\frac{d\varpi}{\sqrt{c+2\beta.n^2.\cos.\varpi}}. \qquad [1238]$$

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whose integral is $\varpi > nt.\sqrt{2\beta}$, supposing ϖ to be nothing at the commencement of the time t. If we suppose T to be the value of the term corresponding to $\varpi=\frac{1}{2}\pi$, the preceding expression will become $\frac{1}{2}\pi > nT.\sqrt{2\beta}$, or $T<\frac{\pi}{2n.\sqrt{2\beta}}$, as in [1237″].

The value of $\frac{\pi}{2n.\sqrt{2\beta}}$, is computed in Book VIII, § 29, [7274], and found to be about [1237a]
401 days, which is considerably less than two years, as is mentioned above.

[1238'] The angle ϖ being always very small, as appears by observations, we may suppose $\cos.\varpi = 1 - \frac{1}{2}\varpi^2$, and the preceding equation* will give by integration,

[1239] $$\varpi = \lambda . \sin.(n\,t . \sqrt{\beta} + \gamma),$$

[1239'] λ and γ being the two arbitrary constant quantities, which can be determined only by observation. No inequality of this kind has yet been discovered; which proves that it must be very small.

From the preceding analysis, we obtain the following results. Since the
[1239''] angle $n\,t - 3\,n't + 2\,n''t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon''$, oscillates about two right angles,
its mean value will be equal to two right angles; therefore if we notice only
[1239'''] the mean motions, we shall have,* $n - 3\,n' + 2\,n'' = 0$; that is, *the mean motion of the first satellite, minus three times that of the second, plus twice*

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* (840) The general value of $\cos.\varpi$, [44] Int., neglecting ϖ^4, and higher powers of ϖ, is $\cos.\varpi = 1 - \frac{1}{2}\varpi^2$; hence [1237] becomes $d\,t = \frac{d\varpi}{\sqrt{c + 2\,\beta . n^2 - \beta . n^2 . \varpi^2}}$. If we put $c + 2\,\beta . n^2 = \beta . n^2 . \lambda^2$, and multiply the preceding value of $d\,t$ by $n . \sqrt{\beta}$, we shall get $n\,d\,t . \sqrt{\beta} = \frac{d\varpi}{\sqrt{\lambda^2 - \varpi^2}}$, whose integral is $n\,t . \sqrt{\beta} + \gamma = \text{arc.}\left(\sin. \frac{\varpi}{\lambda}\right)$, or $\sin.(n\,t . \sqrt{\beta} + \gamma) = \frac{\varpi}{\lambda}$, hence we easily obtain ϖ, [1239].

† (841) As the mean value of the angle $(n - 3\,n' + 2\,n'') . t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon''$, [1239''], is always two right angles, or π, the coefficient of the time t must be nothing, otherwise the angle would constantly increase or decrease, therefore $n - 3\,n' + 2\,n'' = 0$,
[1239a] as in [1239''']; substituting this in $(n - 3\,n' + 2\,n'') . t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon'' = \pi$, we shall get $\varepsilon - 3\,\varepsilon' + 2\,\varepsilon'' = \pi$, [1239$^{\text{v}}$]. The mean *motions* of the satellites m, m', m'', in the time t, are $n\,t$, $n'\,t$, $n''\,t$, respectively, hence the mean motion of the first, minus three times that of the second, plus twice that of the third, is

[1239b] $$n\,t - 3\,n'\,t + 2\,n''\,t = (n - 3\,n' + 2\,n'') . t = 0,$$

as above. The mean longitudes of the same satellites are $n\,t + \varepsilon$, $n'\,t + \varepsilon'$, $n''\,t + \varepsilon''$, hence the mean longitude of the first satellite, minus three times that of the second, plus twice that of the third, is equal to $(n - 3\,n' + 2\,n'') . t + \varepsilon - 3\,\varepsilon' + 2\,\varepsilon''$, which is equal to two right angles, [1239a]. These beautiful laws, discovered by La Place, have been confirmed by Delambre's elaborate computations of several thousands of observations of these satellites.

First law of La Place, relative to the motions of Jupiter's satellites.

that of the third, is accurately and invariably equal to nothing. It is not
necessary that this equality should take place at the commencement of the
motion, which would be highly improbable; it will be sufficient for it to be
nearly correct, so that $n-3n'+2n''$ may, independent of its sign, be less
than $\lambda . n . \sqrt{\beta}$;* and then the mutual attractions of the three satellites, will [1239$^{\text{iv}}$]
render the equation rigorously exact.

We have also $\varepsilon-3\varepsilon'+2\varepsilon''=$ two right angles; hence *the mean longitude* [1239$^{\text{v}}$]
of the first satellite, minus three times that of the second, plus twice that of [1239$^{\text{vi}}$]
the third, is accurately and invariably equal to two right angles. By means
of this theorem, the preceding values of $\delta r'$ and $\delta v'$, are reduced to†

Second law of La Place, relative to the motions of Jupiter's satellites.

$$\delta r' = (m\,G - m''E'') . \cos. (n't - nt + \varepsilon' - \varepsilon);$$

$$\delta v' = (m\,H - m''F'') . \sin. (n't - nt + \varepsilon' - \varepsilon). \qquad [1240]$$

The two inequalities of the motion of m', arising from the actions of m and
m'', are, by this means, reduced to one, and will be always united. It follows
also, from the same theorem, that the three first satellites cannot be eclipsed
at the same instant; neither can they be seen at the same time, all in [1240']
conjunction or in opposition with the sun, when viewed from Jupiter. For

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* (842) Substituting the value of ϖ, [1239], in the equation [1236'''], $\mp\varpi=\pi-V$, we get $V=\pi\pm\lambda . \sin . (nt . \sqrt{\beta}+\gamma)$, whose differential is

$$\frac{dV}{dt}=\pm\lambda n . \sqrt{\beta} . \cos . (nt . \sqrt{\beta}+\gamma).$$

Taking the differential of V, [1234], and substituting the values of $d\zeta$, $d\zeta'$, $d\zeta''$, [1216], we obtain $\frac{dV}{dt}=n-3n'+2n''$; hence

$$n-3n'+2n''=\pm\lambda n . \sqrt{\beta} . \cos . (nt . \sqrt{\beta}+V);$$

and as the cosine of the second member never exceeds unity, the second member, independent of its sign, must be less than $\pm\lambda n . \sqrt{\beta}$, therefore the first member $n-3n'+2n''$. must be less than $\lambda n . \sqrt{\beta}$, to render the equation possible.

† (843) Having $nt-3n't+2n''t+\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, [1239$a$], we get by transposition $2 . (n''t-n't+\varepsilon''-\varepsilon')=\pi+(n't-nt+\varepsilon'-\varepsilon)$, therefore

$$\cos . 2 . (n''t-n't+\varepsilon''-\varepsilon')=-\cos . (n't-nt+\varepsilon'-\varepsilon), \quad \text{and} \qquad [1240a]$$

$$\sin . 2 . (n''t-n't+\varepsilon''-\varepsilon')=-\sin . (n't-nt+\varepsilon'-\varepsilon),$$

hence $\delta r'$, $\delta v'$, [1227], become as in [1240].

the preceding theorems take place relative to the mean synodical motions,
[1240''] and the mean synodical longitudes of the three satellites, as it is easy to
prove.* These two theorems take place, notwithstanding the alterations in
the mean motions which may arise, either from a cause similar to that which
[1240'''] alters the mean motion of the moon, or from the resistance of a very rare
medium [5715]. It is evident that these causes would add to the value of $\frac{ddV}{dt^2}$
[1235], a quantity of the form $\frac{dd\psi}{dt^2}$, which could only become sensible
[1240iv] by the integrations ;† supposing therefore $V=\pi-\varpi$, and ϖ to be very
small, the differential equation in V [1235] will become

[1241] $$0=\frac{dd\varpi}{dt^2}+\beta\,.\,n^2\,.\,\varpi+\frac{dd\psi}{dt^2}.$$

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[*] (844) Let the mean longitude of the sun, as seen from Jupiter, be $\mathcal{N}t+E$, the mean synodical motions of the satellites m, m', m'', $n_{,}t$, $n'_{,}t$, $n''_{,}t$, respectively, put also $\varepsilon=E+\varepsilon_{,}$, $\varepsilon'=E+\varepsilon'_{,}$, $\varepsilon''=E+\varepsilon''_{,}$. Then $n=\mathcal{N}+n_{,}$, $n'=\mathcal{N}+n'_{,}$, $n''=\mathcal{N}+n''_{,}$. Substituting these values of n, n', n'', in the equation [1239b], $(n-3n'+2n'')\,.\,t=0$, it becomes $(n_{,}-3n'_{,}+2n''_{,})\,.\,t=0$, and the values of ε, ε', ε'', being used in $\varepsilon-3\varepsilon'+2\varepsilon''=\pi$, [1239a], we get $\varepsilon_{,}-3\varepsilon'_{,}+2\varepsilon''_{,}=\pi$.
[1240a] Hence $(n_{,}-3n'_{,}+2n''_{,})\,.\,t+\varepsilon_{,}-3\varepsilon'_{,}+2\varepsilon''_{,}=\pi$. From which equations it follows, that the laws of La Place take place, when the *synodical* motions and longitudes are used. The elongation of the satellites from the sun, as seen from Jupiter, being represented by $n_{,}t+\varepsilon_{,}$; $n'_{,}t+\varepsilon'_{,}$; $n''_{,}t+\varepsilon''_{,}$; these quantities cannot be nothing, at the same time, as is evident by substituting nothing for each of them, in the preceding equation [1240a],
[1240b] put under the form $(n_{,}t+\varepsilon_{,})-3\,.\,(n'_{,}t+\varepsilon'_{,})+2\,.\,(n''_{,}t+\varepsilon''_{,})=\pi$, therefore the three satellites cannot be in conjunction at the same time. The same elongations being put equal to π will not satisfy the equation [1240b], therefore the satellites cannot all be in opposition at the same time.

[†] (845) For if V change into $V+\psi$ by means of these secular equations, ddV [1235] will increase by $dd\psi$, whilst the change in $\beta n^2\,.\,\sin.\,V$ may be neglected on account of the smallness of βn^2 and ψ, [1241'].

Otherwise, in Book VII, § 23, [5543], it is shown that the secular equation of the moon is of the form $\alpha't^2+\alpha''t^3$, &c., α', α'', being very small; or rather, as in [1052'] this equation is of the form $\Sigma\,.\,k\,.\,\sin.\,(\alpha t+\beta)$, in which α is very small, and k is a quantity which has been much increased by the integrations. This last expression being developed according to the powers of t, will produce the first. The resistance of a very rare medium would prevent

The period of the angle $nt.\sqrt{\beta}$ being a very small number of years [1237a],
whilst the quantities contained in $\frac{dd\psi}{dt^2}$, are either constant, or include [1241′]
several centuries; we shall have, very nearly, by integrating the preceding
equation,

$$\varpi = \lambda.\sin.(nt.\sqrt{\beta}+\gamma) - \frac{dd\psi}{\beta.n^2.dt^2}. \qquad [1242]$$

Hence the value of ϖ will always be very small, and the secular equations of [1242′]
the mean motions of the three first satellites, will always be modified by the
mutual action of these bodies, so that the secular equation of the first, plus [1242″]
twice that of the third, will be equal to three times that of the second.

The preceding theorems give, between the six constant quantities n, n',
n'', ε, ε', ε'', two equations of condition, which reduce these arbitrary quantities [1242‴]
to four; but the two arbitrary constant quantities λ and γ, of the value of ϖ
[1239], supply their places. This value is apportioned between the three
satellites, so that by putting p, p', p'', for the coefficients of $\sin.(nt.\sqrt{\beta}+\gamma)$,
in the expressions of v, v', v''; these coefficients will be in the ratio of the

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the motion from being uniform, and the change produced might also be expressed by a series proceeding according to the powers of t, connected with very small coefficients. Causes like these operating upon the satellites would produce similar terms in nt, $n't$, $n''t$, or in n, n', n'', or by [1215], in a, a', a'', and their differentials, therefore terms of the form $\alpha' t+\alpha'' t+$&c., or $\Sigma.k.\sin.(\alpha t+\beta)$, would be produced in the values $\frac{dd\zeta}{dt^2}$, $\frac{dd\zeta'}{dt^2}$, $\frac{dd\zeta''}{dt^2}$, [1217, 1222], and also in the values of $\frac{dd\zeta}{dt^2}-3.\frac{dd\zeta'}{dt^2}+2.\frac{dd\zeta''}{dt^2}$, [1232, 1233], or in $\frac{ddV}{dt^2}$, [1234, 1235]; so that the equation [1235] would become

$$\frac{ddV}{dt^2} = \beta.n^2.\sin.V + \Sigma.k.\sin.(\alpha t+\beta).$$

and by putting $V=\pi-\varpi$, it would become

$$0 = \frac{dd\varpi}{dt^2} + \beta.n^2.\sin.\varpi + \Sigma.k.\sin.(\alpha t+\beta).$$

As ϖ is very small, we may write ϖ for $\sin.\varpi$, and we may also put

$$\Sigma.k.\sin.(\alpha t+\beta) = \frac{dd\psi}{dt^2},$$

and then it will become $0=\frac{dd\varpi}{dt^2}+\beta.n^2.\varpi+\frac{dd\psi}{dt^2}$, as in [1241]; whence we may

preceding values of $\frac{d\,d\zeta}{d\,t^2}$, $\frac{d\,d\zeta'}{d\,t^2}$, $\frac{d\,d\zeta''}{d\,t^2}$; moreover, we have
[1242iv] $p-3p'+2p''=\lambda$.* Hence there arises, in the mean motions of the three
Libration of Jupiter's Satellites. first satellites of Jupiter, an inequality which differs for each of them, merely by the coefficient; and which forms, in these motions a species of libration,
[1242v] whose extent is arbitrary. Observations have shown that this libration is insensible.

67. We shall now consider the variations of the excentricities and of the perihelia of the orbits. For this purpose, we shall resume the expressions
[1242vi] of df, df', df'', found in § 64 [1176]; putting r equal to the radius vector of m, projected upon the plane of x and y; v equal to the angle which this
[1242vii] projection makes with the axis of x; and s for the tangent of the latitude of m, above the same plane; we shall then have†

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obtain ϖ, by the formula [865, 871]; any term of $\frac{d\,d\psi}{dt^2}$, represented by $k\,.\,\sin.\,(\alpha t+\beta)$, will produce in ϖ the term $\frac{k}{\alpha^2-\beta\,.\,n^2}\,.\,\sin.\,(\alpha t+\beta)$, [871]; and as α is excessively small this will become nearly $-\frac{k}{\beta\,.\,n^2}\,.\,\sin.\,(\alpha\,t+\beta)$, which is equal to the term of $\frac{d\,d\psi}{dt^2}$, multiplied by $-\frac{1}{\beta\,.\,n^2}$, so that the whole correction to be applied to ϖ will be $-\frac{d\,d\psi}{\beta.n^2.dt^2}$, as in [1242], this quantity being dependent on constant quantities, or angles of a very long period, [1241'].

* (846) Substituting

$$p\,.\,\sin.\,(n\,t\,.\sqrt{\beta}+\gamma),\qquad p'\,.\,\sin.\,(n\,t\,.\sqrt{\beta}+\gamma),\qquad p''\,.\,\sin.\,(n\,t\,.\sqrt{\beta}+\gamma),$$

for the secular librations of v, v', v'', or ζ, ζ', ζ'', the corresponding libration of

$$V=\zeta-3\,\zeta'+2\,\zeta''+\varepsilon-3\,\varepsilon'+2\,\varepsilon'',$$

[1234], will be $(p-3\,p'+2\,p'')\,.\,\sin.\,(n\,t\,.\sqrt{\beta}+\gamma)$, the coefficient of which being put equal to that of the same angle in $V=\pi-\varpi$, [1240iv], or by [1242],

$$V=\pi-\lambda\,.\,\sin.\,(n\,t\,.\sqrt{\beta}+\gamma)+\frac{d\,d\psi}{\beta\,.\,n^2\,.\,dt^2},$$

we shall get $p-3\,p'+2\,p''=-\lambda$, which, independent of its sign, is as stated above.

† (847) These values of x, y, are found as in [371], corresponding to SX, PX, of the figure page 240, P being the projection of the place of m, upon the plane SXP, and the tangent of the angle $m\,SP$ being s, we shall have $m\,P=PS\,.\,\text{tang.}\,m\,S\,P$, or $z=r\,s$.

$$x = r\,.\cos.v\,; \qquad y = r\,.\sin.v\,; \qquad z = rs\,; \qquad [1243]$$

hence it is easy to deduce*

$$x.\left(\frac{dR}{dy}\right) - y.\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dv}\right);$$

$$x.\left(\frac{dR}{dz}\right) - z.\left(\frac{dR}{dx}\right) = (1+s^2).\cos.v.\left(\frac{dR}{ds}\right) - rs.\cos.v.\left(\frac{dR}{dr}\right) + s.\sin.v.\left(\frac{dR}{dv}\right); \qquad [1244]$$

$$y.\left(\frac{dR}{dz}\right) - z.\left(\frac{dR}{dy}\right) = (1+s^2).\sin.v.\left(\frac{dR}{ds}\right) - rs.\sin.v.\left(\frac{dR}{dr}\right) - s.\cos.v.\left(\frac{dR}{dv}\right).$$

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* (848) From the equations [1243] we obtain $r = \sqrt{x^2+y^2}$, $\tan g.\, v = \frac{y}{x}$, $s = \frac{z}{\sqrt{x^2+y^2}}$. This value of r gives

$$\left(\frac{dr}{dx}\right) = \frac{x}{\sqrt{x^2+y^2}} = \frac{r.\cos.v}{r} = \cos.v\,; \qquad \left(\frac{dr}{dy}\right) = \frac{y}{\sqrt{x^2+y^2}} = \frac{r.\sin.v}{r} = \sin.v\,; \qquad \left(\frac{dr}{dz}\right) = 0. \qquad [1243a]$$

The differential of $\tan g.\, v = \frac{y}{x}$ is $\frac{dv}{\cos.^2 v} = \frac{x\,dy - y\,dx}{x^2}$, substituting x, y, [1243], and multiplying by $\cos.^2 v$, we obtain $dv = \frac{dy.\cos.v - dx.\sin.v}{r}$; hence

$$\left(\frac{dv}{dx}\right) = -\frac{\sin.v}{r}, \qquad \left(\frac{dv}{dy}\right) = \frac{\cos.v}{r}; \qquad \left(\frac{dv}{dz}\right) = 0. \qquad [1243b]$$

Lastly $s = \frac{z}{\sqrt{x^2+y^2}}$, gives $ds = \frac{dz}{\sqrt{x^2+y^2}} - \frac{z.(x\,dx + y\,dy)}{(xx+yy)^{\frac{3}{2}}}$, substituting $\sqrt{x^2+y^2} = r$, and the values [1243], we get $ds = \frac{dz}{r} - \frac{dx.s.\cos.v}{r} - \frac{dy.s.\sin.v}{r}$, hence

$$\left(\frac{ds}{dx}\right) = -\frac{s.\cos.v}{r}, \qquad \left(\frac{ds}{dy}\right) = -\frac{s.\sin.v}{r}, \qquad \left(\frac{ds}{dz}\right) = \frac{1}{r}. \qquad [1243c]$$

Now considering R as a function of x, y, z, and then as a function of r, v, s, we shall get

$$\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dr}\right).\left(\frac{dr}{dx}\right) + \left(\frac{dR}{dv}\right).\left(\frac{dv}{dx}\right) + \left(\frac{dR}{ds}\right).\left(\frac{ds}{dx}\right),$$

and by using the preceding values it becomes

$$\left(\frac{dR}{dx}\right) = \cos.v.\left(\frac{dR}{dr}\right) - \frac{\sin.v}{r}.\left(\frac{dR}{dv}\right) - \frac{s.\cos.v}{r}.\left(\frac{dR}{ds}\right);$$

in like manner,

$$\left(\frac{dR}{dy}\right) = \sin.v.\left(\frac{dR}{dr}\right) + \frac{\cos.v}{r}.\left(\frac{dR}{dv}\right) - \frac{s.\sin.v}{r}.\left(\frac{dR}{ds}\right), \qquad [1243d]$$

$$\left(\frac{dR}{dz}\right) = \frac{1}{r}.\left(\frac{dR}{ds}\right).$$

We have also, by § 64 [1171]

[1245] $$x\,dy - y\,dx = c\,dt\,; \qquad x\,dz - z\,dx = c'dt\,; \qquad y\,dz - z\,dy = c''dt\,;$$

therefore the differential equations in f, f', f'', [1176], will become*

[1246]
$$df = -dy.\left(\frac{dR}{dv}\right) - dz.\left\{(1+s^2).\cos.v.\left(\frac{dR}{ds}\right) - rs.\cos.v.\left(\frac{dR}{dr}\right) + s.\sin.v.\left(\frac{dR}{dv}\right)\right\}$$
$$-c\,dt.\left\{\sin.v.\left(\frac{dR}{dr}\right) + \frac{\cos.v}{r}.\left(\frac{dR}{dv}\right) - \frac{s.\sin.v}{r}.\left(\frac{dR}{ds}\right)\right\} - \frac{c'dt}{r}.\left(\frac{dR}{ds}\right);$$
$$df' = dx.\left(\frac{dR}{dv}\right) - dz.\left\{(1+s^2).\sin.v.\left(\frac{dR}{ds}\right) - rs.\sin.v.\left(\frac{dR}{dr}\right) - s.\cos.v.\left(\frac{dR}{dv}\right)\right\}$$
$$+c\,dt.\left\{\cos.v.\left(\frac{dR}{dr}\right) - \frac{\sin.v}{r}.\left(\frac{dR}{dv}\right) - \frac{s.\cos.v}{r}.\left(\frac{dR}{ds}\right)\right\} - \frac{c''dt}{r}.\left(\frac{dR}{ds}\right);$$

Hence

$$x.\left(\frac{dR}{dy}\right) = r.\cos.v.\left(\frac{dR}{dy}\right) = r.\cos.v.\sin.v.\left(\frac{dR}{dr}\right) + \cos.^2v.\left(\frac{dR}{dv}\right) - s.\cos.v.\sin.v.\left(\frac{dR}{ds}\right)$$
$$-y.\left(\frac{dR}{dx}\right) = -r.\sin.v.\left(\frac{dR}{dx}\right) = -r.\cos.v.\sin.v.\left(\frac{dR}{dr}\right) + \sin.^2v.\left(\frac{dR}{dv}\right) + s.\cos.v.\sin.v.\left(\frac{dR}{ds}\right).$$

The sum of these two expressions, putting $\cos.^2 v + \sin.^2 v = 1$, is

$$x.\left(\frac{dR}{dy}\right) - y.\left(\frac{dR}{dx}\right) = \left(\frac{dR}{dv}\right),$$

as in [1244]. Again,

$$x.\left(\frac{dR}{dz}\right) = r.\cos.v.\left(\frac{dR}{dz}\right) = \cos.v.\left(\frac{dR}{ds}\right)$$
$$-z.\left(\frac{dR}{dx}\right) = -rs.\left(\frac{dR}{dx}\right) = -rs.\cos.v.\left(\frac{dR}{dr}\right) + s.\sin.v.\left(\frac{dR}{dv}\right) + s^2.\cos.v.\left(\frac{dR}{ds}\right),$$

whose sum is

$$x.\left(\frac{dR}{dz}\right) - z.\left(\frac{dR}{dx}\right) = (1+s^2).\cos.v.\left(\frac{dR}{ds}\right) - rs.\cos.v.\left(\frac{dR}{dr}\right) + s.\sin.v.\left(\frac{dR}{dv}\right),$$

as in the second of the equations [1244]. The third is found in a similar manner, or more simply, by changing in the preceding equation x into y, which, as in [1243], is the same as changing v into $v - \frac{1}{2}\pi$, that is, $\cos.v$ into $\sin v$, and $\sin.v$ into $-\cos.v$.

* (849) Substituting in [1176] the values [1244, 1245], also the values of $\left(\frac{dR}{dx}\right)$, $\left(\frac{dR}{dy}\right)$, $\left(\frac{dR}{dz}\right)$, found in the last note, we shall obtain, without reduction, the equations [1246].

$$\begin{aligned} df'' = {} & dx \,.\left\{ (1+s^2).\cos.v\,.\left(\frac{dR}{ds}\right) - rs\,.\cos.v\,.\left(\frac{dR}{dr}\right) + s\,.\sin.v\,.\left(\frac{dR}{dv}\right) \right\} \\ & + dy\,.\left\{ (1+s^2).\sin.v\,.\left(\frac{dR}{ds}\right) - rs\,.\sin.v\,.\left(\frac{dR}{dr}\right) - s\,.\cos.v\,.\left(\frac{dR}{dv}\right) \right\} \\ & + c'dt\,.\left\{ \cos.v\,.\left(\frac{dR}{dr}\right) - \frac{\sin.v}{r}\,.\left(\frac{dR}{dv}\right) - \frac{s\,.\cos.v}{r}\,.\left(\frac{dR}{ds}\right) \right\} \\ & + c''dt\,.\left\{ \sin.v\,.\left(\frac{dR}{dr}\right) + \frac{\cos.v}{r}\,.\left(\frac{dR}{dv}\right) - \frac{s\,.\sin.v}{r}\,.\left(\frac{dR}{ds}\right) \right\}. \end{aligned} \qquad [1246']$$

The quantities c', c'', depend, as we have seen in § 64 [1174], upon the inclination of the orbit of m to the fixed plane, so that these quantities would be reduced to nothing if the inclination were nothing;* moreover, it is easy to see, by the nature of R,† that $\left(\frac{dR}{ds}\right)$ is of the order of the inclinations [1246''] of the orbits; neglecting therefore the squares and products of these inclinations, the preceding expressions of df and df' will become

$$\begin{aligned} df & = -dy.\left(\frac{dR}{dv}\right) - c\,dt\,.\left\{ \sin.v\,.\left(\frac{dR}{dr}\right) + \frac{\cos.v}{r}\,.\left(\frac{dR}{dv}\right) \right\}; \\ df' & = dx.\left(\frac{dR}{dv}\right) + c\,dt\,.\left\{ \cos.v\,.\left(\frac{dR}{dr}\right) - \frac{\sin.v}{r}\,.\left(\frac{dR}{dv}\right) \right\}. \end{aligned} \qquad [1247]$$

now we have‡

$$dx = d.(r.\cos.v)\,; \qquad dy = d.(r.\sin.v)\,; \qquad c\,dt = x\,dy - y\,dx = r^2 dv\,; \qquad [1248]$$

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* (850) Putting $\varphi = 0$, in the first equation [1174], it will become $0 = \frac{\sqrt{c'^2 + c''^2}}{c}$, [1245a] and as c is finite, we must have $0 = c'^2 + c''^2$, which cannot be satisfied with any real values of c', c'', except $c' = 0$, $c'' = 0$.

† (851) Substituting $z = rs$, [1243], in R, [913, 914], and then finding $\left(\frac{dR}{ds}\right)$, it will evidently be of the order $m'z'$, $m''z''$, &c., that is, it will be of the order of the disturbing forces, multiplied by the inclinations of the orbits. From the first of the equations [1246a] [1174], c' and c'' will be of the order of the inclination of the orbit of m, and if we neglect terms of the order of the square of the inclinations, in the equations [1246], they will become as in [1247].

‡ (852) The differentials of x, y, [1243], give dx, dy, [1248], or

$$dx = dr\,.\cos.v - r\,dv\,.\sin.v\,; \qquad dy = dr\,.\sin.v + r\,dv\,.\cos.v.$$

Substituting these values of x, y, and their differentials in $c\,dt = x\,dy - y\,dx$, [1245], therefore we shall get

[1249]
$$df = -\{dr\,.\,\sin.v + 2r\,dv\,.\cos.v\}\,.\left(\frac{dR}{dv}\right) - r^2\,dv\,.\sin.v\,.\left(\frac{dR}{dr}\right);$$

$$df' = \{dr\,.\cos.v - 2r\,dv\,.\sin.v\}\,.\left(\frac{dR}{dv}\right) + r^2\,dv\,.\cos.v\,.\left(\frac{dR}{dr}\right).$$

These equations will be more exact, if we take for the fixed plane of x, y, [1249′] the orbit of m at a given epoch; for then c', c'', s, will be of the order of the disturbing forces; therefore the neglected quantities will be of the order of the square of the disturbing forces, multiplied by the square of the mutual inclination of the orbits of m and m'.

The values of r, dr, dv, $\left(\frac{dR}{dr}\right)$, $\left(\frac{dR}{dv}\right)$, evidently remain the same, [1249″] whatever be the position of the point from which the longitudes are counted;* now if we decrease v by a right angle, $\sin.v$ will become $-\cos.v$, and $\cos.v$ will become $\sin.v$, and the expression of df [1249] will, in this manner, change into that of df'; hence it follows, that when we shall have developed df in a series of sines and cosines of angles, increasing in [1249‴] proportion to the time, we may obtain the value of df', by decreasing, in this series, the angles ε, ε', ϖ, ϖ', θ, and θ', by a right angle.†

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we shall get $r^2\,.\,dv$, as in [1248], being the same as in [372*a*]. Substituting these in [1247] we shall get [1249]. Now if we take, for the plane of x, y, the orbit of m at a given epoch, the values of z, dz, s, $\left(\frac{dR}{ds}\right)$, c', c'', will be, as in the last note, of the order $m'z'$, consequently the neglected terms in [1249] will be as the square of that quantity, that is, the neglected terms will be of the order of the square of the disturbing forces, multiplied by the square of the inclinations of the orbit of the disturbing planet, as is remarked above [1249′].

* (852*a*) This follows from the function R, being wholly independent of the plane of x, y, as is remarked in [949′], and in note 668.

† (853) The value of R is composed of terms of the form [958],

$$m'k\,.\cos.(i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - g''\theta - g'''\theta').$$

Substituting this in [930, 931, 932], and for r, $\cos.v$, $\sin.v$, their elliptical values, [669, &c.], and neglecting terms of the order m'^2, we shall get expressions of δr, δv, δs, depending on

The position of the perihelion and the excentricity of the orbit are [1249$^{\mathrm{iv}}$] determined by the quantities f and f'; for we have seen, in § 64 [1178], that

$$\text{tang.}\, I = \frac{f'}{f}\,; \qquad [1250]$$

I being the longitude of the perihelion, referred to the fixed plane [1177′]. When this plane is the primitive orbit of m, we shall have, if we neglect terms [1250′] of the order of the square of the disturbing forces, multiplied by the square of the respective inclinations of the orbits, $I=\varpi$, ϖ being the longitude of the [1250″] perihelion upon the orbit.* We shall then have

$$\text{tang.}\, \varpi = \frac{f'}{f}\,; \qquad [1251]$$

which gives

$$\sin.\, \varpi = \frac{f'}{\sqrt{f^2+f'^2}}\,; \qquad \cos.\, \varpi = \frac{f}{\sqrt{f^2+f'^2}}\,. \qquad [1252]$$

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similar angles, therefore the complete values of r, v, s, will depend on like angles; so that we may assume, for v, an expression of the form

$$v = \Sigma\,.\,K\,.\cos.\,(i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - g''\theta - g'''\theta').$$

This equation must exist, whatever be the origin of the angles [1249″]. Suppose now the origin to be moved forward, by a quantity equal to a right angle, the angles v, ε, ε', ϖ, ϖ', θ, θ', will all be decreased by a right angle; therefore, if in this value of v, we decrease the longitudes ε, ε', ϖ, ϖ', θ, θ', at the epoch, by a right angle, the value of v, resulting in the preceding equation, will also be decreased by a right angle, as is observed in [1249″]. Making these changes in df, we shall obtain df', corresponding to [1249], in the manner [1249a] mentioned in [1249‴].

* (854) The inclination φ of the orbit of m to the fixed plane, at the epoch, being nothing, we shall get from [1032] $p=0$, $q=0$; hence by [1034] s is of the order $m'p$ or $m'q$; therefore the greatest latitude of m, or the greatest inclination to the primitive [1250a] orbit, will be of the same order $m'p'$, or $m'q'$. Substituting this quantity for φ, in [1191], and neglecting quantities of the order $(m'p')^2$, $(m'q')^2$, we shall find $\varpi = I$, and then from [1250] we shall obtain [1251]; whence we easily deduce [1252], from the well known formulas $\sin.\,\varpi = \frac{\text{tang.}\,\varpi}{\sqrt{1+\text{tang.}^2\varpi}}$, $\cos.\,\varpi = \frac{1}{\sqrt{1+\text{tang.}^2\varpi}}$.

We then have, by § 64 [1179, 1184],

[1253] $$\mu\, e=\sqrt{f^2+f'^2+f''^2}\,; \qquad\qquad f''=\frac{f'c'-f\,c''}{c}\,;$$

now since c', c'', are, by the preceding supposition [1246a], of the order of the disturbing forces,* f'' will be of the same order; and if we neglect terms of the order of the square of these forces, we shall have

[1253'] $\mu\, e=\sqrt{f^2+f'^2}$. If we substitute for $\sqrt{f^2+f'^2}$, its value μe, in the expressions of $\sin.\varpi$, and $\cos.\varpi$, [1252], we shall get

[1254] $$\mu\, e\,.\sin.\varpi=f'\,; \qquad\qquad \mu\, e\,.\cos.\varpi=f.$$

From these two equations we may determine the excentricity and the position of the perihelion, and may easily deduce†

[1255] $$\mu^2\,.\,e\,d\,e=f\,d\,f+f'\,d\,f'\,; \qquad\qquad \mu^2\,.\,e^2\,d\,\varpi=f\,d\,f'-f'\,d\,f.$$

[1254'] Taking the orbit of m for the plane of x, y, we shall get, from § 19, 20, when the ellipses are invariable, [1054, 1056, 1057],

[1256] $$r=\frac{a\,.\,(1-e^2)}{1+e\,.\cos.(v-\varpi)}\,; \qquad\qquad d\,r=\frac{r^2\,d\,v\,.\,e\,.\sin.(v-\varpi)}{a\,.\,(1-e^2)}\,;$$

$$r^2\,d\,v=a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}\,;$$

and by § 63, these equations take place when the ellipses are variable;‡ these expressions of $d\,f$, $d\,f'$, will therefore become§

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* (855) From [1246a], c', c'', are of the same order as the inclination of the orbit of m, and from [1250a] this is of the same order as the disturbing force; hence, by the last equation [1253], f'' is of the same order; and if we neglect terms of the order of the square of this force, we may neglect f''^2, in the value of $\mu\, e$ [1253], and we shall get, as

[1253a] in [1253'], $\mu\, e=\sqrt{f^2+f'^2}$. Substituting this in [1252] we shall obtain [1254].

† (856) The differential of $\mu^2\,e^2=f^2+f'^2$, [1253a], gives the first of the equations [1255]. The differential of [1251] is $\frac{d\varpi}{\cos.^2\varpi}=\frac{f\,d\,f'-f'\,d\,f}{f^2}$; multiplying it by $(\mu\, e\,.\cos.\varpi)^2=f^2$, [1254], we shall get the second of the equations [1255].

‡ (857) These equations being either finite, or of the first order, must take place also when the ellipsis is variable, as was observed in [1167''].

§ (858) The equations [1257] were deduced from [1249] by substituting the values [1256]. For the coefficient of $\left(\frac{dR}{dv}\right)$, in the value of $d\,f$, [1249], is

$$-\,d\,r\,.\sin.v-2\,r\,d\,v\,.\cos.v\,;$$

$$\begin{aligned} df &= -\frac{a\,.\,n\,d\,t}{\sqrt{1-e^2}}\,.\,\{2\,.\cos.\,v+\tfrac{3}{2}\,e\,.\cos.\,\varpi+\tfrac{1}{2}\,e\,.\cos.\,(2\,v-\varpi)\}\,.\left(\frac{dR}{dv}\right) \\ &\quad - a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}\,.\sin.\,v\,.\left(\frac{dR}{dr}\right); \\ df' &= -\frac{a\,.\,n\,d\,t}{\sqrt{1-e^2}}\,.\,\{2\,.\sin.\,v+\tfrac{3}{2}\,e\,.\sin.\,\varpi+\tfrac{1}{2}\,e\,.\sin.\,(2\,v-\varpi)\}\,.\left(\frac{dR}{dv}\right) \\ &\quad + a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}\,.\cos.\,v\,.\left(\frac{dR}{dr}\right); \end{aligned}$$ [1257]

therefore*

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and from $d\,r$ [1256], we get $-d\,r\,.\sin.\,v=\dfrac{-r^2\,.\,d\,v\,.\,e\,.\sin.\,(v-\varpi)\,.\sin.\,v}{a\,.(1-e^2)}$, but

$$\sin.\,(v-\varpi)\,.\sin.\,v=\tfrac{1}{2}\cos.\,\varpi-\tfrac{1}{2}\cos.\,(2\,v-\varpi),$$

[17] Int., and $r^2\,d\,v=a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}$, [1256], hence

$$-d\,r\,.\sin.\,v=-\frac{a\,n\,e\,.\,dt}{\sqrt{1-e^2}}\,.\,\{\tfrac{1}{2}\cos.\,\varpi-\tfrac{1}{2}\cos.\,(2\,v-\varpi)\}.$$ [1257a]

Again, from $r^2\,d\,v$, [1256], we get $r\,d\,v=\dfrac{a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}}{r}$. Substituting r, deduced from the first of the equations [1256], it becomes $r\,d\,v=\dfrac{a\,.\,n\,dt}{\sqrt{1-e^2}}\,.\{1+e\,.\cos.\,(v-\varpi)\}$; multiplying this by $-2\cos.\,v$, we get

$$\begin{aligned} -2\,r\,d\,v\,.\cos.\,v &= -\frac{a\,.\,n\,dt}{\sqrt{1-e^2}}\,.\,\{2\,.\cos.\,v+2\,e\,.\cos.\,v\,.\cos.\,(v-\varpi)\} \\ &= -\frac{a\,.\,n\,dt}{\sqrt{1-e^2}}\,.\,\{2\,.\cos.\,v+e\,.\cos.\,\varpi+e\,.\cos.\,(2\,v-\varpi)\}; \end{aligned}$$ [1257b]

adding this to the value of $-d\,r\,.\sin.\,v$, [1257a], we obtain

$$-d\,r\,.\sin.\,v-2\,r\,d\,v\,.\cos.\,v=\frac{-a\,.\,n\,dt}{\sqrt{1-e^2}}\,.\{2\,.\cos.\,v+\tfrac{3}{2}\,e\,.\cos.\,\varpi+\tfrac{1}{2}\,e\,.\cos.\,(2\,v-\varpi)\},$$

which is equal to the coefficient of $\left(\frac{dR}{dv}\right)$ in $d\,f$, [1257]. The coefficient of $\left(\frac{dR}{dr}\right)$ in $d\,f$, [1249], is $-r^2\,d\,v\,.\sin.\,v$, and if we substitute the value of $r^2\,d\,v$, [1256], it becomes as in [1257]. Lastly, if in the coefficients of $\left(\frac{dR}{dv}\right)$, $\left(\frac{dR}{dr}\right)$, we decrease the values of v, ϖ, by a right angle, it will give the corresponding coefficients in $d\,f'$, [1257], as is remarked in [1249a].

* (859) Substituting the values of f, f', [1254] in [1255] we obtain

$$\mu^2\,.\,e\,d\,e=\mu\,e\,.\,\{d\,f\,.\cos.\,\varpi+d\,f'\,.\sin.\,\varpi\},$$ and

$$\mu^2\,e^2\,.\,d\,\varpi=\mu\,e\,.\,\{d\,f'\,.\cos.\,\varpi-d\,f\,.\sin.\,\varpi\}.$$

Dividing by $\mu^2\,e$, we get

$$d\,e=\frac{1}{\mu}\,.\,\{d\,f\,.\cos.\,\varpi+d\,f'\,.\sin.\,\varpi\},\qquad e\,d\,\varpi=\frac{1}{\mu}\,.\,\{d\,f'\,.\cos.\,\varpi-d\,f\,.\sin.\,\varpi\}.$$ [1257c]

$$e\,d\varpi = -\frac{a\,.\,n\,d\,t}{\mu\,.\,\sqrt{1-e^2}}\,.\,\sin.\,(v-\varpi)\,.\,\{2+e\,.\,\cos.\,(v-\varpi)\}\,.\,\left(\frac{dR}{d\,v}\right)$$

$$+\frac{a^2\,.\,n\,d\,t}{\mu}\,.\,\sqrt{1-e^2}\,.\,\cos.\,(v-\varpi)\,.\,\left(\frac{dR}{d\,r}\right);$$

[1258]

$$d\,e = -\frac{a\,.\,n\,d\,t}{\mu\,.\,\sqrt{1-e^2}}\,.\,\{2\,.\,\cos.\,(v-\varpi)+e+e\,.\,\cos.^2(v-\varpi)\}\,.\,\left(\frac{dR}{d\,v}\right)$$

$$-\frac{a^2\,.\,n\,d\,t}{\mu}\,.\,\sqrt{1-e^2}\,.\,\sin.\,(v-\varpi)\,.\,\left(\frac{dR}{d\,r}\right).$$

Before substituting the values of df, df', we shall observe that they will become more symmetrical, if in the coefficient of $\left(\frac{dR}{dr}\right)$, [1257], we write $V-\frac{1}{2}\pi$, for v, by which means the term $-\sin.v$ will become $\cos.V$, $\cos.v$ will become $\sin.V$, and the value of df will consist of terms depending on $\cos.\,v$, $\cos.\,\varpi$, $\cos.\,(2\,v-\varpi)$, $\cos.\,V$; then df' will contain the sines of the *same* angles multiplied by the *same* coefficients, so that if we denote any one of these angles by W, and its coefficient by k, we shall have

[1257*d*] $$df=\Sigma\,.\,k\,.\,\cos.\,W, \qquad df'=\Sigma\,.\,k\,.\,\sin.\,W;$$

substituting these in [1257*c*], we get

$$d\,e=\frac{1}{\mu}\,.\,\Sigma\,.\,k\,.\{\cos.\,W\,.\,\cos.\,\varpi+\sin.\,W\,.\,\sin.\,\varpi\}=\frac{1}{\mu}\,.\,\Sigma\,.\,k\,.\,\cos.\,(W-\varpi),$$

and

$$e\,d\varpi=\frac{1}{\mu}\,.\,\{\sin.\,W\,.\,\cos.\,\varpi-\cos.\,W\,.\,\sin.\,\varpi\}=\frac{1}{\mu}\,.\,\Sigma\,.\,k\,.\,\sin.\,(W-\varpi);$$

hence we may obtain $d\,e$, $e\,d\varpi$, from df, df', [1257], by substituting $v=V-\frac{1}{2}\pi$, in the coefficient of $\left(\frac{dR}{dr}\right)$, then decreasing each of the angles v, ϖ, $2\,v-\varpi$, V, by ϖ, putting $\cos.\,(\varpi-\varpi)=1$, $\sin.\,(\varpi-\varpi)=0$, and dividing by μ. Hence we shall get

$$d\,e=-\frac{a\,.\,n\,d\,t}{\mu\,.\,\sqrt{1-e^2}}\,.\,\{2\,.\,\cos.\,(v-\varpi)+\tfrac{3}{2}\,e+\tfrac{1}{2}\,e\,.\,\cos.\,(2\,v-2\,\varpi)\,.\,\left(\frac{dR}{dv}\right)$$

$$+\frac{a^2\,.\,n\,d\,t}{\mu}\,.\,\sqrt{1-e^2}\,.\,\cos.\,(V-\varpi)\,.\,\left(\frac{dR}{dr}\right),$$

$$e\,d\varpi=-\frac{a\,.\,n\,dt}{\mu\,.\,\sqrt{1-e^2}}\,.\{2\,.\,\sin.\,(v-\varpi)+\tfrac{1}{2}\,e\,.\,\sin.\,(2\,v-2\,\varpi)\}\,.\,\left(\frac{dR}{dv}\right)$$

$$+\frac{a^2\,.\,n\,dt}{\mu}\,.\,\sqrt{1-e^2}\,.\,\sin.\,(V-\varpi)\,.\,\left(\frac{dR}{dr}\right);$$

substituting

$\frac{1}{2}\cos.\,(2\,v-2\,\varpi)=\cos.^2\,(v-\varpi)-\frac{1}{2}$, $\frac{1}{2}\sin.\,(2\,v-2\,\varpi)=\sin.\,(v-\varpi)\,.\,\cos.\,(v-\varpi)$, [6, 31] Int., and $V=\frac{1}{2}\pi+v$, they will become as in [1258].

This expression of de may, in some circumstances, be put under a more convenient form. For this purpose, we shall observe that*

$$dr.\left(\frac{dR}{dr}\right)=\mathrm{d}R-dv.\left(\frac{dR}{dv}\right);$$ [1258′]

substituting for r and dr their preceding values, we shall find

$$r^2dv.e.\sin.(v-\varpi).\left(\frac{dR}{dr}\right)=a.(1-e^2).\mathrm{d}R-a.(1-e^2).dv.\left(\frac{dR}{dv}\right);$$ [1259]

now we have

$$r^2dv=a^2.ndt.\sqrt{1-e^2};\qquad\qquad dv=\frac{ndt.\{1+e.\cos.(v-\varpi)\}^2}{(1-e^2)^{\frac{3}{2}}};$$ [1260]

therefore†

$$\begin{aligned}&a^2.ndt.\sqrt{1-e^2}.\sin.(v-\varpi).\left(\frac{dR}{dr}\right)\\&=\frac{a.(1-e^2)}{e}\mathrm{d}R-\frac{a.ndt}{e.\sqrt{1-e^2}}.\{1+e.\cos.(v-\varpi)\}^2.\left(\frac{dR}{dv}\right);\end{aligned}$$ [1261]

In the appendix to the third volume [5788, 5789], the author reduced these expressions of $ed\varpi$ and de to the following simple forms, which are demonstrated in the appendix,

$$\begin{aligned}ed\varpi&=-a.ndt.\sqrt{1-e^2}.\left(\frac{dR}{de}\right),\\de&=\frac{a.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\mathrm{d}R+\frac{a.\sqrt{1-e^2}}{e}.ndt.\left(\frac{dR}{d\varpi}\right),\end{aligned}$$ [1258a]

* (860) The general value of $\mathrm{d}R$ is $\mathrm{d}R=\left(\frac{dR}{dr}\right).dr+\left(\frac{dR}{dv}\right).dv+\left(\frac{dR}{dz}\right).dz$, and as the plane of x, y, is the orbit of m, at a given time, [1249′], z will be of the order of the disturbing force, and $\left(\frac{dR}{dz}\right)$, will be of the same order [913, 914], hence $dz.\left(\frac{dR}{dz}\right)$, will be of the order of the square of the disturbing force, and if we neglect this term we shall find $\mathrm{d}R=dr.\left(\frac{dR}{dr}\right)+dv.\left(\frac{dR}{dv}\right)$, or, $dr.\left(\frac{dR}{dr}\right)=\mathrm{d}R-dv.\left(\frac{dR}{dv}\right)$, as in [1258′]. Substituting dr, [1256], and multiplying by $a.(1-e^2)$, we shall get [1259]. The value r^2dv, [1260], is the same as in [1256], and if we divide it by the square of r, [1256], we shall obtain dv [1260].

† (861) Dividing [1259] by e, and substituting r^2dv, dv, [1260], we shall get [1261]. Multiplying the value of de, [1258] by e, and substituting the values [1261], we shall get [1262]. For the terms depending on $\cos.(v-\varpi)$, and $\cos.^2(v-\varpi)$, will mutually destroy each other, and the other terms, by reduction will become as in [1262].

hence the preceding expression of de will give

[1262]
$$ede = \frac{a \,.\, n\,dt \,.\, \sqrt{1-e^2}}{\mu} \,.\, \left(\frac{dR}{dv}\right) - \frac{a \,.\, (1-e^2)}{\mu} \,.\, \mathrm{d}R.$$

We may also find this formula in the following very simple manner. In § 64 we have*

[1263]
$$\frac{dc}{dt} = y \,.\, \left(\frac{dR}{dx}\right) - x \,.\, \left(\frac{dR}{dy}\right) = -\left(\frac{dR}{dv}\right);$$

[1263′] but by the same article,† $c = \sqrt{\mu a.(1-e^2)}$, which gives

[1264]
$$dc = \frac{da \,.\, \sqrt{\mu a.(1-e^2)}}{2a} - \frac{e\,de \,.\, \sqrt{\mu a}}{\sqrt{1-e^2}};$$

therefore

[1265]
$$ede = \frac{a \,.\, n\,dt \,.\, \sqrt{1-e^2}}{\mu} \,.\, \left(\frac{dR}{dv}\right) + a \,.\, (1-e^2) \,.\, \frac{da}{2a^2};$$

we then have, by § 64,

[1266]
$$\frac{\mu \,.\, da}{2a^2} = -\mathrm{d}R;$$

hence we shall get, for ede, the same expression as above [1262].

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* (862) The first of the equations [1173], compared with the first of [1244] gives [1263].

† (863) c' and c'' are of the order m', [1246a]; and if we neglect quantities of the order m'^2, the equation [1175] will become $\mu a \,.\, (1-e^2) = c^2$, hence $c = \sqrt{\mu a.(1-e^2)}$, as in [1263′]; its differential gives [1264], and if in this we substitute the value

$$dc = -dt \,.\, \left(\frac{dR}{dv}\right), \qquad [1263], \text{ it will become}$$

$$-dt \,.\, \left(\frac{dR}{dv}\right) = \frac{da \,.\, \sqrt{\mu a.(1-e^2)}}{2a} - \frac{e\,de \,.\, \sqrt{\mu a}}{\sqrt{1-e^2}},$$

multiplying this by $\frac{\sqrt{1-e^2}}{\sqrt{\mu a}}$, we get $ede = \frac{\sqrt{1-e^2}.dt}{\sqrt{\mu a}} \,.\, \left(\frac{dR}{dv}\right) + (1-e^2) \,.\, \frac{da}{2a}$. Substituting $\frac{1}{\sqrt{\mu}} = \frac{n\,a^{\frac{3}{2}}}{\mu}$ [605′], in the coefficient of $\left(\frac{dR}{dv}\right)$, and putting $a \,.\, \frac{da}{2a^2}$ for $\frac{da}{2a}$, in the other term, it becomes $ede = \frac{a.n\,dt.\sqrt{1-e^2}}{\mu} \,.\, \left(\frac{dR}{dv}\right) + a \,.\, (1-e^2) \,.\, \frac{da}{2a^2}$, as in [1265], but from [1177] we easily obtain $\frac{\mu\,da}{2a^2} = -\mathrm{d}R$. Substituting this we finally obtain the expression [1262].

68. We have seen, in § 65 [1197], that if we neglect the square of the
disturbing forces, the variations of the transverse axes, and of the mean [1266']
motion, will contain only periodical equations, depending on the configurations
of the bodies m, m', m'', &c. This is not the case with the variations of the
excentricities and of the inclinations; their differential expressions contain
terms, independent of this configuration, which, if they were rigorously
constant, would produce, by integration, terms proportional to the time.
This would at length render the orbits very excentrical, and make them [1266'']
very much inclined to each other; therefore the preceding approximations,
founded upon the smallness of the excentricities, and of the inclinations of
the orbits to each other, would be defective, and might be wholly inaccurate.
But the terms which enter the differential expressions of the excentricities
and inclinations, though they appear to be constant, are really functions of
the elements of the orbits, varying with extreme slowness, by reason of [1266''']
the changes thus introduced. Hence we see that there may result, in these
elements, considerable inequalities, independent of the mutual configuration
of the bodies of the system, and the periods of these inequalities will
depend upon the ratios of the masses m, m', &c., to the mass M. These
inequalities are those we have before named *secular equations*, which we [1266^iv]
have considered in Chapter VII. To determine them by this method, we
shall resume the value of df of the preceding article [1257],

Secular equations.

$$df = -\frac{a\,.\,n\,d\,t}{\sqrt{1-e^2}}\,.\,\{2\,.\,\cos.\,v + \tfrac{3}{2}\,e\,.\,\cos.\,\varpi + \tfrac{1}{2}\,e\,.\,\cos.\,(2\,v - \varpi)\}\,.\left(\frac{d\,R}{d\,v}\right)$$
$$-\,a^2\,.\,n\,d\,t\,.\sqrt{1-e^2}\,.\,\sin.\,v\,.\left(\frac{d\,R}{d\,r}\right). \qquad [1267]$$

We shall neglect, in the development of this equation, the squares and
products of the excentricities and inclinations of the orbits; and among the
terms depending on the excentricities and the inclinations, we shall retain [1267']
only those which are constant; we shall then suppose, as in § 48,
[952, 953],

$$r = a\,.\,(1+u_{,})\,; \qquad r' = a'\,.\,(1+u'_{,})\,;$$
$$v = n\,t + \varepsilon + v_{,}\,; \qquad v' = n'\,t + \varepsilon' + v'_{,}. \qquad [1268]$$

This being premised, we shall substitute for R, its value, found in § 48, [957], observing that by the same article [962],

[1269] $$\left(\frac{dR}{dr}\right)=\frac{a}{r}\cdot\left(\frac{dR}{da}\right)=(1-u_{,})\cdot\left(\frac{dR}{da}\right);$$

we shall also substitute, for $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, their values, given in § 22 [1010],

[1269'] $$u_{,}=-e\,.\cos.\,(nt+\varepsilon-\varpi)\,; \qquad u'_{,}=-e'.\cos.\,(n't+\varepsilon'-\varpi')\,;$$
$$v_{,}=2\,e\,.\sin.\,(nt+\varepsilon-\varpi)\,; \qquad v'_{,}=2\,e'\,.\sin.\,(n't+\varepsilon'-\varpi')\,;$$

retaining only the constant terms depending on the first power of the excentricities, and neglecting the squares of the excentricities, and of the inclinations; hence we shall find*

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* (864) The terms $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, are of the order of the excentricities; if we neglect the squares of these quantities, we shall have, as in [60, 61], Int.,

$$\sin.\,v=\sin.\,(nt+\varepsilon)+v_{,}.\cos.\,(nt+\varepsilon)\,; \qquad \cos.\,v=\cos.\,(nt+\varepsilon)-v_{,}.\sin.\,(nt+\varepsilon)\,;$$
$$e\,.\cos.\,(2\,v-\varpi)=e\,.\cos.\,(2\,nt+2\,\varepsilon-\varpi)\,; \quad \text{also} \quad \left(\frac{dR}{dr}\right)=\{1-u_{,}\}\cdot\left(\frac{dR}{da}\right).$$

Substituting these in [1267] we shall get

[1266a] $$df=-a.ndt.\{2\cos.\,(nt+\varepsilon)-2\,v_{,}.\sin.\,(nt+\varepsilon)+\tfrac{3}{2}e\,.\cos.\,\varpi+\tfrac{1}{2}\,e\,.\cos.\,(2nt+2\,\varepsilon-\varpi)\}.\left(\frac{dR}{dv}\right)$$
$$-a^2\,.\,n\,d\,t\,.\{\sin.\,(nt+\varepsilon)+v_{,}.\cos.\,(nt+\varepsilon)-u_{,}.\sin.\,(nt+\varepsilon)\}.\left(\frac{dR}{da}\right),$$

using the values of $u_{,}$, $v_{,}$, and reducing by [17—20] Int., we shall have

$$-2v_{,}.\sin.(nt+\varepsilon)=-4\,e\,.\sin.\,(nt+\varepsilon).\sin.\,(nt+\varepsilon-\varpi)=-2\,e.\cos.\varpi+2e\,.\cos.(2nt+2\varepsilon-\varpi)\,;$$
$$v_{,}\,.\cos.\,(nt+\varepsilon)=2\,e.\cos.\,(nt+\varepsilon)\,.\sin.\,(nt+\varepsilon-\varpi)=e\,.\sin.\,(2nt+2\varepsilon-\varpi)-e\,.\sin.\,\varpi\,;$$
$$-u_{,}.\sin.\,(nt+\varepsilon)=e\,.\sin.(nt+\varepsilon)\,.\cos.\,(nt+\varepsilon-\varpi)=\tfrac{1}{2}\,e\,.\sin.\,(2\,nt+2\,\varepsilon-\varpi)+\tfrac{1}{2}\,e\,.\sin.\,\varpi)\,;$$

hence [1226*a*] becomes

[1267a] $$df=-a\,n\,d\,t\,.\{2\,.\cos.\,(nt+\varepsilon)-\tfrac{1}{2}\,e\,.\cos.\,\varpi+\tfrac{5e}{2}\,.\cos.\,(2\,nt+2\,\varepsilon-\varpi)\}\cdot\left(\frac{dR}{dv}\right)$$
$$-a^2\,n\,d\,t\,.\{\sin.\,(nt+\varepsilon)-\tfrac{1}{2}\,e\,.\sin.\,\varpi+\tfrac{3e}{2}\,.\sin.\,(2\,nt+2\,\varepsilon-\varpi)\}\cdot\left(\frac{dR}{da}\right).$$

If we now substitute the above values of $u_{,}$, $u'_{,}$, $v_{,}$, $v'_{,}$, in R, [957], it will become by means of [954*c*, 955*a*],

$$R=\tfrac{m'}{2}.\Sigma\,.A^{(i)}.\cos.\,i.(n't-nt+\varepsilon'-\varepsilon)-\tfrac{m'}{2}.e\,.\Sigma.a\,.\left(\frac{d\,A^{(i)}}{d\,a}\right).\cos.\,\{i\,.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi)$$
$$-\tfrac{m'}{2}\,.e'\,.\Sigma\,.a'\,.\left(\frac{d\,A^{(i)}}{d\,a'}\right).\cos.\,\{i\,.(n'\,t-n\,t+\varepsilon'-\varepsilon)+n'\,t+\varepsilon'-\varpi'\}$$
$$-\tfrac{m'}{2}\,.2\,e\,.\Sigma\,.i\,.A^{(i)}.\cos.\,\{i\,.(n'\,t-n\,t+\varepsilon'-\varepsilon)+n\,t+\varepsilon-\varpi\}$$
$$+\tfrac{m'}{2}\,.2\,e'\,.\Sigma\,.i\,.A^{(i)}.\cos.\,\{i\,.(n'\,t-n\,t+\varepsilon'-\varepsilon)+n'\,t+\varepsilon'-\varpi'\},$$

$$df = \frac{am'.ndt}{2}.e.\sin.\varpi.\left\{a.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}a^2.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}$$

$$+am'.ndt.e'.\sin.\varpi'.\left\{A^{(1)}+\tfrac{1}{2}a.\left(\frac{dA^{(1)}}{da}\right)+\tfrac{1}{2}a'.\left(\frac{dA^{(1)}}{da'}\right)+\tfrac{1}{4}aa'.\left(\frac{ddA^{(1)}}{dada'}\right)\right\} \qquad [1270]$$

$$-am'.ndt.\Sigma.\left\{i.A^{(i)}+\tfrac{1}{2}a.\left(\frac{dA^{(i)}}{da}\right)\right\}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\};$$

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and if we connect the terms depending on the same angles

$$R = \tfrac{m'}{2}.\Sigma.A^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-m'e.\Sigma.\left\{iA^{(i)}+\tfrac{1}{2}a.\left(\frac{dA^{(i)}}{da}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$+m'e'.\Sigma.\left\{iA^{(i)}-\tfrac{1}{2}a'.\left(\frac{dA^{(i)}}{da'}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\}.$$

Hence we may find $\left(\frac{dR}{da}\right)$, and also $\left(\frac{dR}{dv}\right)$; observing, as in [963], that $\left(\frac{dR}{dv}\right)=\left(\frac{dR}{d\varepsilon}\right)$, supposing $\varepsilon-\varpi$ and $\varepsilon-\theta$ to be constant, in the differential relative to ε,

$$\left(\frac{dR}{dv}\right) = \tfrac{m'}{2}.\Sigma.iA^{(i)}.\sin.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-m'e.\Sigma.\left\{iA^{(i)}+\tfrac{1}{2}a.\left(\frac{dA^{(i)}}{da}\right)\right\}.i.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\} \qquad [1267b]$$

$$+m'e'.\Sigma.\left\{iA^{(i)}-\tfrac{1}{2}a'.\left(\frac{dA^{(i)}}{da'}\right)\right\}.i.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\};$$

$$\left(\frac{dR}{da}\right) = \tfrac{m'}{2}.\Sigma.\left(\frac{dA^{(i)}}{da}\right).\cos.i.(n't-nt+\varepsilon'-\varepsilon)$$

$$-m'e.\Sigma.\left\{(i+\tfrac{1}{2}).\left(\frac{dA^{(i)}}{da}\right)+\tfrac{1}{2}a.\left(\frac{d^2A^{(i)}}{da^2}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi\}$$

$$+m'e'.\Sigma.\left\{i.\left(\frac{dA^{(i)}}{da}\right)-\tfrac{1}{2}a'.\left(\frac{d^2A^{(i)}}{dada'}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi'\}.$$

Substituting these values in df, [1267*a*], and reducing by [18, 19] Int., there will arise terms depending on $\sin\varpi$, $\cos.\varpi$, of the order e, which are to be retained, also other terms depending on angles containing the time t, which have coefficients containing terms independent of e, e', connected with terms depending on e, e', but these last being much smaller than the others may be neglected. Moreover, we shall, in the terms depending on $\sin.\varpi$, $\sin.\varpi'$, retain only the terms depending on the first power of e, e'. Therefore, in the value of df, we may, in the terms multiplied by e, take only the first term of $\left(\frac{dR}{dv}\right).\left(\frac{dR}{da}\right)$ instead of their whole values, by this means we shall have

[1270′] the sign Σ in this expression, as in that of R, § 48 [954′], includes all integral positive and negative values of i, also the value $i=0$.

We shall obtain, as in the preceding article [1249‴], the value of df',

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[1267c]
$$\begin{aligned}df= & -andt.2.\cos.(nt+\varepsilon).\left(\frac{dR}{dv}\right)\\ & -andt.\{-\tfrac{1}{2}e.\cos.\varpi+\tfrac{5}{2}e.\cos.(2nt+2\varepsilon-\varpi)\}.\{\tfrac{m'}{2}.\Sigma.i\mathcal{A}^{(i)}.\sin.i.(n't-nt+\varepsilon'-\varepsilon)\}\\ & -a^2ndt.\sin.(nt+\varepsilon).\left(\frac{dR}{da}\right)\\ & -a^2ndt.\{-\tfrac{1}{2}e.\sin.\varpi+\tfrac{3}{2}e.\sin.(2nt+2\varepsilon-\varpi)\}.\left\{\tfrac{m'}{2}.\Sigma.\left(\frac{d\mathcal{A}^{(i)}}{da}\right).\cos.i.(n't-nt+\varepsilon'-\varepsilon)\right\}.\end{aligned}$$

In the terms of this expression containing e, we must, as in [1267′], retain only such as are independent of the time, corresponding to $i=0$, which makes $i\mathcal{A}^{(i)}=0$, and

$$\tfrac{1}{2}m'.\left(\frac{d\mathcal{A}^{(i)}}{da}\right).\cos.i.(n't-nt+\varepsilon'-\varepsilon)=\tfrac{1}{2}m'.\left(\frac{d\mathcal{A}^{(0)}}{da}\right);$$

and we must also neglect the term multiplied by $\tfrac{3}{2}e.\sin.(2nt+2\varepsilon-\varpi)$, because it will contain the time t, hence we shall get, from [1267c],

[1270a]
$$\begin{aligned}df= & -andt.2.\cos.(nt+\varepsilon).\left(\frac{dR}{dv}\right)\\ & -a^2ndt.\sin.(nt+\varepsilon).\left(\frac{dR}{da}\right)+\frac{am'.ndt}{2}.e.\sin.\varpi.\tfrac{1}{2}a.\left(\frac{d\mathcal{A}^{(0)}}{da}\right).\end{aligned}$$

If we substitute in this the values of $\left(\frac{dR}{dv}\right)$, $\left(\frac{dR}{da}\right)$, [1267b], the terms of the order e, e', will depend on the sine or cosine of the angles

$$i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi, \qquad i.(n't-nt+\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi',$$

and as these are multiplied, in [1270a], by $\cos.(nt+\varepsilon)$, or $\sin.(nt+\varepsilon)$, they cannot produce terms independent of t, except i is taken, so as to make the former angles depend on $nt+\varepsilon$, now this is done, by putting in the first angle,

$$i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon-\varpi, \qquad i=0,$$

and in the second $\quad i.(n't-nt-\varepsilon'-\varepsilon)+n't+\varepsilon'-\varpi', \qquad i=-1,$

and as $\mathcal{A}^{(-1)}=\mathcal{A}^{(1)}$, we may use, instead of $\left(\frac{dR}{dv}\right)$, $\left(\frac{dR}{da}\right)$, [1267b], the following values,

$$\left(\frac{dR}{dv}\right)=\tfrac{1}{2}m'.\Sigma.i\mathcal{A}^{(i)}.\sin.i.(n't-nt+\varepsilon'-\varepsilon)+m'e'.\left\{\mathcal{A}^{(1)}+\tfrac{1}{2}a'.\left(\frac{d\mathcal{A}^{(1)}}{da'}\right)\right\}.\sin.(nt+\varepsilon-\varpi'),$$

$$\begin{aligned}\left(\frac{dR}{da}\right)= & \tfrac{1}{2}m'.\Sigma.\left(\frac{d\mathcal{A}^{(i)}}{da}\right).\cos.i.(n't-nt+\varepsilon'-\varepsilon)-m'e.\left\{\tfrac{1}{2}.\left(\frac{d\mathcal{A}^{(0)}}{da}\right)+\tfrac{1}{2}a.\left(\frac{d^2\mathcal{A}^{(0)}}{da^2}\right)\right\}\\ & \times\cos.(nt+\varepsilon-\varpi)+m'e'.\left\{-\left(\frac{d\mathcal{A}^{(1)}}{da}\right)-\tfrac{1}{2}a'.\left(\frac{dd\mathcal{A}^{(1)}}{da\,da'}\right)\right\}.\cos.(nt+\varepsilon-\varpi').\end{aligned}$$

from decreasing the angles ε, ε', ϖ, ϖ', by a right angle, in the value of df; hence we shall get [1270'']

$$df' = -\frac{am'.ndt}{2}.e.\cos.\varpi.\left\{a.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}a^2.\left(\frac{ddA^{(0)}}{da^2}\right)\right\}$$

$$-am'.ndt.e'.\cos.\varpi'.\left\{A^{(1)}+\tfrac{1}{2}a.\left(\frac{dA^{(1)}}{da}\right)+\tfrac{1}{2}a'.\left(\frac{dA^{(1)}}{da'}\right)+\tfrac{1}{4}aa'.\left(\frac{ddA^{(1)}}{da\,da'}\right)\right\} \qquad [1271]$$

$$+am'.ndt.\Sigma.\left\{i.A^{(i)}+\tfrac{1}{2}a.\left(\frac{dA^{(i)}}{da}\right)\right\}.\cos.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}.$$

We shall, for brevity, put X equal to the part of the expression of df [1270], contained under the sign Σ; and Y for the part of the expression of df' [1271], contained under the same sign. We shall also put, as in § 55, [1073], [1271']

$$(0,1) = -\frac{m'.n}{2}.\left\{a^2.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}a^3.\left(\frac{ddA^{(0)}}{da^2}\right)\right\};$$

$$\boxed{0,1} = \frac{m'.n}{2}.\left\{aA^{(1)}-a^2.\left(\frac{dA^{(1)}}{da}\right)-\tfrac{1}{2}a^3.\left(\frac{ddA^{(1)}}{da^2}\right)\right\}. \qquad [1272]$$

We shall then observe that the coefficient of $e'dt.\sin.\varpi'$, in the expression of df [1270], would become $\boxed{0,1}$, if we should substitute, for the partial [1272']

Multiplying these expressions by $-andt.2\cos.(nt+\varepsilon)$, $-a^2ndt.\sin.(nt+\varepsilon)$, we shall get, by means of [955*b*, *c*, &c.], and retaining only the angles and terms as abovementioned,

$$-andt.2\cos.(nt+\varepsilon).\left(\frac{dR}{dv}\right) = -am'.ndt.\Sigma.iA^{(i)}.\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}$$

$$+am'ndt.e'.\sin.\varpi'.\left\{A^{(1)}+\tfrac{1}{2}a'.\left(\frac{dA^{(1)}}{da'}\right)\right\},$$

$$-a^2ndt.\sin.(nt+\varepsilon).\left(\frac{dR}{da}\right) = -am'ndt.\Sigma.\tfrac{1}{2}a.\left(\frac{dA^{(i)}}{da}\right).\sin.\{i.(n't-nt+\varepsilon'-\varepsilon)+nt+\varepsilon\}$$

$$+\frac{am'ndt}{2}.e.\sin.\varpi.\left\{\tfrac{1}{2}a.\left(\frac{dA^{(0)}}{da}\right)+\tfrac{1}{2}a^2.\left(\frac{d^2A^{(0)}}{da^2}\right)\right\}$$

$$+am'ndt.e'.\sin.\varpi'.\left\{\tfrac{1}{2}a.\left(\frac{dA^{(1)}}{da}\right)+\tfrac{1}{4}aa'.\left(\frac{d^2A^{(1)}}{da\,da'}\right)\right\},$$

substituting these values in [1270*a*], we shall obtain df, [1270].

differentials of $A^{(1)}$, in terms of a', their values in partial differentials relative to a;* lastly, we shall suppose, as in § 50 [1022],

[1273]
$$e\,.\,\sin.\varpi = h\,; \qquad e'\,.\,\sin.\varpi' = h'\,;$$
$$e\,.\,\cos.\varpi = l\,; \qquad e'\,.\,\cos.\varpi' = l'\,;$$

which gives, by the preceding article, $f=\mu\,l$, $f'=\mu\,h$; or simply, $f=l$. [1273'] $f'=h$; taking M for the unity of mass, and neglecting m with respect to M, we shall find

[1274]
$$\frac{d\,h}{d\,t} = (0,1)\,.\,l - \boxed{0,1}\,.\,l' + a\,m'\,.\,n\,.\,Y\,;$$
$$\frac{d\,l}{d\,t} = -(0,1)\,.\,h + \boxed{0,1}\,.\,h' - a\,m'\,.\,n\,.\,X.$$

[1274'] Hence it is easy to conclude, that if we put (Y) for the sum of the terms analogous to $a\,m'.n.Y$, arising from the action of each of the bodies m', m'', [1274''] &c., upon m; (X) for the sum of the terms analogous to $-a\,m'\,.\,n\,.\,X$,

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* (866) From the two first of the equations [1003], we get

$$\tfrac{1}{2}\,a'\,.\left(\frac{d\,A^{(1)}}{d\,a'}\right) = -\tfrac{1}{2}\,A^{(1)} - \tfrac{1}{2}\,a\,.\left(\frac{d\,A^{(1)}}{d\,a}\right)$$
$$\tfrac{1}{4}\,a\,a'\,.\left(\frac{d\,d\,A^{(1)}}{d\,a\,d\,a'}\right) = -\tfrac{1}{2}\,a\,.\left(\frac{d\,A^{(1)}}{d\,a}\right) - \tfrac{1}{4}\,a^2\,.\left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right);$$

substituting these in the term depending on $\sin.\varpi'$, [1270], it becomes

$$a\,m'\,n\,d\,t\,.\,e'\,.\,\sin.\varpi'\,.\left\{\tfrac{1}{2}\,A^{(1)} - \tfrac{1}{2}\,a\,.\left(\frac{d\,A^{(1)}}{d\,a}\right) - \tfrac{1}{4}\,a^2\,.\left(\frac{d\,d\,A^{(1)}}{d\,a^2}\right)\right\},$$

which by [1272] is equal to $e'\,.\,\sin.\varpi'\,.\,d\,t\,.\,\boxed{0,1}$, and if we substitute, in the term depending on $\sin\varpi$, [1270], the value of $(0, 1)$, [1272], and put

$$-a\,m'\,n\,d\,t\,.\,\Sigma\,.\left\{i\,A^{(i)} + \tfrac{1}{2}\,a\,.\left(\frac{d\,A^{(i)}}{d\,a}\right)\right\}.\,\sin.\{i\,.\,(n't - n\,t + \varepsilon' - \varepsilon) + n\,t + \varepsilon\} = -a\,m'\,n\,d\,t\,.\,X,$$

the value of $d\,f$, [1270], will become

[1272a]
$$d\,f = -(0,1)\,.\,d\,t\,.\,e\,.\,\sin.\varpi + \boxed{0,1}\,.\,d\,t\,.\,e'\,.\,\sin.\varpi' - a\,m'\,n\,d\,t\,.\,X,$$

and in like manner $d\,f'$, [1271] will become

[1272b]
$$d\,f' = (0,1)\,.\,d\,t\,.\,e\,.\,\cos.\varpi - \boxed{0,1}\,.\,d\,t\,.\,e'\,.\,\cos.\varpi' + a\,m'\,n\,d\,t\,.\,Y.$$

Substituting in [1254], the values of $e\,.\,\sin.\varpi$. $e\,.\,\cos.\varpi$, [1273], we shall obtain $\mu\,l=f$, $\mu\,h=f'$, and if we put, as in [1273'], $M=1$, and neglect m, m', &c., in comparison with M, we shall get $\mu=M+m=1$, [914']; hence $l=f$, $h=f'$, consequently $d\,l=d\,f$, $d\,h=d\,f'$. Substituting these in $d\,f$, $d\,f'$, [1272a, b], also the values of $e\,.\,\sin.\varpi$, &c., [1273], we shall obtain [1274], which, by generalization, produces [1275].

arising from the same forces; and mark successively, with one accent, two accents, &c., what the quantities (X), (Y), h, l, become, relative to the [1274'''] bodies m', m'', &c.; we shall obtain the following system of differential equations,

$$\frac{dh}{dt} = \{(0,1)+(0,2)+\&c.\}.l - \boxed{0,1}.l' - \boxed{0,2}.l'' - \&c. + (Y);$$
$$\frac{dl}{dt} = -\{(0,1)+(0,2)+\&c.\}.h + \boxed{0,1}.h' + \boxed{0,2}.h'' + \&c. + (X);$$
$$\frac{dh'}{dt} = \{(1,0)+(1,2)+\&c.\}.l' - \boxed{1,0}.l - \boxed{1,2}.l'' - \&c. + (Y');$$
$$\frac{dl'}{dt} = -\{(1,0)+(1,2)+\&c.\}.h' + \boxed{1,0}.h + \boxed{1,2}.h'' + \&c. + (X');$$
&c. [1275]

To integrate these equations, we shall observe, that each of the quantities h, l, h', l', &c., is composed of two parts; the one depending on the mutual [1275'] configuration of the bodies m, m', &c.; the other independent of that configuration, and which contains the secular variations of these quantities. We shall obtain the first part from the consideration, that if we notice that part only, h, l, h', l', &c., will be of the order of the disturbing masses; [1275''] consequently, $(0,1).h$, $(0,1).l$, &c., will be of the order of the squares of the masses; and if we neglect quantities of this order, we shall have,*

$$\frac{dh}{dt} = (Y); \qquad \frac{dl}{dt} = (X);$$
$$\frac{dh'}{dt} = (Y'); \qquad \frac{dl'}{dt} = (X');$$ [1276]

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* (867) Denoting the *periodical* parts of h, l, h', l', &c., by $h_{,}$, $l_{,}$, $h'_{,}$, &c., the *secular* parts by $h_{,,}$, $l_{,,}$, $h'_{,,}$, &c., we shall have $h = h_{,} + h_{,,}$, $l = l_{,} + l_{,,}$; $l' = l'_{,} + l'_{,,}$, &c. These being substituted in [1275], produce equations of the form

$$\frac{dh_{,}+dh_{,,}}{dt} = \{(0,1)+(0,2)+\&c.\}.(l_{,}+l_{,,}) - \boxed{0,1}.(l'_{,}+l'_{,,}) - \boxed{0,2}.(l''_{,}+l''_{,,}) - \&c. + (Y),$$ [1276a]
&c.

Now these equations are linear in h, h', l, l', &c., and the secular and periodical parts will be satisfied separately. The periodical parts become

$$\frac{dh_{,}}{dt} = \{(0,1)+(0,2)+\&c.\}.l_{,} - \boxed{0,1}.l'_{,} - \boxed{0,2}.l''_{,} - \&c. + (Y), \quad \&c.,$$
$$\frac{dh'_{,}}{dt} = \{(1,0)+(1,2)+\&c.\}.l'_{,} - \boxed{1,0}.l_{,} - \boxed{1,2}.l''_{,} - \&c. + (Y'), \quad \&c.,$$

therefore

[1277] $$h=\int(Y)\,.\,dt\,; \qquad l=\int(X)\,.\,dt\,; \qquad h'=\int(Y')\,.\,dt\,; \qquad \&c.$$

If we take these integrals, without noticing the variableness of the elements [1277′] of the orbits, and put Q for what $\int(Y)\,.\,dt$ then becomes; also δQ for the variation of Q, arising from the variation of the elements, we shall have,*

[1278] $$\int(Y)\,.\,dt=Q-\int\delta Q\,;$$

now Q being of the order of the disturbing masses, and the variations of the elements of the orbits being of the same order, δQ will be of the order of the square of these masses; if we neglect quantities of this order, we shall have

[1279] $$\int(Y)\,.\,dt=Q.$$

Therefore we may take the integrals $\int(Y).dt$, $\int(X)\,.\,dt$, $\int(Y').dt$, &c., [1279′] supposing the elements of the orbits to be constant, and then consider these elements as variable in the integrals; we shall thus obtain, in a very simple manner, the periodical parts of the expressions h, l, h', l', &c.

To obtain the parts of these expressions which contain the secular inequalities, we shall observe, that they are given by the integration of the

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and as (Y), (Y'), &c., are of the order of the disturbing forces, it will follow from these equations that $h_{,}$, $h'_{,}$, &c., $l_{,}$, $l'_{,}$, &c., are of the same order; and since $(0,1)$, $(0,2)$, &c., $\boxed{0,1}$, $\boxed{0,2}$, &c., [1272], are also of the same order, the terms $(0,1)\,.\,l_{,}$, $(0,2)\,.\,l_{,}$, &c., $\boxed{0,1}\,.\,l'_{,}$, &c., must be of the order of the squares of the disturbing forces, and, if we neglect terms of this order, we shall find $\frac{dh_{,}}{dt}=(Y)$, $\frac{dh'_{,}}{dt}=(Y')$, &c., as in [1276], whose integrals give [1277].

* (868) Let $d'Q$ be the differential of Q, supposing the elements of the orbit to be constant, δQ the differential of Q, supposing these elements only to be variable, and dQ the complete differential, we shall have $dQ=d'Q+\delta Q$, or $d'Q=dQ-\delta Q$. Then [1277a] as the integral of $(Y)\,.\,dt$, taken upon the supposition that the elements are constant, is Q, [1277′], we shall find, if we again take its differential relative to d', $(Y)\,.\,dt=d'Q$, hence $(Y)\,.\,dt=dQ-\delta Q$, whose integral, considering all the quantities as variable, is $\int(Y)\,.\,dt=Q-\int\delta Q$, as in [1278].

preceding differential equations, deprived of their last terms,* (Y), (X), &c.; for it is evident that the substitution of the periodical parts of h, l, [1279''] h', l', &c., will make these terms disappear. But if we deprive these equations of their last terms, they will become like the differential equations (A), § 55 [1089], which we have considered before in a very full manner.

69. We have observed, in § 65 [1197'] that if the mean motions nt, $n't$, of two bodies m and m', are nearly in the ratio of i' to i, so that $i'n'-in$ [1279'''] is a very small quantity, there may result, in the mean motions of these bodies, very sensible inequalities. This ratio of the mean motions may also produce sensible variations in the excentricities of the orbits, and in the positions of their perihelia. To determine these variations, we shall resume the equation found in § 67 [1262],

$$ede = \frac{a\,.\,n\,dt\,.\sqrt{1-e^2}}{\mu}\,.\left(\frac{dR}{dv}\right) - \frac{a\,.\,(1-e^2)}{\mu}\,.\,\mathrm{d}R. \qquad [1280]$$

It follows from what has been said in § 48 [961, 926'], that *if we take the orbit of* m, *at a given epoch, for the fixed plane, we may neglect in* R, *the inclination* φ *of the orbit* m *to this plane*; then all the terms of the expression [1280'] of R depending on the angle $i'n't-int$, will be comprised in the following form [958],†

$$m'k\,.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'); \qquad [1281]$$

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* (869) This is evident from [1276a, &c.], where we find

$$\frac{dh_{\prime\prime}}{dt} = \{(0,1)+(0,2)+\&c.\}\,.\,l_{\prime\prime} - \boxed{0,1}\,.\,l_{\prime\prime}' - \&c.;$$

$$\frac{dl_{\prime\prime}}{dt} = -\{(0,1)+\&c.\}\,.\,h_{\prime\prime} + \boxed{0,1}\,.\,h_{\prime\prime}' + \boxed{0,2}\,.\,h_{\prime\prime}'' + \&c.;$$

&c.;

which are of the same form as [1275], neglecting the last terms (Y), (X), &c.

† (870) The term θ, of the expression [958], is to be neglected, because the orbit of m, at the origin, is taken for the fixed plane, [1280'], hence g'', [958] is to be put equal to nothing, and if we change g''' into g'', that expression will be as in [1281]; moreover, the equation [959], if we make the same changes, will be $0=i'-i-g-g'-g''$, as above. If the same changes be made in the coefficient of [961], putting also Q for H, it will become $e^g\,.\,e'^{g'}\,.\,(\text{tang.}\,\tfrac{1}{2}\varphi')^{g''}\,.\,Q$, the term $\text{tang.}\,\tfrac{1}{2}\varphi$ being neglected for the same reason that θ was omitted.

i', i, g, g', g'', being integral numbers, such that $0=i'-i-g-g'-g''$,
[1281′] [960]. The coefficient k has for its factor $e^{g}.e'^{g'}.(\text{tang.}\,\frac{1}{2}\varphi')^{g''}$, g, g', g'', being positive in the exponents [961′]; again, if we suppose i and i' to be positive, and i' greater than i, the terms of R which depend on the angle $i'n't-int$, will be of the order $i'-i$, or of an order superior by two, by four, &c., as has been shown in § 48 [957^{viii}]; therefore, if we notice only
[1281″] terms of the order $i'-i$, k will be of the form $e^{g}.e'^{g'}.(\text{tang.}\,\frac{1}{2}\varphi')^{g''}.Q$, Q being a function independent of the excentricities, and of the respective inclinations of the orbits. The numbers g, g', g'', contained under the sign *cos.*, are then *positive*; for if one of them, for example, g, were negative,
[1281‴] and equal to $-f$, k would be of the order $f+g'+g''$; but the equation $0=i'-i-g-g'-g''$, [1281′], would then become $0=i'-i+f-g'-g''$, whence $f+g'+g''=i'-i+2f$; therefore k would be of a higher order than $i'-i$, which is contrary to the supposition. This being premised, we
[1281^{iv}] shall have, by § 48, [963], $\left(\frac{dR}{dv}\right)=\left(\frac{dR}{d\varepsilon}\right)$, provided we suppose $\varepsilon-\varpi$ to be constant in the last differential; the term of $\left(\frac{dR}{dv}\right)$, corresponding to the preceding term of R [1281], will therefore become*

[1282] $$m'.(i+g).k.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta').$$

The corresponding term of dR is

[1283] $$m'.ink.dt.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta');$$

[1283′] noticing therefore only these terms, and neglecting e^2, in comparison with unity, the preceding expression of $e\,d\,e$ [1280] will give

[1284] $$d\,e=\frac{m'a.n\,dt}{\mu}.\frac{gk}{e}.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta');$$

now we have†

[1285] $$\frac{gk}{e}=g.e^{g-1}.e'^{g'}.(\text{tang.}\,\tfrac{1}{2}\varphi')^{g''}.Q=\left(\frac{dk}{de}\right);$$

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* (871) Putting $i+g-g$ for i, in [1281], it becomes
$$m'k.\cos.\{i'n't-(i+g).nt+i'\varepsilon'-(i+g).\varepsilon+g.(nt+\varepsilon-\varpi)-g'\varpi'-g''\theta'\},$$
and if we take the differential relative to ε, supposing $g.(nt+\varepsilon-\varpi)$, to be constant, we shall get the expression [1282].

† (872) In [1281″] we have $k=e^{g}.e'^{g'}.(\text{tang.}\,\frac{1}{2}\varphi)^{g''}.Q$, hence
$$\frac{gk}{e}=g.e^{g-1}.e'^{g'}.(\text{tang.}\,\tfrac{1}{2}\varphi)^{g''}.Q,$$

therefore we shall have, by integration,

$$e = -\frac{m'.an}{\mu.(i'n'-in)}.\left(\frac{dk}{de}\right).\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'). \qquad [1286]$$

Now the sum of all the terms of R, which depend on the angle $i'n't-int$, being represented by the following quantity,*

$$m'.P.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)+m'.P'.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon), \qquad [1287]$$

the corresponding part of e will be

$$\frac{-m'.an}{\mu.(i'n'-in)}.\left\{\left(\frac{dP}{de}\right).\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)+\left(\frac{dP'}{de}\right).\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)\right\}. \qquad [1288]$$

This inequality may become very sensible, if the coefficient $i'n'-in$ be very small, as is the case in the theory of Jupiter and Saturn. It is true [1288'] that the divisor is only of the first power of $i'n'-in$, whereas the mean motion has the second power of that quantity for its divisor, as we have seen in § 65 [1197]; but $\left(\frac{dP}{de}\right)$ and $\left(\frac{dP'}{de}\right)$, being of a lower order

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and from the same value of k we get $\left(\frac{dk}{de}\right)=g.e^{g-1}.e'^{g'}.(\text{tang.}\tfrac{1}{2}\varphi)^{g''}.Q$, therefore $\frac{gk}{e}=\left(\frac{dk}{de}\right)$, as in [1285]. This value being substituted in [1284], we get

$$de=\frac{m'.andt}{\mu}.\left(\frac{dk}{de}\right).\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'), \qquad [1284a]$$

whose integral is [1286].

* (873) Supposing for brevity

$$i'n't-int+i'\varepsilon'-i\varepsilon=T; \qquad g\varpi+g'\varpi'+g''\theta'=W, \qquad [1286a]$$

the value of R, depending on the angle T will be represented by $R=m'k.\cos.(T-W)$, [1281], or $R=m'k.\cos.W.\cos.T+m'k.\sin.W.\sin.T$, [24] Int., and if we put $k.\sin.W=P$, $k.\cos.W=P'$, we shall get $R=m'P.\sin.T+m'P'.\cos.T$, [1286b] as in [1287]. Applying the same notation to the expression [1286], it will become

$$e=-\frac{m'.an}{\mu.(i'n'-in)}.\left(\frac{dk}{de}\right).\{\cos.T.\cos.W+\sin.T.\sin.W\},$$

moreover, if we use the values P, P', [1286b], which give $\left(\frac{dk}{de}\right).\sin.W=\left(\frac{dP}{de}\right)$, [1286c] $\left(\frac{dk}{de}\right).\cos.W=\left(\frac{dP'}{de}\right)$, it will become as in [1288].

[1288″] than P, P', the inequality of the excentricity may be considerable, and even exceed that of the mean motion, if the excentricities e and e' are very small; we shall see some examples of this in the theory of the satellites of Jupiter.

We shall now determine the corresponding inequality of the motion of the perihelion. For this purpose, we shall resume the two equations,

[1289] $$e\,d\,e=\frac{f\,df+f'\,df'}{\mu^2}\,; \qquad e^2\,d\varpi=\frac{f\,df'-f'\,df}{\mu^2}\,;$$

which we have found in § 67 [1255]. These equations give*

[1290] $$df=\mu\,.\,de\,.\,\cos.\varpi-\mu\,e\,.\,d\varpi\,.\,\sin.\varpi\,;$$

hence if we notice only the angle $i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'$, we shall find

[1291] $$df=m'.a.ndt.\left(\frac{dk}{de}\right).\cos.\varpi\,.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta')-\mu e.d\varpi.\sin.\varpi.$$

We shall put

[1292] $$-m'.andt.\left\{\left(\frac{dk}{de}\right)+k'\right\}.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'),$$

for the part of $\mu e\,.\,d\varpi$ depending on the same angle,† and we shall get

[1293] $$df=m'.andt.\left\{\left(\frac{dk}{de}\right)+\tfrac{1}{2}k'\right\}.\sin.\{i'n't-int+i'\varepsilon'-i\varepsilon-(g-1).\varpi-g'\varpi'-g''\theta'\}$$
$$-\frac{m'.andt}{2}.k'.\sin.\{i'n't-int+i'\varepsilon'-i\varepsilon-(g+1).\varpi-g'\varpi'-g''\theta'\}.$$

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* (874) Multiplying the values of $e\,d\,e$, $e^2\,d\varpi$, [1289] by f and $-f'$, respectively, and adding the products we shall obtain $f\,e\,d\,e-f'\,e^2\,d\varpi=\frac{f^2+f'^2}{\mu^2}\,.\,df$, but by [1253′, 1254], $f^2+f'^2=(\mu\,e)^2$, $f=\mu\,e\,.\,\cos.\varpi$, $f'=\mu\,e\,.\,\sin.\varpi$, hence

$$\mu\,e^2\,.\,d\,e\,.\,\cos.\varpi-\mu\,e^3\,.\,d\varpi\,.\,\sin.\varpi=\frac{(\mu\,e)^2}{\mu^2}\,.\,df;$$

dividing by e^2, and reducing, we shall obtain [1290]. Substituting in it the value of $d\,e$, [1284a] we shall get [1291].

† (875) It will be seen that this substitution tends to simplify the computation, by finally rejecting the term k', on account of its smallness, [1293′]. If we put

$$T'=i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta',$$

It is evident, from the last of the expressions of df, given in § 67 [1257], that the coefficient of this last sine has for a factor* $e^{g+1} \,.\, e'^{g'} \,.\, (\text{tang.}\, \tfrac{1}{2} \varphi)^{g''}$; [1293′] k' is therefore of a higher order by two than $\left(\frac{dk}{de}\right)$, and if we neglect it, in comparison with $\left(\frac{dk}{de}\right)$, we shall have

$$-\frac{m' \,.\, a n d t}{\mu} \,.\, \left(\frac{dk}{de}\right) \,.\, \cos.(i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - g''\theta'). \qquad [1294]$$

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and substitute the value of $\mu e d\varpi$ [1292] in [1291], we shall get

$$df = m' \,.\, a n d t \,.\, \left(\frac{dk}{de}\right) . \cos.\varpi \,.\, \sin.\, T' + m' \,.\, a n d t \,.\, \left\{ \left(\frac{dk}{de}\right) + k' \right\} . \sin.\varpi \,.\, \cos.\, T';$$

putting, as in [18, 19] Int.,

$$\cos.\varpi \,.\, \sin.\, T' = \tfrac{1}{2} \sin.\,(T' + \varpi) + \tfrac{1}{2} \sin.\,(T' - \varpi),$$

$$\sin.\varpi \,.\, \cos\; T' = \tfrac{1}{2} \sin.\,(T' + \varpi) - \tfrac{1}{2} \sin.\,(T' - \varpi),$$

and reducing, we get

$$df = m' \,.\, a n d t \,.\, \left\{ \left(\frac{dk}{de}\right) + \tfrac{1}{2} k' \right\} . \sin.\,(T' + \varpi) - \frac{m' \,.\, a n d t}{2} \,.\, k' \,.\, \sin.\,(T' - \varpi),$$

as in [1293].

[*] (876) Having $\left(\frac{dR}{dr}\right) = (1 + u_{\prime})^{-1} \,.\, \left(\frac{dR}{da}\right)$, [962, 952], $\left(\frac{dR}{dv}\right) = \left(\frac{dR}{d\varepsilon}\right)$, [963], [1292a] and $v = nt + \varepsilon + v_{\prime}$, [953], we get by substitution in df, [1257],

$$df = -\frac{a n d t}{\sqrt{1-e^2}} \,.\, \left\{ 2 \cos.\,(nt + \varepsilon + v_{\prime}) + \tfrac{3}{2} e .\, \cos.\varpi + \tfrac{1}{2} e \,.\, \sin.\,(2nt + 2\varepsilon - \varpi + 2v_{\prime}) \right\} . \left(\frac{dR}{d\varepsilon}\right)$$
$$- a^2 n d t \,.\, \sqrt{1-e^2} \,.\, \sin.\,(nt + \varepsilon + v_{\prime}) \,.\, (1 + u_{\prime})^{-1} \,.\, \left(\frac{dR}{da}\right), \qquad [1293a]$$

in which the terms containing ϖ explicitly, are multiplied by e. If we now develop the terms containing $v_{\prime}$, according to the powers of $v_{\prime}$, as in [678a], and then substitute the values of $u_{\prime}$, $v_{\prime}$, deduced from [669], writing $1 + u_{\prime}$ for $\frac{r}{a}$, as in [952], and $nt + \varepsilon + v_{\prime}$, for v, supposing the series [669] to be continued, as in [659, 668], to higher powers of e, and to multiples of the angle ϖ, it will appear that wherever the angle ϖ occurs, in the *coefficient* of $\left(\frac{dR}{d\varepsilon}\right)$, or $\left(\frac{dR}{da}\right)$, [1293a], it will be multiplied by the quantity e; the term depending on [1293b] the angle 2ϖ will be multiplied by a coefficient of the order e^2, and in general the angle $g\varpi$, will be connected with a coefficient of the order e^g. Again, from [961], the value R, as for the term of $e\,d\varpi$, which corresponds to the term

[1295] $$m'k\,.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta')\,;$$

of the expression of R. Hence it follows, that the part of ϖ, which corresponds to the part of R, expressed by [1287],

[1296] $$m'.P.\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)+m'.P'.\cos.(i'n't-int+i'\varepsilon'-i\varepsilon),$$

is equal to

[1297] $$\frac{m'.an}{\mu.(i'n'-in).e}.\left\{\left(\frac{dP}{de}\right).\cos.(i'n't-int+i'\varepsilon'-i\varepsilon)-\left(\frac{dP'}{de}\right).\sin.(i'n't-int+i'\varepsilon'-i\varepsilon)\right\}.$$

We shall therefore have, in a very simple manner, the variations of the excentricity and of the perihelion, depending on the angle $i'n't-int+i'\varepsilon'-i\varepsilon$. [1297′] They are connected with the corresponding variation of the mean motion ζ, in such a manner that the variation of the excentricity is*

[1298] $$\frac{1}{3in}.\left(\frac{dd\zeta}{de\,dt}\right);$$

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well as those of $\left(\frac{dR}{da}\right)$, $\left(\frac{dR}{d\varepsilon}\right)$, must also have the same property, therefore df, [1293a], must also have the same property, and the term of [1293],

$$-\frac{m'.andt}{2}.k'.\sin.\{i'n't-int+i'\varepsilon'-i\varepsilon-(g+1).\varpi-g'\varpi'-g''\theta'\},$$

which contains the angle $(g+1).\varpi$, must have for its coefficient a term of the order e^{g+1}, and as e' and $\text{tang.}\,\frac{1}{2}\,\varphi'$ do not occur in the value of df, [1293a], except as they are found in R, these quantities must be of the same form in df as in R, [961], namely, $e'^{g'}.(\text{tang.}\,\frac{1}{2}\,\varphi')^{g''}$, according to the form assumed in [1293′]. Hence the value of k' must contain the factor $e^{g+1}.e'^{g'}.(\text{tang.}\,\frac{1}{2}\,\varphi')^{g''}$. The other term of df, [1293], has for a factor $\left(\frac{dk}{dc}\right)$, and this, by [1285], is of the order $e^{g-1}.e'^{g'}.(\text{tang.}\,\frac{1}{2}\,\varphi')^{g''}$, in which the exponent of e is less by *two* than in the other term k'; therefore we may neglect k', and the value of $\mu\,e\,d\varpi$ [1292], will give $e\,d\varpi$, as in [1294], and if we use T, W, [1286a], we shall get

$$e\,d\varpi=-\frac{m'andt}{\mu}.\left(\frac{dk}{de}\right).\cos.(T-W)=-\frac{m'andt}{\mu}.\left(\frac{dk}{de}\right).(\cos.T.\cos.W+\sin.T.\sin.W)$$
$$=\frac{mandt}{\mu}.\left\{-\left(\frac{dP}{de}\right).\sin.T-\left(\frac{dP'}{de}\right).\cos.T\right\},$$

[1286c]. Dividing by e and integrating, we get ϖ, [1297].

* (877) Putting in ζ, [1197], $A=i'\varepsilon'-i\varepsilon-g\varpi-g'\varpi'-g''\theta'$, to agree with [1281], and using T, W, P, P', [1286a, b, c], we shall find

and the variation of the longitude of the perihelion is

$$\frac{i'n'-in}{3in.e}\cdot\left(\frac{d\zeta}{de}\right). \qquad [1299]$$

The corresponding variation of the excentricity of the orbit of m', arising from the action of m, will be*

$$-\frac{1}{3i'n'}\cdot\left(\frac{dd\zeta'}{de'dt}\right); \qquad [1300]$$

and the variation of the longitude of the perihelion of m', will be

$$-\frac{(i'n'-in)}{3i'n'.e'}\cdot\left(\frac{d\zeta'}{de'}\right), \qquad [1301]$$

and as we have by § 65 [1208], $\zeta'=-\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.\zeta$, these variations will be

$$\frac{m.\sqrt{a}}{3i'n'.m'.\sqrt{a'}}\cdot\left(\frac{dd\zeta}{de'dt}\right) \quad\text{and}\quad \frac{(i'n'-in).m.\sqrt{a}}{3i'n'.e'.m'.\sqrt{a'}}\cdot\left(\frac{d\zeta}{de'}\right). \qquad [1302]$$

$$\zeta=-\frac{3i.m'an^2}{\mu.(i'n'-in)^2}.k.\sin.(T-W)=-\frac{3i.m'an^2}{\mu.(i'n'-in)^2}.k.\{\sin.T.\cos.W-\cos.T.\sin.W\} \qquad [1298a]$$

$$=\frac{3i.m'an^2}{\mu.(i'n'-in)^2}.\{P.\cos.T-P'.\sin.T\}. \qquad [1298b]$$

From this last we obtain

$$\left(\frac{d\zeta}{de}\right)=\frac{3i.m'an^2}{\mu.(i'n'-in)^2}.\left\{\left(\frac{dP}{de}\right).\cos.T-\left(\frac{dP'}{de}\right).\sin T\right\},$$

$$\left(\frac{dd\zeta}{dedt}\right)=-\frac{3i.m'an^2}{\mu.(i'n'-in)}.\left\{\left(\frac{dP}{de}\right).\sin.T+\left(\frac{dP'}{de}\right).\cos.T\right\},$$

hence we easily deduce

$$\frac{1}{3in}\cdot\left(\frac{dd\zeta}{dedt}\right)=\frac{-m'an}{\mu.(i'n'-in)}.\left\{\left(\frac{dP}{de}\right).\sin.T+\left(\frac{dP'}{de}\right).\cos.T\right\};$$

$$\frac{(i'n'-in)}{3ine}\cdot\left(\frac{d\zeta}{de}\right)=\frac{m'an}{\mu.(i'n'-in).e}.\left\{\left(\frac{dP}{de}\right).\cos.T-\left(\frac{dP'}{de}\right).\sin.T\right\};$$

and the second members of these expressions are equal respectively to those of [1288, 1297], representing the corresponding variations of e, ϖ; the first members must therefore represent those variations, which is conformable to [1298, 1299].

* (878) The formulas [1300, 1301] are deduced from [1298, 1299], by changing n, e, ζ, ε, &c., into n', e', ζ', ε', &c., and the contrary, considering m' as the disturbed planet, and m as the disturbing planet; and in order that the angle T may remain unchanged, it will also be necessary to write $-i'$ for i, and $-i$ for i'. [1300a]

When the quantity $i'n' - in$ is very small, the inequality depending on the
[1302'] angle $i'n't - int$, produces another sensible equation in the expression of the mean motion, among the terms depending on the square of the disturbing masses; we have given the analysis of it in § 65 [1213, 1214]. The same inequality produces, in the expressions of de and $d\varpi$, some terms of the order of the square of these masses, which are functions of the elements of the orbits only, and have a sensible influence on the secular variations of these
[1302''] elements. For if we take into consideration the expression of de, depending upon the angle $i'n't - int$, we shall find, by what has been said*

[1303] $$de = -\left\{\frac{m'a.ndt}{\mu}.\left(\frac{dP}{de}\right).\cos.(i'n't - int + i'\varepsilon' - i\varepsilon) - \left(\frac{dP'}{de}\right).\sin.(i'n't - int + i'\varepsilon' - i\varepsilon)\right\}.$$

From § 65, the mean motion of nt ought to be increased by†

[1304] $$\frac{3m'.an^2i}{(i'n' - in)^2.\mu}.\{P.\cos.(i'n't - int + i'\varepsilon' - i\varepsilon) - P'.\sin.(i'n't - int + i'\varepsilon' - i\varepsilon)\},$$

and the mean motion $n't$ ought to be increased by

[1305] $$-\frac{3m'.an^2i}{(i'n' - in)^2.\mu}.\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.\{P.\cos.(i'n't - int + i'\varepsilon' - i\varepsilon) - P'.\sin.(i'n't - int + i'\varepsilon' - i\varepsilon)\}.$$

In consequence of these increments, the value of de will be increased by the function‡

[1306] $$-\frac{3m'.a^2in^3.dt}{2\mu^2.\sqrt{a'}.(i'n' - in)^2}.\{i.m'.\sqrt{a'} + i'.m.\sqrt{a}\}.\left\{P.\left(\frac{dP'}{de}\right) - P'.\left(\frac{dP}{de}\right)\right\};$$

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* (879) The differential of the part of e [1288], relative to dt, gives de, [1303].

† (880) In [1212] the increment of nt is $-\frac{3i.m'an^2}{\mu.(i'n' - in)^2}.k.\sin.(i'n't - int + \mathcal{A})$, which is the same as ζ, [1298a], and this was in [1298b], reduced to the form

$$\frac{3i.m'an^2}{\mu.(i'n' - in)^2}.\{P.\cos.T - P'.\sin.T\},$$

as in [1304]; Multiplying it by $-\frac{m.\sqrt{a}}{m'.\sqrt{a'}}$, [1208], we shall get ζ', [1305].

‡ (881) Multiplying [1305] by i', [1304] by $-i$, and adding the products we shall get the increment $\mathcal{A}$ of the angle $T = i'n't - int + i'\varepsilon' - i\varepsilon$, [1286a], arising from these terms, which will be

and the value of $d\varpi$ will be increased by the function*

$$\frac{3m'.a^2in^3.dt}{2\mu^2.\sqrt{a'}.(i'n'-in)^2.e}.\{i.m'.\sqrt{a'}+i'.m.\sqrt{a}\}.\left\{P.\left(\frac{dP}{de}\right)+P'.\left(\frac{dP'}{de}\right)\right\}; \qquad [1307]$$

$$\mathcal{A}=-\frac{3m'an^2.ii'}{(i'n'-in)^2.\mu}.\frac{m.\sqrt{a}}{m'.\sqrt{a'}}.\{P.\cos.T-P'.\sin.T\}-\frac{3m'an^2.ii}{(i'n'-in)^2.\mu}.\{P.\cos.T-P'.\sin.T\}$$

$$=\frac{-3an^2.i.\{i'm.\sqrt{a}+im'.\sqrt{a'}\}}{(i'n'-in)^2.\mu.\sqrt{a'}}.\{P.\cos.T-P'.\sin.T\}. \qquad [1306a]$$

Now if we increase the angle T by $\mathcal{A}$ in the expression [1303], it becomes.

$$de=\frac{-m'an.dt}{\mu}.\left\{\left(\frac{dP}{de}\right).\cos.(T+\mathcal{A})-\left(\frac{dP'}{de}\right).\sin.(T+\mathcal{A})\right\}; \qquad [1306b]$$

from [60, 61] Int., neglecting $\mathcal{A}^2$, we get

$$\cos.(T+\mathcal{A})=\cos.T-\mathcal{A}.\sin.T; \qquad \sin.(T+\mathcal{A})=\sin.T+\mathcal{A}.\cos.T; \qquad [1306c]$$

substituting this in [1306b], and retaining only the terms depending on $\mathcal{A}$, we shall find

$$de=\frac{m'an.dt}{\mu}.\mathcal{A}.\left\{\left(\frac{dP}{de}\right).\sin.T+\left(\frac{dP'}{de}\right).\cos.T\right\}.$$

Substituting now the value of $\mathcal{A}$, [1306a], and reducing the angles, retaining only the terms independent of T, so that $\sin.T.\{P.\cos.T-P'.\sin.T\}$, produces $-\frac{1}{2}P'$, and $\cos.T.\{P.\cos.T-P'.\sin.T\}$, becomes $\frac{1}{2}P$, we shall get

$$de=\frac{m'an.dt}{\mu}.\left\{\frac{-3an^2.i.(i'm.\sqrt{a}+im'.\sqrt{a'})}{(i'n'-in)^2.\mu.\sqrt{a'}}\right\}.\left\{-\tfrac{1}{2}P'.\left(\frac{dP}{de}\right)+\tfrac{1}{2}P.\left(\frac{dP'}{de}\right)\right\},$$

and if we arrange the terms differently it will become as in [1306].

* (882) Taking the differential of ϖ, [1297], we get

$$\frac{m'andt}{\mu e}.\left\{-\left(\frac{dP}{de}\right).\sin.T-\left(\frac{dP'}{de}\right).\cos.T\right\},$$

and if we change, as in the last note, T into $T+\mathcal{A}$, and develop the expressions as in [1306c], retaining only the terms depending on $\mathcal{A}$, it becomes

$$\frac{m'andt}{\mu e}.\mathcal{A}.\left\{-\left(\frac{dP}{de}\right).\cos.T+\left(\frac{dP'}{de}\right).\sin.T\right\};$$

substituting the value of $\mathcal{A}$, [1306a], reducing the angles and retaining only the terms independent of T, in the same manner as in the last note it becomes

$$\frac{m'andt}{\mu e}.\left\{\frac{-3an^2i.\{i'm.\sqrt{a}+im'.\sqrt{a'}\}}{(i'n'-in)^2.\mu.\sqrt{a'}}\right\}.\left\{-\tfrac{1}{2}P.\left(\frac{dP}{de}\right)-\tfrac{1}{2}P'.\left(\frac{dP'}{de}\right)\right\},$$

which is evidently equal to the expression [1307].

we shall likewise find that the value of de' will be increased by the function*

[1308] $$-\frac{3m\,.\,a^2\,.\,\sqrt{a}\,.\,i\,n^3\,.\,dt}{2\,\mu^2\,.\,a'\,.\,(i'n'-in)^2}\,.\,\{i\,.\,m'\,.\,\sqrt{a'}+i'\,.\,m\,.\,\sqrt{a}\}\,.\left\{P\,.\left(\frac{dP'}{de'}\right)-P'\,.\left(\frac{dP}{de'}\right)\right\};$$

and that the value of $d\varpi'$ will be increased by the function

[1309] $$\frac{3m\,.\,a^2\,.\,\sqrt{a}\,.\,i\,n^3\,.\,dt}{2\,\mu^2\,.\,a'\,.\,(i'n'-in)^2\,.\,e'}\,.\,\{i\,.\,m'\,.\,\sqrt{a'}+i'\,.\,m\,.\,\sqrt{a}\}\,.\left\{P\,.\left(\frac{dP}{de'}\right)+P'\,.\left(\frac{dP'}{de'}\right)\right\}.$$

[1309'] These different terms are sensible in the theory of Jupiter and Saturn, and in that of the satellites of Jupiter. The variations of e, e', ϖ, and ϖ', relative to the angle $i'n't-int$, may also introduce some constant terms, [1309''] of the order of the square of the disturbing masses, in the differentials de, de', $d\varpi$, and $d\varpi'$, depending on the variations of e, e', ϖ, ϖ', relative to the same angle; it will be easy to take notice of them by the preceding analysis. Lastly, it will be easy, by this analysis, to find the terms of the expressions [1309'''] of e, ϖ, e', and ϖ', which depend on the angle $i'n't-int+i'\varepsilon'-i\varepsilon$, but have not $i'n'-in$ for a divisor, and those which depend on the same angle, and on the double of that angle, which are of the order of the square of the [1309$^{\text{iv}}$] disturbing forces. These terms, in the theory of Jupiter and Saturn, are of

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* (883) These increments of de', $d\varpi'$, corresponding to the part of R, [1296], may be deduced from those of de, $d\varpi$, by making the changes mentioned in [1300a], hence we get

[1307a]
$$de'=-\frac{3m\,.\,a'^2\,.\,i'n'^3\,.\,dt}{2\,\mu^2\,.\,\sqrt{a}\,.\,(i'n'-in)^2}\,.\,\{im'\,.\,\sqrt{a'}+i'm\,.\,\sqrt{a}\}\,.\left\{P\,.\left(\frac{dP'}{de'}\right)-P'\,.\left(\frac{dP}{de'}\right)\right\},$$
$$d\varpi'=\frac{3m\,.\,a'^2\,.\,i'n'^3\,.\,dt}{2\,\mu^2\,.\,\sqrt{a}\,.\,(i'n'-in)^2\,.\,e'}\,.\,\{im'\,.\,\sqrt{a'}+i'm\,.\,\sqrt{a}\}\,.\left\{P\,.\left(\frac{dP}{de'}\right)+P'\,.\left(\frac{dP'}{de'}\right)\right\},$$

in which terms the factor $\frac{a'^2i'n'^3}{\sqrt{a}}$, may be put under another form; for $n^2a^3=\mu=M+m$, [605'], and if we neglect m in comparison with M, and put $M=1$, we shall get $n^2a^3=1=n'^2a'^3$, hence

[1308a] $$\frac{a'^2i'n'^3}{\sqrt{a}}=\frac{i'n'.(a'^3n'^2)}{a'\,.\,\sqrt{a}}=\frac{i'n'.(a^3n^2)}{a'\,.\,\sqrt{a}}=\frac{i'n'\,.\,a^2\,.\,\sqrt{a}\,.\,n^2}{a'},$$

and as $i'n'-in$ is very small, we may put $i'n'=in$, and it becomes

$$\frac{a'^2\,.\,i'n'^3}{\sqrt{a}}=\frac{in\,.\,a^2\,.\,\sqrt{a}\,.\,n^2}{a'}=\frac{a^2\,.\,\sqrt{a}\,.\,in^3}{a'}.$$

Substituting this in [1307a], we get for de', $d\varpi'$, the values [1308, 1309].

sufficient importance to be noticed; we shall develop them as far as it shall be found necessary, when we shall treat of that theory.

70. We shall now determine the variations of the nodes, and of the inclinations of the orbits; for this purpose, we shall resume the equations of § 64 [1173],

$$dc = dt \cdot \left\{ y \cdot \left(\frac{dR}{dx}\right) - x \cdot \left(\frac{dR}{dy}\right) \right\};$$
$$dc' = dt \cdot \left\{ z \cdot \left(\frac{dR}{dx}\right) - x \cdot \left(\frac{dR}{dz}\right) \right\};$$
$$dc'' = dt \cdot \left\{ z \cdot \left(\frac{dR}{dy}\right) - y \cdot \left(\frac{dR}{dz}\right) \right\}.$$
[1310]

If we take notice only of the action of m', the value of R, § 46 [913], will give,*

$$y \cdot \left(\frac{dR}{dx}\right) - x \cdot \left(\frac{dR}{dy}\right) = m' \cdot \{x'y - xy'\} \cdot \left\{ \frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}} \right\};$$
$$z \cdot \left(\frac{dR}{dx}\right) - x \cdot \left(\frac{dR}{dz}\right) = m' \cdot \{x'z - xz'\} \cdot \left\{ \frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}} \right\};$$
$$z \cdot \left(\frac{dR}{dy}\right) - y \cdot \left(\frac{dR}{dz}\right) = m' \cdot \{y'z - yz'\} \cdot \left\{ \frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}} \right\}.$$
[1311]

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* (884) The expression of R, [913, 914], depending on m' is

$$R = \frac{m' \cdot (xx' + yy' + zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}};$$ hence

$$\left(\frac{dR}{dx}\right) = \frac{m'x'}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{m' \cdot (x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$
[1310a]

and if we put

$$W = \frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}} - \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$
$$V = \frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$
[1310b]

it becomes $\left(\frac{dR}{dx}\right) = m'x' \cdot W + m'x \cdot V$, in like manner if we change x, x', into y, y',

If we now put

[1312] $$\frac{c''}{c}=p\,; \qquad \frac{c'}{c}=q\,;$$

the two variable quantities p and q will give, as in § 64, the tangent of the inclination φ of the orbit of m, and the longitude θ of its node, by means of the equations*

[1313] $$\text{tang.}\,\varphi=\sqrt{p^2+q^2}\,; \qquad \text{tang.}\,\theta=\frac{p}{q}.$$

We shall put p', q', p'', q'', &c., for what p, q, become, relative to the bodies m', m'', &c., and we shall find, from § 64,†

[1314] $$z=q\,y-p\,x\,; \qquad z'=q'\,y'-p'\,x'\,; \qquad \&c.$$

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and the contrary, $\left(\frac{dR}{dy}\right)=m'\,y'\,.\,W+m'\,y\,.\,V$; changing also x, x', into z, z', and the contrary, we find $\left(\frac{dR}{dz}\right)=m'\,z'\,.\,W+m'\,z\,.\,V$. Substituting these in the first members of the equations [1311], and then using [1310], we obtain

[1311a] $$\frac{dc}{dt}=m'\,.\,(x'\,y-x\,y')\,.\,W\,; \qquad \frac{dc'}{dt}=m'\,.\,(x'\,z-x\,z')\,.W\,; \qquad \frac{dc''}{dt}=m'\,.\,(y'\,z-y\,z')\,.\,W\,;$$

the terms depending on V destroying each other. The remaining terms agree with the second members of the equations [1311, 1311a].

* (885) The assumed values of p, q, [1312], give $c''=p\,c$, $c'=q\,c$, substituting these in [1174] we get $\text{tang.}\,\varphi=\sqrt{p^2+q^2}$, $\text{tang.}\,\theta=\frac{p}{q}$, as in [1313], which might also be reduced to the form of those in [1032], $\text{tang.}\,\varphi\,.\,\sin.\,\theta=p$, $\text{tang.}\,\varphi\,.\,\cos.\,\theta=q$.
[1312a] For the sum of the squares of these last gives $\text{tang.}\,\varphi=\sqrt{p^2+q^2}$, and if we divide the first by the second we get $\text{tang.}\,\theta=\frac{p}{q}$, therefore the values of p, q, [1312], are equivalent to those in [1032].

† (886) The equation [579], $0=c''x-c'y+cz$, being divided by c, gives $z=\frac{c'}{c}\,.\,y-\frac{c''}{c}\,.\,x\,;$ and if we substitute the values of $\frac{c''}{c}$, $\frac{c'}{c}$, [1312], it becomes $z=q\,y-p\,x$, as in [1314], and from this we get $z'=q'\,y'-p'\,x'$, &c., by accenting the letters in the usual manner.

The differential of the preceding value of p [1312] gives*

$$\frac{dp}{dt}=\frac{1}{c}\cdot\left\{\frac{d\,c''-p\,d\,c}{d\,t}\right\}; \qquad [1315]$$

substituting for $d\,c$, $d\,c''$, their values, we shall get†

$$\frac{dp}{dt}=\frac{m'}{c}\cdot\{(q-q').yy'+(p'-p).x'y\}.\left\{\frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}-\frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}\right\}; \qquad [1316]$$

we shall also have‡

$$\frac{dq}{dt}=\frac{m'}{c}\cdot\{(p'-p).xx'+(q-q').xy'\}.\left\{\frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}-\frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}\right\}. \qquad [1317]$$

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* (887) The differential of the first equation $\frac{c''}{c}=p$, divided by $d\,t$ is,

$$\frac{dp}{dt}=\frac{1}{c}\cdot\left\{\frac{d\,c''}{dt}-\frac{c''}{c}\cdot\frac{d\,c}{d\,t}\right\},$$

and if we substitute the value of $\frac{c''}{c}=p$, it becomes as in [1315].

† (888) The values of $\frac{d\,c}{d\,t}$, $\frac{d\,c''}{d\,t}$, [1311*a*], being substituted in [1315] it becomes $\frac{dp}{dt}=\frac{m'}{c}\cdot\{y'z-y\,z'-p\,.\,(x'y-xy')\}\,.\,W$, and by using the values z, z', [1314], $\frac{dp}{dt}=\frac{m'}{c}\cdot\{y'\,.\,(q\,y-p\,x)-y\,.\,(q'y'-p'x')-p\,.\,(x'y-x\,y')\}\,.\,W$, or by reduction

$$\frac{dp}{d\,t}=\frac{m'}{c}\cdot\{(q-q')\,.\,y\,y'+(p'-p)\,.\,x'y\}\,.\,W, \qquad [1316a]$$

as in [1316].

‡ (889) The differential of the value of q, [1312], divided by $d\,t$, gives

$$\frac{d\,q}{dt}=\frac{1}{c}\cdot\left\{\frac{dc'}{dt}-\frac{c'}{c}\cdot\frac{dc}{dt}\right\}=\frac{1}{c}\cdot\left\{\frac{dc'}{dt}-q\cdot\frac{d\,c}{d\,t}\right\},$$

[1312]. Substituting the values of $\frac{dc'}{dt}$, $\frac{dc}{d\,t}$, [1311*a*], we get

$$\frac{d\,q}{d\,t}=\frac{m'}{c}\cdot\{x'z-x\,z'-q\,.\,(x'y-x\,y')\}\,.\,W;$$

and the values of z, z', [1314], make it

$$\frac{d\,q}{d\,t}=\frac{m'}{c}\cdot\{x'\,.\,(q\,y-p\,x)-x\,.\,(q'y'-p'x')-q\,.\,(x'y-x\,y')\}\,.\,W;$$

which by reduction becomes

$$\frac{d\,q}{d\,t}=\frac{m'}{c}\cdot\{(p'-p)\,.\,x\,x'+(q-q')\,.\,x\,y'\}\,.\,W; \qquad [1316b]$$

as in [1317].

If we substitute, for x, y, x', y', their values [1243], $r.\cos.v$, $r.\sin.v$, $r'.\cos.v'$, $r'.\sin.v'$, we shall find*

$$(q-q')\cdot yy'+(p'-p)\cdot x'y=\left(\frac{q'-q}{2}\right).rr'.\{\cos.(v'+v)-\cos.(v'-v)\}$$

$$+\left(\frac{p'-p}{2}\right).rr'.\{\sin.(v'+v)-\sin.(v'-v)\};$$

[1318]

$$(p'-p)\cdot xx'+(q-q')\cdot xy'=\left(\frac{p'-p}{2}\right).rr'.\{\cos.(v'+v)+\cos.(v'-v)\}$$

$$+\left(\frac{q-q'}{2}\right).rr'.\{\sin.(v'+v)+\sin.(v'-v)\}.$$

Neglecting the excentricities and inclinations of the orbits, we get†

[1319] $$r=a; \qquad v=nt+\varepsilon; \qquad r'=a'; \qquad v'=n't+\varepsilon';$$

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* (890) The values $x=r.\cos.v$, $y=r.\sin.v$, [1243], and the similar values $x'=r'.\cos.v'$; $y'=r'.\sin.v'$; give

$$yy'=rr'.\sin.v.\sin.v'=\tfrac{1}{2}rr'.\{\cos.(v'-v)-\cos.(v'+v)\},$$

and in like manner

$$x'y=rr'.\cos.v'.\sin.v=\tfrac{1}{2}rr'.\{\sin.(v'+v)-\sin.(v'-v)\};$$

[1317a] $$xx'=rr'.\cos.v.\cos.v'=\tfrac{1}{2}rr'.\{\cos.(v'+v)+\cos.(v'-v)\};$$

$$xy'=rr'.\cos.v.\sin.v'=\tfrac{1}{2}rr'.\{\sin.(v'+v)+\sin.(v'-v)\}.$$

Substituting these in the first members of [1318], we shall obtain the second members of the same equations.

† (891) The expressions [1319] may be deduced from [1268], neglecting $u_{,}$. $u'_{,}$, $v_{,}$, $v'_{,}$, which are of the same order as the excentricities. Neglecting z, z', which are of the order of the inclinations, and substituting the values of x, y, x', y', of the last note, we shall get, as in [1224b], $x'^2+y'^2+z'^2=x'^2+y'^2=r'^2$, and

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=(x'-x)^2+(y'-y)^2=r^2-2rr'.\cos.(v'-v)+r'^2,$$

and if we use the values [1319], we may put $x'^2+y'^2+z'^2=a'^2$,

$$(x'-x)^2+(y'-y)^2+(z'-z)^2=a^2-2aa'.\cos.(n't-nt+\varepsilon'-\varepsilon)+a'^2,$$

substituting these in the first member of [1320], or in W, [1310b], it becomes like the second member of that expression; hence if we use the values [1319, 1321], this expression

[1320a] of W will become $W=\frac{1}{a'^3}-\tfrac{1}{2}\Sigma.B^{(i)}.\cos.i.(n't-nt+\varepsilon'-\varepsilon)$.

hence we obtain

$$\frac{1}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}-\frac{1}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}=\frac{1}{a'^3}-\frac{1}{\{a^2-2aa'.\cos.(n't-nt+\varepsilon'-\varepsilon)+a'^2\}^{\frac{3}{2}}}$$ [1320]

we have also, by § 48 [956],

$$\frac{1}{\{a^2-2\,a\,a'.\cos.\,(n'\,t-n\,t+\varepsilon'-\varepsilon)+a'^2\}^{\frac{3}{2}}}=\tfrac{1}{2}\,\Sigma\,.\,B^{(i)}.\cos.\,i\,.(n'\,t-n\,t+\varepsilon'-\varepsilon).$$ [1321]

The symbol Σ includes all integral values of i, positive or negative, also the value $i=0$; therefore, if we neglect the terms of the order of the squares and products of the excentricities and inclinations of the orbits, we shall find,* [1321']

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* (892) The terms p, q, p', q', being of the same order, as the inclination of the orbits [1313], we may, if we neglect the square of these quantities, substitute in [1318], the values r, v, r', v', [1319], and if we put $T=n'\,t-n\,t+\varepsilon'-\varepsilon$, we shall find

$$v'+v=n'\,t+n\,t+\varepsilon'+\varepsilon=T+2\,n\,t+2\,\varepsilon, \qquad v'-v=n'\,t-n\,t+\varepsilon'-\varepsilon=T.$$

and the expressions [1318] will become

$$(q-q').\,y\,y'+(p'-p).\,x'\,y=\tfrac{1}{2}.(q'-q).\,a\,a'.\{\cos.\,(T+2\,n\,t+2\,\varepsilon)-\cos.\,T\}$$
$$+\tfrac{1}{2}.(p'-p).\,a\,a'.\{\sin.\,(T+2\,n\,t+2\,\varepsilon)-\sin.\,T\};$$ [1320b]

$$(p'-p).\,x\,x'+(q-q').\,x\,y'=\tfrac{1}{2}.(p'-p).\,a\,a'.\{\cos.\,(T+2\,n\,t+2\,\varepsilon)+\cos.\,T\}$$
$$+\tfrac{1}{2}.(q-q').\,a\,a'.\{\sin.\,(T+2\,n\,t+2\,\varepsilon)+\sin.\,T\};$$ [1320c]

substituting these in [1316a, b], and using W, [1320a], we shall get,

$$\frac{dp}{dt}=\left(\frac{q'-q}{2c}\right).m'.\,a\,a'.\{\cos.\,(T+2\,n\,t+2\,\varepsilon)-\cos.\,T\}.\left\{\frac{1}{a'^3}-\tfrac{1}{2}\,\Sigma\,.\,B^{(i)}.\cos.\,i\,T\right\}$$
$$+\left(\frac{p'-p}{2c}\right).m'.\,a\,a'.\{\sin.\,(T+2\,n\,t+2\,\varepsilon)-\sin.\,T\}.\left\{\frac{1}{a'^3}-\tfrac{1}{2}\,\Sigma\,.\,B^{(i)}.\cos.\,i\,T\right\};$$

$$\frac{dq}{dt}=\left(\frac{p'-p}{2c}\right).m'.\,a\,a'.\{\cos.\,(T+2\,n\,t+2\,\varepsilon)+\cos.\,T\}.\left\{\frac{1}{a'^3}-\tfrac{1}{2}\,\Sigma\,.\,B^{(i)}.\cos.\,i\,T\right\}$$
$$+\left(\frac{q-q'}{2c}\right).m'.\,a\,a'.\{\sin.\,(T+2\,n\,t+2\,\varepsilon)+\sin.\,T\}.\left\{\frac{1}{a'^3}-\tfrac{1}{2}\,\Sigma\,.\,B^{(i)}.\cos.\,i\,T\right\}.$$

The factor $\frac{1}{a'^3}$ of the expression of W, produces in these values of $\frac{dp}{dt}$, $\frac{dq}{dt}$, the same terms as are found in [1322], independent of $B^{(i)}$. The terms depending on $B^{(i)}$ can be simplified, observing that by the formulas [954c, b], we get

$$\begin{aligned}
\frac{dp}{dt} &= \frac{(q'-q)}{2c}\cdot\frac{m'a}{a'^2}.\{\cos.(n't+nt+\varepsilon'+\varepsilon)-\cos.(n't-nt+\varepsilon'-\varepsilon)\} \\
&+\frac{(p'-p)}{2c}\cdot\frac{m'a}{a'^2}.\{\sin.(n't+nt+\varepsilon'+\varepsilon)-\sin.(n't-nt+\varepsilon'-\varepsilon)\} \\
&+\frac{(q'-q)}{4c}.m'.aa'.\Sigma.B^{(i)}.\left\{\begin{array}{l}\cos.[(i+1).(n't-nt+\varepsilon'-\varepsilon)] \\ -\cos.[(i+1).(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\end{array}\right\} \\
&+\frac{(p'-p)}{4c}.m'.aa'.\Sigma.B^{(i)}.\left\{\begin{array}{l}\sin.[(i+1).(n't-nt+\varepsilon'-\varepsilon)] \\ -\sin.[(i+1).(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\end{array}\right\};
\end{aligned}$$

[1322]

$$\begin{aligned}
\frac{dq}{dt} &= \frac{(p'-p)}{2c}\cdot\frac{m'a}{a'^2}.\{\cos.(n't+nt+\varepsilon'+\varepsilon)+\cos.(n't-nt+\varepsilon'-\varepsilon)\} \\
&+\frac{(q'-q)}{2c}\cdot\frac{m'a}{a'^2}.\{\sin.(n't+nt+\varepsilon'+\varepsilon)+\sin.(n't-nt+\varepsilon'-\varepsilon)\} \\
&+\frac{(p-p')}{4c}.m'.aa'.\Sigma.B^{(i)}.\left\{\begin{array}{l}\cos.[(i+1).(n't-nt+\varepsilon'-\varepsilon)] \\ +\cos.[(i+1).(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\end{array}\right\} \\
&+\frac{(q'-q)}{4c}.m'.aa'.\Sigma.B^{(i)}.\left\{\begin{array}{l}\sin.[(i+1).(n't-nt+\varepsilon'-\varepsilon)] \\ +\sin.[(i+1).(n't-nt+\varepsilon'-\varepsilon)+2nt+2\varepsilon]\end{array}\right\}.
\end{aligned}$$

The value $i=-1$, gives, in the expression of $\frac{dp}{dt}$, the constant quantity* [1322'] $\frac{(q'-q)}{4c}.m'.aa'.B^{(-1)}$; all the other terms of the expression of $\frac{dp}{dt}$, are

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$$\cos.(T+2nt+2\varepsilon).\tfrac{1}{2}\Sigma.B^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.B^{(i)}.\cos.\{(i+1).T+2nt+2\varepsilon\};$$

$$\cos.T.\tfrac{1}{2}\Sigma.B^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.B^{(i)}.\cos.(i+1).T;$$

$$\sin.(T+2nt+2\varepsilon).\tfrac{1}{2}\Sigma.B^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.B^{(i)}.\sin.\{(i+1).T+2nt+2\varepsilon\};$$

$$\sin.T.\tfrac{1}{2}\Sigma.B^{(i)}.\cos.iT=\tfrac{1}{2}\Sigma.B^{(i)}.\sin.(i+1).T;$$

hence, by substitution, we obtain the terms of [1322], depending on $B^{(i)}$.

* (893) Because the term $\cos.(i+1).(n't-nt+\varepsilon'-\varepsilon)$, then becomes 1, and the part of $\frac{1}{2}\Sigma.B^{(i)}.\cos.(i+1).T$, becomes $\frac{1}{2}B^{(-1)}$, or $\frac{1}{2}B^{(1)}$, which produces in $\frac{dp}{dt}$, $\frac{dq}{dt}$, [1322], the parts depending on $B^{(1)}$ in [1323, 1324].

periodical: if we denote their sum by P, and observe, that $B^{(-1)}=B^{(1)}$, [1322''] § 48 [956'], we shall get

$$\frac{dp}{dt}=\frac{(q'-q)}{4c}\,.\,m'\,.\,a\,a'\,.\,B^{(1)}+P. \qquad [1323]$$

In the same manner if we denote by Q, the sum of all the periodical terms of [1323'] the expression of $\frac{dq}{dt}$, we shall find

$$\frac{dq}{dt}=\frac{(p-p')}{4c}\,.\,m'\,.\,a\,a'\,.\,B^{(1)}+Q. \qquad [1324]$$

If we neglect the squares of the excentricities and of the inclinations of the orbits, we shall get, from § 64,* $c=\sqrt{\mu a}$; then supposing $\mu=1$, we shall [1324'] have $n^2a^3=1$; hence $c=\frac{1}{an}$, and the quantity $\frac{m'\,.\,a\,a'\,.\,B^{(1)}}{4c}$ will become $\frac{m'\,.\,a^2\,a'\,.\,n\,.\,B^{(1)}}{4}$, which, in § 59 [1130], is equal to $(0,1)$; therefore we shall get,

$$\frac{dp}{dt}=(0,1)\,.\,(q'-q)+P\,;$$
$$\frac{dq}{dt}=(0,1)\,.\,(p-p')+Q. \qquad [1325]$$

Hence it follows, that if we put (P) and (Q) to denote the sums of all the [1325'] functions P and Q, relative to the action of the different bodies m', m'', &c., on m; if we likewise put (P'), (Q'), (P''), (Q''), &c., for what (P) and (Q) [1325''] become, by changing successively the quantities relative to m, into those relative to m', m'', &c., and the contrary; we shall have, to determine the variable quantities, p, q, p', q', p'', q'', &c., the following system of differential equations,

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* (894) From the first equation [1313], p, q, are of the order of the inclinations, therefore c', c'', [1312], are of the same order, and if we neglect quantities of the order of the square of the excentricities and of the inclinations, the equation [1175] will become $\mu\,a=c^2$, or $c=\sqrt{\mu a}$, as in [1324'], and if we put $\mu=1$, we shall get $c=\sqrt{a}$; but [1324a] from [605'], $\sqrt{a}=\frac{1}{an}$, hence $c=\frac{1}{an}$, as above; substituting this in $\frac{m'.a\,a'.B^{(1)}}{4c}$, it will become $\frac{m'\,.\,a^2\,a'\,.\,n\,B^{(1)}}{4}=(0,1)$, [1130], and the equations [1323, 1324], will become as in [1325].

[1326]

$$\begin{aligned}
\frac{dp}{dt} &= -\{(0,1)+(0,2)+\&c.\}.q+(0,1).q'+(0,2).q''+\&c.+(P);\\
\frac{dq}{dt} &= \{(0,1)+(0,2)+\&c.\}.p-(0,1).p'-(0,2).p''-\&c.+(Q);\\
\frac{dp'}{dt} &= -\{(1,0)+(1,2)+\&c.\}.q'+(1,0).q+(1,2).q''+\&c.+(P');\\
\frac{dq'}{dt} &= \{(1,0)+(1,2)+\&c.\}.p'-(1,0).p-(1,2).p''-\&c.+(Q');
\end{aligned}$$

&c.

The analysis of § 68, gives, for the periodical parts of p, q, p', q', &c.,*

[1327]

$$p=\int (P).dt; \qquad q=\int (Q).dt;$$
$$p'=\int (P').dt; \qquad q'=\int (Q').dt;$$

we may obtain the secular parts of the same quantities, from the integration of the preceding differential equations, after effacing the last terms (P), (Q),
[1327'] (P'), &c.; by which means they will become like the equations (C), § 59, [1132], which we have already discussed, with much care, so that it will not be necessary to say more on the subject.

71. We shall resume the equations § 64 [1174],

[1328]

$$\text{tang.}\,\varphi=\frac{\sqrt{c'^2+c''^2}}{c}; \qquad \text{tang.}\,\theta=\frac{c''}{c'};$$

hence we shall obtain†

[1329]

$$\frac{c'}{c}=\text{tang.}\,\varphi.\cos.\theta; \qquad \frac{c''}{c}=\text{tang.}\,\varphi.\sin.\theta;$$

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* (895) If in the equations [1275], we change h, h', h'', &c., into q, q', q'', &c., l, l', l'', &c., into p, p', p'', &c., (X), (Y), (X'), &c., into (P), (Q), (P'), &c., they will become like [1326]; and if we make the same changes in the equations [1277], we shall obtain [1327]. The method of finding the secular equations [1279''] is the same as in [1327'].

† (896) Already found in [590, 591].

taking the differentials, we get*

$$d\,.\,\text{tang.}\,\varphi = \frac{1}{c}.\{d\,c'\,.\,\cos.\,\theta + d\,c''\,.\,\sin.\,\theta - d\,c\,.\,\text{tang.}\,\varphi\}\,;$$

[1330]

$$d\,\theta\,.\,\text{tang.}\,\varphi = \frac{1}{c}.\{d\,c''\,.\,\cos.\,\theta - d\,c'\,.\,\sin.\,\theta\}.$$

If we substitute, in these equations, for $\frac{dc}{dt}$, $\frac{dc'}{dt}$, $\frac{dc''}{dt}$, their values [1310],

$$y\,.\left(\frac{dR}{dx}\right) - x\,.\left(\frac{dR}{dy}\right),\qquad z\,.\left(\frac{dR}{dx}\right) - x\,.\left(\frac{dR}{dz}\right),\qquad z\,.\left(\frac{dR}{dy}\right) - y\,.\left(\frac{dR}{dz}\right);$$ [1330′]

and for these last quantities, their values given in § 67 [1244]; observing also that $s = \text{tang.}\,\varphi\,.\,\sin.\,(v-\theta)$ [679], we shall get† [1330″]

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* (897) The differentials of [1328] are

$$d\,.\,\text{tang.}\,\varphi = \frac{1}{c}.\left\{\frac{c'}{\sqrt{c'^2+c''^2}}.d\,c' + \frac{c''}{\sqrt{c'^2+c''^2}}.d\,c'' - \frac{\sqrt{c'^2+c''^2}}{c}.\,d\,c\right\},$$

[1329a]

$$\frac{d\,\theta}{\cos.^2\,\theta} = \frac{d\,c''}{c'} - \frac{c''\,d\,c'}{c'^2}.$$

Multiplying the first member of this last equation by $\cos.^2\theta\,.\,\text{tang.}\,\varphi$, and the second member by $\frac{c'}{c}\,.\,\cos.\,\theta$, which is equal to it, by the first of the equations [1329], we get

$$d\,\theta\,.\,\text{tang.}\,\varphi = \frac{1}{c}.\left\{d\,c''\,.\,\cos.\,\theta - \frac{c''}{c'}\,.\,\cos.\,\theta\,.\,d\,c'\right\};$$

but the second of the equations [1328], gives $\frac{c''}{c'}.\cos.\,\theta = \text{tang.}\,\theta\,.\,\cos.\,\theta = \sin.\,\theta$; substituting this we get the second of the equations [1330]. Again, if we substitute in the equations [1329], the value of $\text{tang.}\,\varphi = \frac{\sqrt{c'^2+c''^2}}{c}$, [1328], they become $\frac{c'}{c} = \frac{\sqrt{c'^2+c''^2}}{c}\,.\,\cos.\,\theta$, $\frac{c''}{c} = \frac{\sqrt{c'^2+c''^2}}{c}\,.\,\sin.\,\theta$; hence $\frac{c'}{\sqrt{c'^2+c''^2}} = \cos.\,\theta$, and $\frac{c''}{\sqrt{c'^2+c''^2}} = \sin.\,\theta$, these values, and that of $\frac{\sqrt{c'^2+c''^2}}{c}$, given in the first of the equations [1328], being substituted in $d\,.\,\text{tang.}\,\varphi$, [1329a], it becomes like the first equation of [1330].

* (898) If we substitute in the equations [1310], divided by $d\,t$, their values computed in [1244], we shall obtain

$$\frac{d\,c}{d\,t} = -\left(\frac{dR}{dv}\right);$$

$$\frac{d\,c'}{d\,t} = -(1+s^2)\,.\,\cos.\,v\,.\left(\frac{dR}{ds}\right) + r\,s\,.\,\cos.\,v\,.\left(\frac{dR}{dr}\right) - s\,.\,\sin.\,v\,.\left(\frac{dR}{dv}\right);$$ [1330a]

$$\frac{d\,c''}{d\,t} = -(1+s^2)\,.\,\sin.\,v\,.\left(\frac{dR}{ds}\right) + r\,s\,.\,\sin.\,v\,.\left(\frac{dR}{dr}\right) + s\,.\,\cos.\,v\,.\left(\frac{dR}{dv}\right);$$

[1331]

$$d\,.\,\text{tang.}\,\varphi=\frac{dt\,.\,\text{tang.}\,\varphi\,.\,\text{cos.}\,(v-\theta)}{c}\,.\,\left\{r\,.\left(\frac{dR}{dr}\right).\text{sin.}(v-\theta)+\left(\frac{dR}{dv}\right).\text{cos.}\,(v-\theta)\right\}$$

$$-\frac{(1+s^2)\,.\,dt}{c}\,.\,\text{cos.}\,(v-\theta)\,.\left(\frac{dR}{ds}\right);$$

$$d\theta\,.\,\text{tang.}\,\varphi=\frac{dt\,.\,\text{tang.}\,\varphi\,.\,\text{sin.}\,(v-\theta)}{c}\,\left\{r\,.\left(\frac{dR}{dr}\right).\,\text{sin.}(v-\theta)+\left(\frac{dR}{dv}\right).\,\text{cos.}\,(v-\theta)\right\}$$

$$-\frac{(1+s^2)\,.\,dt}{c}\,.\,\text{sin.}\,(v-\theta)\,.\left(\frac{dR}{ds}\right).$$

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substituting these in $\frac{dc''}{dt}\,.\,\text{cos.}\,\theta-\frac{dc'}{dt}\,.\,\text{sin.}\,\theta$, and connecting together the terms depending on $\left(\frac{dR}{ds}\right)$, $\left(\frac{dR}{dr}\right)$, $\left(\frac{dR}{dv}\right)$, we shall get

$$\frac{dc''}{dt}\,.\,\text{cos.}\,\theta-\frac{dc'}{dt}\,.\,\text{sin.}\,\theta=-(1+s^2)\,.\left(\frac{dR}{ds}\right).\,\{\text{sin.}\,v\,.\,\text{cos.}\,\theta-\text{cos.}\,v\,.\,\text{sin.}\,\theta\}$$

$$+rs\,.\left(\frac{dR}{dr}\right).\,\{\text{sin.}\,v\,.\,\text{cos.}\,\theta-\text{cos.}\,v\,.\,\text{sin.}\,\theta\}+s\,.\left(\frac{dR}{dv}\right).\{\text{cos.}\,v\,.\,\text{cos.}\,\theta+\text{sin.}\,v\,.\,\text{sin.}\,\theta\};$$

multiplying by dt, and reducing, by means of [22, 24] Int., we get

[1331*a*]

$$dc''\,.\,\text{cos.}\,\theta-dc'\,.\,\text{sin.}\,\theta=-(1+s^2)\,.\,dt\,.\,\text{sin.}\,(v-\theta)\,.\left(\frac{dR}{ds}\right)$$

$$+s\,dt\,.\,\text{sin.}\,(v-\theta)\,.\,r\,.\left(\frac{dR}{dr}\right)+s\,dt\,.\left(\frac{dR}{dv}\right).\,\text{cos.}\,(v-\theta),$$

and as dc', dc'', [1330*a*], do not contain θ explicitly, we may change θ into $\theta-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ being a right angle; hence we shall obtain

[1331*b*]

$$dc''\,.\,\text{sin.}\,\theta+dc'\,.\,\text{cos.}\,\theta=-(1+s^2)\,.\,dt\,.\,\text{cos.}\,(v-\theta)\,.\left(\frac{dR}{ds}\right)$$

$$+s\,dt\,.\,\text{cos.}\,(v-\theta)\,.\,r\,.\left(\frac{dR}{dr}\right)-s\,dt\,.\left(\frac{dR}{dv}\right).\,\text{sin.}\,(v-\theta).$$

s is the tangent of the latitude [1242^{vii}], which is equal to $\text{tang.}\,\varphi\,.\,\text{sin.}\,(v-\theta)$, [679]. Substituting this in the two last terms of [1331*a*], and the result in the second of the equations [1330], we shall get the second of [1331]. The same value of s being substituted in the two last terms of [1331*b*], and the resulting value, together with that of $\frac{dc}{dt}=-\left(\frac{dR}{dv}\right)$, [1330*a*], being substituted in the first of the equations [1330], we shall get

$$d\,.\,\text{tang.}\,\varphi=\frac{1}{c}\,.\left\{\begin{array}{l}-(1+s^2)\,.\,dt\,.\,\text{cos.}\,(v-\theta)\,.\left(\frac{dR}{ds}\right)+dt\,.\,\text{tang.}\,\varphi\,.\,\text{sin.}\,(v-\theta)\\ \times\,\text{cos.}\,(v-\theta)\,.\,r\,.\left(\frac{dR}{dr}\right)+\left(\frac{dR}{dv}\right).\,dt\,.\,\text{tang.}\,\varphi\,.\,[-\text{sin.}^2\,(v-\theta)+1]\end{array}\right\},$$

and if we put $-\text{sin.}^2\,(v-\theta)+1=\text{cos.}^2\,(v-\theta)$, it will become like the first equation [1331].

From these two differential equations we may determine directly the inclination of the orbit, and the motion of the nodes; they give*

$$\sin.(v-\theta).d.\text{tang.}\,\varphi - d\theta.\cos.(v-\theta).\text{tang.}\,\varphi = 0\,; \qquad [1332]$$

an equation which may also be obtained from $s = \text{tang.}\,\varphi.\sin.(v-\theta)$ [1330″]. For this equation being finite, we may, as in § 63 [1167″], take its differential,
supposing φ and θ to be constant, or we may consider both these quantities to [1332′]
be variable; therefore the differential of this value of s, supposing φ and θ only to be variable, must be nothing; hence results the preceding differential equation.

Suppose now the inclination of the fixed plane to the orbit of m to be extremely small, so that we may neglect the squares of s and of $\text{tang.}\,\varphi$, we shall find†

$$d.\text{tang.}\,\varphi = -\frac{dt}{c}.\cos.(v-\theta).\left(\frac{dR}{ds}\right);$$
$$d\theta.\text{tang.}\,\varphi = -\frac{dt}{c}.\sin.(v-\theta).\left(\frac{dR}{ds}\right); \qquad [1333]$$

and if we again put, as in [1312*a*],

$$p = \text{tang.}\,\varphi.\sin.\theta\,; \qquad q = \text{tang.}\,\varphi.\cos.\theta\,; \qquad [1334]$$

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* (899) Multiplying the first of the equations [1331] by $\sin.(v-\theta)$, the second by $-\cos.(v-\theta)$, and adding the products, the terms of the second member destroy each other, producing the equation [1332].

† (901) Substituting $z = rs$, [1243], in [951], we get

$$R = \frac{m'.\{rr'.\cos.(v'-v) + rsz'\}}{(r'^2+z'^2)^{\frac{3}{2}}} - \frac{m'}{\{r^2 - 2rr'.\cos.(v'-v) + r^2 + (z'-rs)^2\}^{\frac{1}{2}}}, \qquad [1333a]$$

so that $\left(\frac{dR}{dr}\right)$, $\left(\frac{dR}{dv}\right)$, are of the order m'; these quantities are multiplied, in [1331], by $\text{tang.}\,\varphi$, which may also be considered as of the order m', the fixed plane being the primitive orbit of m, therefore these products are of the order m'^2; and if we neglect them, and also the quantity s^2, which is of the same order, the two equations [1331] will change into [1333].

we shall obtain the following equations, instead of the two preceding [1333],*

[1335]
$$dq = -\frac{dt}{c} . \cos. v . \left(\frac{dR}{ds}\right);$$
$$dp = -\frac{dt}{c} . \sin. v . \left(\frac{dR}{ds}\right).$$

[1335′] Now we have† $s = q . \sin. v - p . \cos. v$, which gives

[1336]
$$\left(\frac{dR}{ds}\right) = \frac{1}{\sin. v} . \left(\frac{dR}{dq}\right); \qquad \left(\frac{dR}{ds}\right) = -\frac{1}{\cos. v} . \left(\frac{dR}{dp}\right);$$

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* (902) The differentials of [1334] are $dp = \sin. \theta . d . \text{tang.}\, \varphi + d\theta . \cos. \theta . \text{tang.}\, \varphi$, $dq = -d\theta . \sin. \theta . \text{tang.}\, \varphi + \cos. \theta . d . \text{tang.}\, \varphi$, which, by means of [1333], and using [21, 23] Int., become

$$dp = -\frac{dt}{c} . \{\sin. \theta . \cos. (v-\theta) + \cos. \theta . \sin. (v-\theta)\} . \left(\frac{dR}{ds}\right)$$
$$= -\frac{dt}{c} . \sin. \{\theta + (v-\theta)\} . \left(\frac{dR}{ds}\right) = -\frac{dt}{dc} . \sin. v . \left(\frac{dR}{ds}\right),$$
$$dq = -\frac{dt}{c} . \{-\sin. \theta . \sin. (v-\theta) + \cos. \theta . \cos. (v-\theta)\} . \left(\frac{dR}{ds}\right)$$
$$= -\frac{dt}{c} . \cos. \{\theta + (v-\theta)\} . \left(\frac{dR}{ds}\right) = -\frac{dt}{c} . \cos. v . \left(\frac{dR}{ds}\right),$$

as in [1335].

† (903) The value assumed above for s, [1330″], is

$$s = \text{tang.}\, \varphi . \sin. (v-\theta) = \text{tang.}\, \varphi . \{\sin. v . \cos. \theta - \cos. v . \sin. \theta\},$$

which by means of [1334] becomes $s = q . \sin. v - p . \cos. v$, as in [1335′]. Now p, q, do not occur in R, [1331a], except through s, therefore

$$\left(\frac{dR}{dq}\right) = \left(\frac{dR}{ds}\right) . \left(\frac{ds}{dq}\right); \qquad \left(\frac{dR}{dp}\right) = \left(\frac{dR}{ds}\right) . \left(\frac{ds}{dp}\right);$$

and the preceding value of s gives $\left(\frac{ds}{dq}\right) = \sin. v$; $\left(\frac{ds}{dp}\right) = -\cos. v$; hence

$$\left(\frac{dR}{dq}\right) = \left(\frac{dR}{ds}\right) . \sin. v; \qquad \left(\frac{dR}{dp}\right) = -\left(\frac{dR}{ds}\right) . \cos. v;$$

as in [1336]; substituting these in [1335] we get [1337]. If we take the primitive orbit of m for the fixed plane, we may, as in [1824a], neglect c'^2, c''^2, being of the order of the square of the disturbing mass, and the expression [1175] will give $c = \sqrt{\mu a} . \sqrt{1-e^2}$, but
[1335a] from [605′], $n^2 a^3 = \mu$, hence $\sqrt{\mu a} = n a^2 = \frac{n^2 a^3}{na} = \frac{\mu}{na}$, and if we put $\mu = 1$,

therefore

$$dq = \frac{dt}{c} \cdot \left(\frac{dR}{dp}\right);$$

$$dp = -\frac{dt}{c} \cdot \left(\frac{dR}{dq}\right).$$ [1337]

We have seen, in § 48 [949′], that the function R is independent of the position of the fixed plane of x and y; supposing therefore all the angles of that function to be referred to the orbit of m, it is evident that R will be a function of these angles, and of the inclination of the two orbits to each other, which inclination we shall denote by $\varphi_{,}'$. Let $\theta_{,}'$ be the longitude of [1337′] the node of the orbit of m', upon the orbit of m; suppose also that

$$m'k \, . \, (\text{tang.}\ \varphi_{,}')^{g} \, . \, \cos.\, (i'n't - int + A - g\,\theta_{,}'),$$ [1337″]

is a term of R, depending upon the angle $i'n't - int$;* we shall have, by

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it will become $\sqrt{\mu a} = \frac{1}{an}$, hence $c = \frac{1}{an} \, . \, \sqrt{1-e^2}$. This value being substituted in [1337a] [1337], we shall get $\frac{dp}{dt} = -\frac{andt}{\sqrt{1-e^2}} \, . \left(\frac{dR}{dq}\right)$; $\frac{dq}{dt} = \frac{andt}{\sqrt{1-e^2}} \, . \left(\frac{dR}{dp}\right)$; [1337b] which are the same as the formulas [5790, 5791], in the appendix to the third volume, being accurate in terms of the first order of the disturbing forces.

* (904) This term of R is deduced from [961], observing that $g'' = 0$, because the fixed plane is the primitive orbit of m, so that the inclination and longitude of the node of m, upon the fixed plane, must disappear, if we neglect terms of the order of the square of the disturbing masses; and the term of R becomes

$$He^{g} \, . \, e'^{g'} \, . \, (\text{tang.}\ \tfrac{1}{2}\,\varphi')^{g'''} \, . \, \cos.\, \{i'n't - int + i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' - g'''\theta'\}.$$

To conform to the preceding notation we must change φ' into $\varphi_{,}'$, and θ' into $\theta_{,}'$ and if we put $He^{g} \, . \, e'^{g'} = 2^{g'''} \, . \, m'k$, $i'\varepsilon' - i\varepsilon - g\varpi - g'\varpi' = A$, this term will become, [1337c]

$$m'k \, . \, (2 \, . \, \text{tang.}\ \tfrac{1}{2}\,\varphi_{,}')^{g'''} \, . \, \cos.\, \{i'n't - int + A - g'''\theta_{,}'\},$$

and by neglecting, as above, the cube of φ, we may put $2 \, . \, \text{tang.}\ \tfrac{1}{2}\,\varphi_{,}' = \text{tang.}\ \varphi_{,}'$; lastly to simplify the notation, we may put g instead of g''', and it will become

$$m'k \, . \, (\text{tang.}\ \varphi_{,}')^{g} \, . \, \cos.\, (i'n't - int + A - g\,\theta_{,}'),$$

as in [1337″].

§ 60 [1144],*

[1338] $\text{tang.}\,\varphi'_{,}\,.\,\sin.\,\theta'_{,} = p'-p\,;\qquad \text{tang.}\,\varphi'_{,}\,.\,\cos.\,\theta'_{,} = q'-q\,;$

whence we deduce†

[1339]

$$(\text{tang.}\,\varphi'_{,})^{g}\,.\,\sin.\,g\,\theta'_{,} = \frac{\{q'-q+(p'-p).\sqrt{-1}\}^{g}-\{q'-q-(p'-p).\sqrt{-1}\}^{g}}{2\,.\sqrt{-1}}\,;$$

$$(\text{tang.}\,\varphi'_{,})^{g}\,.\,\cos.\,g\theta'_{,} = \frac{\{q'-q+(p'-p).\sqrt{-1}\}^{g}+\{q'-q-(p'-p).\sqrt{-1}\}^{g}}{2}.$$

Noticing therefore only the preceding value of R, we shall have‡

[1340]

$$\left(\frac{dR}{dp}\right) = -g\,.\,(\text{tang.}\,\varphi'_{,})^{g-1}\,.\,m'k\,.\,\sin.\,\{i'n't-int+A-(g-1)\,.\,\theta'_{,}\}\,;$$

$$\left(\frac{dR}{dq}\right) = -g\,.\,(\text{tang.}\,\varphi'_{,})^{g-1}\,.\,m'k\,.\,\cos.\,\{i'n't-int+A-(g-1)\,.\,\theta'_{,}\}.$$

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* (905) The equations [1144], $\text{tang.}\,\varphi'_{,} = \sqrt{(p'-p)^2+(q'-q)^2}\,;$ $\text{tang.}\,\theta'_{,} = \frac{p'-p}{q'-q}$, are of the same form as those in [1313], and may be derived from them, by changing p, q, φ, θ, into $p'-p$, $q'-q$, $\varphi'_{,}$, $\theta'_{,}$, respectively. The same changes being made in the equations $\text{tang.}\,\varphi\,.\,\sin.\,\theta = p$, $\text{tang.}\,\varphi\,.\,\cos.\,\theta = q$, deduced in [1312a], from the equations [1313], we get the expressions [1338].

† (906) Multiplying the first of the equations [1338] by $\pm\sqrt{-1}$, and adding the product to the second, we shall get

$$(\text{tang.}\,\varphi'_{,})\,.\,\{\cos.\,\theta'_{,}\pm\sin.\,\theta'_{,}\,.\sqrt{-1}\} = q'-q\pm(p'-p)\,.\sqrt{-1}\,;$$

raising this to the power g, and using [15, 16] Int., we shall find

$$(\text{tang.}\,\varphi'_{,})^{g}\,.\{\cos.\,g\,\theta'_{,}\pm\sqrt{-1}\,.\,\sin.\,g\,\theta'_{,}\} = \{q'-q\pm(p'-p)\,.\sqrt{-1}\}^{g}\,;$$

taking the sum and difference of these two equations, depending on the different signs $\pm$, we shall find

$$(\text{tang.}\,\varphi'_{,})^{g}\,.\,2\,.\cos.\,g\,\theta'_{,} = \{q'-q+(p'-p)\,.\sqrt{-1}\}^{g}+\{q'-q-(p'-p)\,.\sqrt{-1}\}^{g}\,;$$

$$(\text{tang.}\,\varphi'_{,})^{g}\,.\,2\,.\sqrt{-1}\,.\,\sin.\,g\,\theta'_{,} = \{q'-q+(p'-p)\,.\sqrt{-1}\}^{g}-\{q'-q-(p'-p)\,.\sqrt{-1}\}^{g}\,;$$

dividing these by 2, and $2\,.\sqrt{-1}$, respectively, we shall get the expressions [1339].

[1340a] ‡ (907) If we put $T = i'n't-int+A$, we shall get, from [24] Int.,

$$\cos.\,(i'n't-int+A-g\,\theta'_{,}) = \cos.\,(T-g\,\theta'_{,}) = \cos.\,T\,.\cos.\,g\,\theta'_{,}+\sin.\,T\,.\,\sin.\,g\,\theta'_{,},$$

hence the term of R, [1337″], will become

$$R = m'k\,.(\text{tang.}\,\varphi'_{,})^{g}\,.\{\cos.\,T\,.\cos.\,g\,\theta'_{,}+\sin.\,T\,.\,\sin.\,g\,\theta'_{,}\}.$$

If we substitute these values in the preceding expressions of dp and dq, observing that we have, very nearly,* $c=\frac{\mu}{an}$, we shall obtain [1340′]

Substituting the values [1339], we shall get,

$$R=\tfrac{1}{2}\,.\,m'k\,.\,\{[q'-q+(p'-p)\,.\,\sqrt{-1}]^{g}+[q'-q-(p'-p)\,.\,\sqrt{-1}]^{g}\}\,.\,\text{cos.}\,T$$
$$+\frac{m'k}{2\,.\,\sqrt{-1}}\,.\,\{[q'-q+(p'-p)\,.\,\sqrt{-1}]^{g}-[(q'-q)-(p'-p)\,.\,\sqrt{-1}]^{g}\}\,.\,\text{sin.}\,T\,;$$

hence

$$\left(\frac{dR}{dp}\right)=\frac{-\sqrt{-1}\,.\,m'kg}{2}\,.\,\{[q'-q+(p'-p)\,.\sqrt{-1}]^{g-1}-[q'-q-(p'-p)\,.\sqrt{-1}]^{g-1}\}\,.\,\text{cos.}\,T$$
$$-\tfrac{1}{2}m'kg\,.\,\{[q'-q+(p'-p)\,.\,\sqrt{-1}]^{g-1}+[q'-q-(p'-p)\,.\,\sqrt{-1}]^{g-1}\}\,.\,\text{sin.}\,T,$$

and

$$\left(\frac{dR}{dq}\right)=-\tfrac{1}{2}m'kg\,.\,\{[q'-q+(p'-p)\,.\,\sqrt{-1}]^{g-1}+[q'-q-(p'-p)\,.\,\sqrt{-1}]^{g-1}\}\,.\,\text{cos.}\,T$$
$$-\frac{m'kg}{2\,.\,\sqrt{-1}}\,.\,\{[q'-q+(p'-p)\,.\,\sqrt{-1}]^{g-1}-[q'-q-(p'-p)\,.\,\sqrt{-1}]^{g-1}\}\,.\,\text{sin.}\,T.$$

Changing g into $g-1$, in the formulas [1339], we shall obtain the values of

$$\{q'-q+(p'-p)\,.\,\sqrt{-1}\}^{g-1}\pm\{q'-q-(p'-p)\,.\,\sqrt{-1}\}^{g-1},$$

hence by substitution we shall get

$$\left(\frac{dR}{dp}\right)=m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{sin.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{cos.}\,T-m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{cos.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{sin.}\,T,$$
$$\left(\frac{dR}{dq}\right)=-m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{cos.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{cos.}\,T-m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{sin.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{sin.}\,T.$$

If in the first we substitute for $\text{sin.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{cos.}\,T-\text{cos.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{sin.}\,T$, its value $-\text{sin.}\,\{T-(g-1)\,.\,\theta'_{\prime}\}$, [22] Int., and in like manner, in the second, for

$$-\text{cos.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{cos.}\,T-\text{sin.}\,(g-1)\,.\,\theta'_{\prime}\,.\,\text{sin.}\,T,$$

its value $-\text{cos.}\,\{T-(g-1)\,.\,\theta'_{\prime}\}$, [24] Int. they will become

$$\left(\frac{dR}{dp}\right)=-m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{sin.}\,\{T-(g-1)\,.\,\theta'_{\prime}\},$$
$$\left(\frac{dR}{dq}\right)=-m'kg\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{cos.}\,\{T-(g-1)\,.\,\theta'_{\prime}\},$$

resubstituting $T=i'n't-int+A$, [1340a], they will become as in [1340].

* (908) Neglecting the square of the excentricity, we shall have, as in [1335a], $c=\frac{\mu}{an}$; substituting this, and $\left(\frac{dR}{dp}\right)$, $\left(\frac{dR}{dq}\right)$, [1340] in [1337] we get

$$dq=-\frac{g\,.\,m'k\,.\,an\,.\,dt}{\mu}\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{sin.}\,\{i'n't-int+A-(g-1)\,.\,\theta'_{\prime}\}\,;$$
$$dp=\frac{g\,.\,m'k\,.\,an\,.\,dt}{\mu}\,.\,(\text{tang.}\,\varphi'_{\prime})^{g-1}\,.\,\text{cos.}\,\{i'n't-int+A-(g-1)\,.\,\theta'_{\prime}\}\,;$$ [1341a]

whose integrals give q, p, [1341].

[1341]
$$p=\frac{g\,.\,m'k\,.\,a\,n}{\mu\,.\,(i'\,n'-i\,n)}\,.\,(\text{tang.}\;\varphi'_{,})^{g-1}\,.\,\sin.\,\{i'\,n'\,t-i\,n\,t+A-(g-1)\,.\,\theta'_{,}\}\;;$$
$$q=\frac{g\,.\,m'k\,.\,a\,n}{\mu\,.\,(i'\,n'-i\,n)}\,.\,(\text{tang.}\;\varphi'_{,})^{g-1}\,.\,\cos.\,\{i'\,n'\,t-i\,n\,t+A-(g-1)\,.\,\theta'_{,}\}.$$

Substituting these values in the equation $s=q\,.\sin.\,v-p\,.\cos.\,v$ [1335′], we shall get*

[1342]
$$s=\frac{-g\,.\,m'k\,.\,a\,n}{\mu\,.\,(i'\,n'-i\,n)}\,.\,(\text{tang.}\;\varphi'_{,})^{g-1}\,.\,\sin.\{i'\,n'\,t-i\,n\,t-v+A-(g-1)\,.\,\theta'_{,}\}.$$

This expression of s is the variation of the latitude, corresponding to the preceding term of R [1337″], and it is evident that it is the same, whatever be the fixed plane to which we refer the motions of m and m', provided the [1342′] inclinations of the planes of the orbits to the fixed plane be small; we shall thus obtain the part of the expression of the latitude, which becomes sensible by means of the smallness of the divisor $i'n'-in$. It is true, that this inequality of the latitude contains only the first power of that divisor, and it [1342″] must, on this account, be less sensible than the corresponding inequality of the mean longitude, which contains the square of that divisor; but on the other hand, $\text{tang.}\,\varphi'_{,}$ is not raised to so high a power by unity; which is analogous to the remark we have made in § 69 [1288′], upon the [1342‴] corresponding inequality of the excentricities of the orbits. Hence we see that all these inequalities are connected with each other, and with the corresponding part of R, by very simple relations.

If we take the differentials of the preceding expressions of p and q [1341], and then, in the values of $\frac{dp}{dt}$, $\frac{dq}{dt}$, thus obtained, augment the angles nt, and $n't$, by the inequalities of the mean motions, depending on the angle [1342iv] $i'n't-int$; there will be produced, in these differentials, some quantities,

---

* (909) The values p, q, [1341], being substituted in s, [1335′], it becomes.

$$s=\frac{-g\,.\,m'k\,.\,a\,n}{\mu\,.\,(i'\,n'-i\,n)}\,.\,(\text{tang.}\;\varphi'_{,})^{g-1}\,.\left\{\begin{array}{c}\sin.\{i'\,n'\,t-i\,n\,t+A-(g-1)\,.\,\theta'_{,}\}\,.\cos.\,v\\ -\cos.\{i'\,n'\,t-i\,n\,t+A-(g-1)\,.\,\theta'_{,}\}\,.\sin.\,v\end{array}\right\};$$

the terms between the braces may be reduced to $\sin.\,\{i'\,n'\,t-i\,n\,t+A-v-(g-1)\,.\,\theta'_{,}\}$, [22] Int., by which means the value of s becomes as in [1342].

which will be functions of the elements of the orbits only,* and which may have a sensible influence upon the secular variations of the inclinations and of the nodes, although they are of the order of the square of the disturbing masses; which is analogous to what we have said in § 69 [1302′, &c.], upon the secular variations of the excentricities and of the aphelia.

72. It now remains to consider the variation of the longitude ε of the epoch. We have, by § 64 [1188],

$$\begin{aligned} d\varepsilon = {} & de \, . \left\{ \left(\frac{dE^{(1)}}{de}\right) . \sin. (v-\varpi) + \tfrac{1}{2} . \left(\frac{dE^{(2)}}{de}\right) . \sin. 2 . (v-\varpi) + \&c. \right\} \\ & - d\varpi . \{ E^{(1)} . \cos. (v-\varpi) + E^{(2)} . \cos. 2 . (v-\varpi) + \&c. \}; \end{aligned} \qquad [1343]$$

substituting for $E^{(1)}$, $E^{(2)}$, &c., their values in series arranged according to the powers of e, which series may easily be deduced from the general expression of $E^{(i)}$, § 16 [541],† we shall find

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* (910) Substituting the value of A, [1337c], in the angle $i'n't - int + A - (g-1) . \theta_{,}'$, it becomes $(i'n't - int + i'\varepsilon' - i\varepsilon) - \{(g-1) . \theta_{,}' + g\varpi + g'\varpi'\}$, and if we put $i'n't - int + i'\varepsilon' - i\varepsilon = T$, $(g-1) . \theta_{,}' + g\varpi + g'\varpi' = W$, it will change into $T - W$, and since

$$\sin. (T-W) = \cos. W . \sin. T - \sin. W . \cos. T,$$
$$\cos. (T-W) = \cos. T . \cos. W + \sin. T . \sin. W,$$

the values of dq, dp, [1341a], will become, by putting,

$$\left(\frac{dP''}{de}\right) = gk . (\text{tang.}\, \varphi_{,}')^{g-1} . \cos. W; \qquad \left(\frac{dP'''}{de}\right) = gk . (\text{tang.}\, \varphi_{,}')^{g-1} . \sin. W;$$

$$dq = \frac{m'andt}{\mu} . \left\{ \left(\frac{dP'''}{de}\right) . \cos. T - \left(\frac{dP''}{de}\right) . \sin. T \right\};$$

$$dp = \frac{m'andt}{\mu} . \left\{ \left(\frac{dP''}{de}\right) . \cos. T + \left(\frac{dP'''}{de}\right) . \sin. T \right\};$$

which are of a similar form to that of de, [1303], and if we increase the angles nt, $n't$, by the expressions [1304, 1305], respectively, the increment of the angle T will produce, in the preceding values of dq, dp, terms similar to those in [1306], which depend upon the elements of the orbits only.

† (911) If we neglect e^4 we may put $\sqrt{1-e^2} = 1 - \frac{1}{2}e^2$, in [541], hence

$$E^{(i)} = \pm \frac{2e^i . \{1 + i - \frac{1}{2}ie^2\}}{(2 - \frac{1}{2}e^2)^i};$$

[1344]
$$
\begin{aligned}
d\varepsilon = & -2\,de\,.\,\sin.\,(v-\varpi) + 2\,e\,.\,d\varpi\,.\,\cos.\,(v-\varpi) \\
& + e\,de\,.\{\tfrac{3}{2}+\tfrac{1}{2}e^2+\&\text{c.}\}.\sin.2.(v-\varpi) - e^2.d\varpi.\{\tfrac{3}{2}+\tfrac{1}{4}e^2+\&\text{c.}\}.\cos.2.(v-\varpi) \\
& - e^2 de.\{1+\&\text{c.}\}.\sin.3.(v-\varpi) + e^3.d\varpi.\{1+\&\text{c.}\}.\cos.3.(v-\varpi) \\
& + \&\text{c.}
\end{aligned}
$$

If we substitute, for de and $e\,d\varpi$, their values, given in § 67 [1258], we shall find, by retaining only quantities of the order e^2 inclusively,*

[1345]
$$
\begin{aligned}
d\varepsilon = & \frac{a^2.\,n\,dt}{\mu}\,.\sqrt{1-e^2}\,.\{2-\tfrac{3}{2}\,e\,.\,\cos.\,(v-\varpi)+e^2\,.\,\cos.\,2\,.\,(v-\varpi)\}\,.\left(\frac{dR}{dr}\right) \\
& - \frac{a\,.\,n\,dt}{\mu\,.\sqrt{1-e^2}}\,.\,e\,.\,\sin.\,(v-\varpi)\,.\{1+\tfrac{1}{2}\,e\,.\,\cos.\,(v-\varpi)\}\,.\left(\frac{dR}{dv}\right).
\end{aligned}
$$

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and the denominator $\frac{1}{(2-\frac{1}{2}e^2)^i}=\frac{1}{2^i.(1-\frac{1}{4}e^2)^i}=\frac{1}{2^i}.(1+\frac{1}{4}i\,e^2)$; therefore

$$E^{(i)}=\frac{\pm e^i.(i+1)\pm e^{i+2}.\frac{1}{4}i.(i-1)}{2^{i-1}};$$

putting successively $i=1,\ 2,\ 3,\ 4$, &c., and using the signs, as in [541, &c.], we obtain $E^{(1)}=-2\,e-\&\text{c.}$; $E^{(2)}=\frac{3}{2}e^2+\frac{1}{4}e^4-\&\text{c.}$; $E^{(3)}=-e^3-\&\text{c.}$; hence $\left(\frac{dE^{(1)}}{de}\right)=-2$, &c. $\frac{1}{2}.\left(\frac{dE^{(2)}}{de}\right)=\frac{3}{2}e+\frac{1}{2}e^3$, &c. $\frac{1}{3}.\left(\frac{dE^{(3)}}{de}\right)=-e^2$, &c. substituting these in [1343) we get [1344].

* (912) Neglecting terms of the order e^4, in [1344], we get

$$
\begin{aligned}
d\varepsilon = de\,.\{&-2\,.\,\sin.\,(v-\varpi)+\tfrac{3}{2}\,e\,.\,\sin.\,2\,.\,(v-\varpi)-e^2\,.\,\sin.\,3\,.\,(v-\varpi)\} \\
+\,e\,d\varpi\,.\{&2\,.\,\cos.\,(v-\varpi)-\tfrac{3}{2}\,e\,.\,\cos.\,2\,.\,(v-\varpi)+e^2\,.\,\cos.\,3\,.\,(v-\varpi)\}.
\end{aligned}
$$

Substituting the values of de, $e\,d\varpi$, [1258], we find in $d\varepsilon$, terms multiplied by

$$-\frac{a\,n\,dt}{\mu.\sqrt{1-e^2}}.\left(\frac{dR}{dv}\right), \quad \text{and} \quad \frac{a^2\,n\,dt}{\mu}.\sqrt{1-e^2}.\left(\frac{dR}{dr}\right).$$

The factor of $-\frac{a\,n\,dt}{\mu.\sqrt{1-e^2}}.\left(\frac{dR}{dv}\right)$, as it first appears, without reduction, and putting for brevity $x=v-\varpi$, is

$$
\begin{aligned}
&(2\,.\,\cos.\,x+e+e\,.\,\cos.^2 x)\,.\,(-2\,.\sin.x+\tfrac{3}{2}\,e\,.\,\sin.\,2\,x-e^2\,.\,\sin.\,3\,x) \\
&+(2\,.\,\sin.\,x+e\,.\,\sin.\,x\,.\,\cos.\,x)\,.\,(2\,.\,\cos.\,x-\tfrac{3}{2}\,e\,.\,\cos.\,2\,x+e^2\,.\,\cos.\,3\,x),
\end{aligned}
$$

The general expression of $d\varepsilon$ contains some terms of the form

$$m'k\,.\,n\,d\,t\,.\cos.\,(i'n't-i\,n\,t+A)\;;$$

therefore the expression of ε contains terms of the form

$$\frac{m'kn}{i'n'-in}\,.\sin.\,(i'n't-i\,n\,t+A)\;;$$ [1345']

---

multiplying these factors together, and arranging the terms according to the powers of e, observing that the terms independent of e mutually destroy each other, the product will become

$$e\,.\{3\,.\,(\sin.2x\,.\cos.x-\cos.2x\,.\sin.x)-2\sin.x\,.\,(1+\cos.^2x-\cos.^2x)\}$$

$$+e^2.\{-2.(\sin.3x\,.\cos.x-\cos.3x\,.\sin.x)+\tfrac{3}{2}.\cos.x\,.(\sin.2x\,.\cos.x-\cos.2x.\sin.x)+\tfrac{3}{2}.\sin.2x\}$$

$$=e.\{3\,.\sin.(2x-x)-2\,.\sin.x\}+e^2.\{-2\,.\sin.(3x-x)+\tfrac{3}{2}\,.\cos.x\,.\sin.(2x-x)+\tfrac{3}{2}\,.\sin.2x\}$$

$$=e.\{3\,.\sin.x-2\,.\sin.x\}+e^2.\{-2\,.\sin.2x+\tfrac{3}{2}\,.\cos.x\,.\sin.x+\tfrac{3}{2}\,.\sin.2x\},$$

and since $\sin.2x=2\,.\sin.x\,.\cos.x,$ this will finally become

$$e\,.\sin.x+\tfrac{1}{2}e^2\,.\sin.x\,.\cos.x,$$

resubstituting for x its value $v-\varpi$, we shall get the coefficient of $-\frac{a\,n\,d\,t}{\mu.\sqrt{1-e^2}}\,.\left(\frac{dR}{dv}\right)$, as in [1345].

Again, the factor of $\frac{a^2n\,d\,t}{\mu}\,.\sqrt{1-e^2}\,.\left(\frac{dR}{dr}\right)$, in the expressions of $d\varepsilon$ abovementioned is

$$-\sin.x\,.(-2\,.\sin.x+\tfrac{3}{2}e\,.\sin.2x-e^2.\sin.3x)+\cos.x\,.(2.\cos.x-\tfrac{3}{2}e\,.\cos.2x+e^2.\cos.3x),$$

which, being arranged according to the powers of e, is

$$2.(\sin.^2x+\cos.^2x)-\tfrac{3}{2}e\,.(\cos.x\,.\cos.2x+\sin.x\,.\sin.2x)+e^2.(\cos.3x\,.\cos.x+\sin.3x.\sin.x)$$

$$=2-\tfrac{3}{2}e\,.\cos.(2x-x)+e^2.\cos.(3x-x)=2-\tfrac{3}{2}e\,.\cos.x+e^2.\cos.2x$$

$$=2-\tfrac{3}{2}e\,.\cos.(v-\varpi)+e^2\,.\cos.2\,.(v-\varpi),$$

as in [1345].

In the appendix to Vol. III. [5787], it is shown that $d\varepsilon$ is expressed by the following formula, which includes all terms of the first order of the disturbing force, μ being equal to unity,

$$d\varepsilon=-\frac{a\,n\,d\,t.\sqrt{1-e^2}}{e}.(1-\sqrt{1-e^2}).\left(\frac{dR}{de}\right)+2\,a^2.\left(\frac{dR}{da}\right).n\,d\,t,$$ [1344a]

but it is evident, that the coefficient k, in these terms, is of the order* $i'-i$, therefore these terms are of the same order as those of the mean longitude, depending on the same angle, and as these have for a divisor the square of
[1345‴] $i'n'-in$, it is evident that we may neglect the corresponding terms of ε, in comparison with them, if $i'n'-in$ be a very small quantity.

If in the terms of the expression of $d\varepsilon$, which are functions of the elements of the orbits only,† we substitute, for these elements, the secular parts of

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* (913) Substituting in [1345], for $\left(\frac{dR}{dr}\right)$ and $\left(\frac{dR}{dv}\right)$, their values [1292$a$], $(1+u_{,})^{-1}.\left(\frac{dR}{da}\right)$, and $\left(\frac{dR}{d\varepsilon}\right)$, also $v=nt+\varepsilon+v_{,}$, it becomes

[1345a]
$$d\varepsilon=\frac{a^2ndt}{\mu}.\sqrt{1-e^2}.\{2-\tfrac{3}{2}.e.\cos.(nt+\varepsilon-\varpi+v_{,})+e^2.\cos.(2nt+2\varepsilon-2\varpi+2v_{,})\}.(1+u_{,})^{-1}.\left(\frac{dR}{da}\right)$$
$$-\frac{andt}{\mu.\sqrt{1-e^2}}.e.\sin.(nt+\varepsilon-\varpi+v_{,}).\{1+\tfrac{1}{2}e.\cos.(nt+\varepsilon-\varpi+v_{,})\}.\left(\frac{dR}{d\varepsilon}\right).$$

This expression of $d\varepsilon$ is similar to that of df, [1293a], as it regards the order of the coefficients, and it will appear, from what is said [1293b], that this value of $d\varepsilon$ has the same property, relative to the form and order of the terms, as the value R; now in [957$^{\text{viii}}$], it is shown, that if the value of R is composed of terms of the form $m'k.\cos.(i'n't-int+A)$, the coefficient k will be of the order $i'-i$, therefore the part of $d\varepsilon$, depending on this angle will be of the same order, and its integral will give ε, of the order $i'-i$, with the divisor $i'n'-in$. But the part of ζ, [1197], depending on the same angle, is of the order $i'-i$, and has the divisor $(i'n'-in)^2$, therefore this part of ζ must be much larger than the corresponding part of ε, if $i'n'-in$ be very small.

† (914) To give an example of the manner in which such terms are formed, we may take in the value of $d\varepsilon$ [1345a], the term

[1345b]
$$\frac{a^2ndt}{\mu}.\sqrt{1-e^2}.\{\tfrac{3}{2}e.\cos.(nt+\varepsilon-\varpi+v_{,})\}.u_{,}.\left(\frac{dR}{da}\right);$$

in which $\cos.(nt+\varepsilon-\varpi+v_{,})$, being developed [61] Int., has for its chief term $\cos.(nt+\varepsilon-\varpi)$; multiplying this by $u_{,}=-e.\cos.(nt+\varepsilon-\varpi)$, [1010], we shall get $-\frac{1}{2}e-\frac{1}{2}e.\cos.2.(nt+\varepsilon-\varpi)$, and if we retain only $-\frac{1}{2}e$, and put 1 for $\sqrt{1-e^2}$, in [1345b], we shall obtain the term $-\frac{a^2ndt}{\mu}.\frac{3}{4}e^2.\left(\frac{dR}{da}\right)$. Now the first term of $\left(\frac{dR}{da}\right)$, deduced in [957], from $i=0$, is $\frac{m'}{2}.\left(\frac{dA^{(0)}}{da}\right)$, therefore $d\varepsilon$ contains

their values, it is evident that there will be produced some constant terms,
and other terms depending on the sines and cosines of the angles, on which $[1345''']$
the secular variations of the excentricities and inclinations of the orbits
depend. The constant terms will produce, in the expression of ε, some
terms proportional to the time, which are included in the mean motion of $[1345^{\rm iv}]$
m. As it respects the terms affected with the sines and cosines, they will
acquire by integration, in the expression of ε, very small divisors, of the same
order as the disturbing forces; and as these terms are both multiplied and
divided by these forces, they may become sensible, although they are of the $[1345^{\rm v}]$
order of the squares and of the products of the excentricities and inclinations.
We shall find, in the theory of the planets, that these terms are insensible;
but they are very sensible in the theory of the moon [5543], and of the
satellites of Jupiter, [Book VIII], and it is upon these terms that their $[1345^{\rm vi}]$
secular equations depend.

We have seen in § 65 [1195], that the mean motion of *m* has for
expression $\frac{3}{\mu}\int\int a\,.\,n\,d\,t\,.\,\mathrm{d}R$, and if we notice only terms of the order of $[1345^{\rm vii}]$
the first power of the disturbing masses, $\mathrm{d}\,R$ will contain only periodical
quantities [1197′]. But if we consider the squares and products of these

---

the term $-\frac{a^2\,n\,d\,t}{\mu}\,.\,\frac{3}{8}\,e^2\,.\,m'\,.\left(\frac{d\,A^{(0)}}{d\,a}\right)$, which is a function of the elements of the orbit $[1345c]$
only; A being a function of a, a', [954]. If we substitute in $e^2=h^2+l^2$, [1108], the
values of h, l, &c., [1102, 1102*a*], we shall obtain $e^2=E^2+\Sigma\,.\,C\,.\cos.(g\,t+\beta)$;
E^2 being the constant term of e^2, and g a term of the same order as g, g_1, &c., [1102], or $[1345d]$
of the order of the disturbing forces. Substituting this in [1345*c*], we shall obtain in $d\,\varepsilon$,
the terms $-\frac{a^2\,n\,d\,t}{\mu}\,.\,\frac{3}{8}\,m'\,.\left(\frac{d\,A^{(0)}}{d\,a}\right).\{E^2+\Sigma\,.\,C\,.\cos.(g\,t+\beta)\}$, and by integration $[1345e]$
we shall get the corresponding part of

$$\varepsilon=-\frac{3\,a^2\,m'\,n}{8\,\mu}\,.\left(\frac{d\,A^{(0)}}{d\,a}\right).\,E^2\,t-\frac{3\,a^2\,m'\,n}{8}\,.\left(\frac{d\,A^{(0)}}{d\,a}\right).\,\Sigma\,.\,\frac{C}{g}\,.\sin.(g\,t+\beta),$$

in which the first term is proportional to t, and is comprised in the mean motion; the last
terms are divided by g, which is of the order m', [1345*d*], so that they are both multiplied $[1345f]$
and divided by terms of the order m', which might render them sensible. It is shown
however, in Book VI, that in those planets where this ought to be most sensible, they are not
of any importance.

[1345$^{\text{viii}}$] masses, this differential may contain terms which are functions of the elements of the orbits only.* If we substitute in them the secular parts of the values of these elements, there will result some terms affected with the sines and cosines of the angles, upon which the secular variations of the orbits depend. These terms will acquire, by the double integration, in the expression of the mean motion, very small divisors, which will be of the order of the squares and of
[1345$^{\text{ix}}$] the products of the disturbing masses; and being both multiplied and divided by the squares and products of these masses, they may become sensible, although they are of the order of the squares and of the products of the
[1345$^{\text{x}}$] excentricities and of the inclinations of the orbits. We shall also find that these terms are insensible in the theory of the planets.

73. The elements of the orbit of *m*, being determined in the preceding manner, we must substitute them in the expressions of the radius vector, longitude, and latitude, which we have given in § 22 [659, 668, &c.]; we
[1345$^{\text{xi}}$] shall thus obtain the values of these three variable quantities, by which astronomers usually determine the positions of the heavenly bodies. If we develop these expressions in terms of sines and cosines, we shall get a series of inequalities, which we may arrange in tables, and by this means we may compute the position of *m*, at any given time.

This method, founded upon the variation of the parameters, is very useful in the investigation of those inequalities which in certain ratios of the mean
[1345$^{\text{xii}}$] motions of the bodies of the system, acquire great divisors, and on that account become very sensible. Inequalities of this kind chiefly affect the elliptical elements of the orbits; therefore if we determine the variations of the elements arising from these inequalities, and substitute them in the
[1345$^{\text{xiii}}$] expression of the elliptical motion, we shall obtain, in the most simple manner, all the inequalities which these divisors render sensible.

---

† (916) That is, terms similar to the quantity computed in [1345*c*], which may be reduced, in like manner as in [1345*d*, *e*], and the double integral being taken, the coefficient of the term depending on the angles of the form $(gt+\beta)$, will be of the order of the square of the masses, divided, on account of the double integration, by g^2, which divisor is also of the order of the square of the disturbing masses, [1345*f*]. These terms, notwithstanding the smallness of their divisors, are insensible, as is shown in Book VI.

The preceding method is also useful in the theory of comets. These
bodies are visible only in a small portion of their path, and observations
furnish merely that part of the ellipsis which coincides with the arc of the [1345^{xiv}]
orbit they describe during their appearance ; therefore if we determine the
nature of the orbit, considered as a variable ellipsis, we shall obtain the
changes in this ellipsis during the interval of two successive appearances of
the same comet ;* and we shall thus be enabled to predict its return, and [1345^{xv}]
upon its reappearance, we may compare its theory with the observations.

Having thus given the methods and formulas, to determine, by successive
approximations, the motions of the centres of gravity of the heavenly bodies,
it now remains to apply these formulas to the different bodies of the solar
system ; but the ellipticity of the heavenly bodies having a sensible influence [1345^{xvi}]
upon the motions of several of them, it is proper, before we make the
numerical calculations to examine into the forms of these bodies ; a subject
which is quite as interesting as the theory of their motions.

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* (917) This method is explained in Book IX.

APPENDIX, BY THE TRANSLATOR.

DEMONSTRATIONS OF THE FORMULAS OF THE INTRODUCTION.

The formulas [49—59] are found in almost every book treating of the differential calculus, and they may be easily demonstrated. Thus if we put x for the cosine, y the sine, and t the tangent, of an arch of a circle z, whose radius is 1, we shall have $x^2+y^2=1$, whose differential gives $dx=-\frac{y\,dy}{x}$. Substituting this in $\sqrt{dx^2+dy^2}$, which expresses the differential of an arch of any plane curve, whose rectangular ordinates are x, y, we shall get, [1345^1] by reduction, the expression [49]. The substitution of $dy=-\frac{x\,dx}{y}$, gives in like manner [50]; the negative sign being prefixed, because x decreases when z increases. The expressions [49, 50] are equivalent to [52, 53]. Substituting these values of $d.\sin.z$. $d.\cos.z$, in the differential of $\text{tang}.z=\frac{\sin.z}{\cos.z}$, we get

$$d.\text{tang}.z=\frac{\cos.z.d.\sin.z-\sin.z.d.\cos.z}{\cos.^2z}=\frac{dz.(\cos.^2z+\sin.^2z)}{\cos.^2z}=\frac{dz}{\cos.^2z}=dz.(1+\text{tang}.^2z),$$

as in [54]; hence $dz=\frac{d.\text{tang}.z}{1+\text{tang}.^2z}=\frac{dt}{1+tt}$, as in [51]. If we develop the formulas [49, 50, 51], according to the powers of y, x, t, respectively, and take the integrals, so as to commence with $z=0$, we shall obtain [46—48].

If $x=\log.y$, we shall have $x+dx=\log.(y+dy)$. Subtracting the first from the second, and observing that the difference of the logarithms of two numbers is equal to the logarithm of their ratio, we shall have

$$dx=\log.(y+dy)-\log.y=\log.\left(\frac{y+dy}{y}\right)=\log.\left(1+\frac{dy}{y}\right).$$

Developing this, by Taylor's theorem [617], according to the powers of $\frac{dy}{y}$, and retaining [1345^2] only the first power, it will be of the form $dx=(\log.1)+a.\frac{dy}{y}$, a being a constant quantity, representing the modulus of the logarithms, and as $\log.1=0$, it will be $dx=a.\frac{dy}{y}$. If $a=1$, it will become $dx=\frac{dy}{y}$, [59], corresponding to hyperbolic logarithms.

If we put $y=1+x$, in [59], we shall get

$$d\,.\,\text{hyp. log.}\,(1+x)=\frac{d\,x}{1+x}=d\,x-x\,d\,x+x^2\,d\,x-\&\text{c.},$$

[1345^3] whose integral is [58]. If we suppose $\text{hyp. log.}\,y=z$, and multiply it by $1=\text{hyp. log.}\,c$, we shall get $\text{hyp. log.}\,y=z\,\text{hyp. log.}\,c=\text{hyp. log.}\,c^z$, hence $y=c^z$, and $dy=d\,.\,c^z$, but from [59] we have $d\,y=y\,.\,d\,.\,\text{hyp. log.}\,y=y\,d\,z=c^z\,d\,z$, or $d\,.\,c^z=c^z\,d\,z$, as in [57].

Changing x into $\pm z$ in [607*c*], we shall obtain [55, 56]. From [607*d*, *e*] we get [43, 44]. Dividing [43] by [44], the resulting equation is [45]. Putting $\pm z\sqrt{-1}$, for x in [607*c*], we get

[1345^4]

$$c^{z\,.\sqrt{-1}}=1+z\,.\sqrt{-1}-\frac{z^2}{1.2}-\frac{z^3}{1.2.3}\,.\sqrt{-1}+\frac{z^4}{1.2.3.4}+\&\text{c.};$$

$$c^{-z\,.\sqrt{-1}}=1-z\,.\sqrt{-1}-\frac{z^2}{1.2}+\frac{z^3}{1.2.3}\,.\sqrt{-1}+\frac{z^4}{1.2.3.4}+\&\text{c.}$$

Hence,

[1345^5]

$$\frac{c^{z\,.\sqrt{-1}}-c^{-z\,.\sqrt{-1}}}{2\,.\sqrt{-1}}=z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2.3.4.5}-\&\text{c.}=\sin.\,z, \quad [607d];$$

$$\frac{c^{z\,.\sqrt{-1}}+c^{-z\,.\sqrt{-1}}}{2}=1-\frac{z^2}{1.2}+\frac{z^4}{1.2.3.4}-\&\text{c.}=\cos.\,z, \quad [607e].$$

These expressions of $\sin.\,z$, $\cos.\,z$, are as in [11, 12]. If we multiply [11] by $\pm\sqrt{-1}$, and add the product to [12], we shall get the formulas [13, 14]. Raising these to the power n, we shall obtain [15, 16].

If n be an integral number, and we raise the formula [12] to the power n, and connect together the positive and negative powers of c, which have the same exponent, we shall obtain an expression of the following form,

[1345^6]

$$\cos.^n z=A\,.\tfrac{1}{2}\,.\left\{c^{nz\,.\sqrt{-1}}+c^{-nz\,.\sqrt{-1}}\right\}+B\,.\,\tfrac{1}{2}\,.\,\left\{c^{(n-2)\,.\,z\,.\sqrt{-1}}+c^{-(n-2)\,.\,z\,.\sqrt{-1}}\right\}$$

$$+\,C\,.\tfrac{1}{2}\,.\left\{c^{(n-4)\,.\,z\,.\sqrt{-1}}+c^{-(n-4)\,.\,z\,.\sqrt{-1}}\right\}+\&\text{c.}$$

$$=A\,.\cos.\,n\,z+B\,.\cos.\,(n-2)\,.\,z+C\,.\cos.\,(n-4)\,.\,z+\&\text{c.},$$

and if we put successively $n=2,\ 3,\ 4,\ 5,\ 6,$ we shall get the formulas [6—10]. If in these we change z into $z-\frac{1}{2}\pi$, π being the semi-circumference of a circle whose radius is unity, we shall obtain the formulas [1—5].

From [11] we get $\sin.\,a=\dfrac{c^{a\,.\sqrt{-1}}-c^{-a\,.\sqrt{-1}}}{2\,.\sqrt{-1}}$, $\sin.\,b=\dfrac{c^{b\,.\sqrt{-1}}-c^{-b\,.\sqrt{-1}}}{2\,\sqrt{-1}}$.

The product of these two expressions is

$$\sin.a\,.\,\sin.b=\tfrac{1}{2}\,.\left\{\frac{c^{(a-b)\,.\,\sqrt{-1}}+c^{-(a-b)\,.\,\sqrt{-1}}}{2}\right\}-\tfrac{1}{2}\,.\left\{\frac{c^{(a+b)\,.\,\sqrt{-1}}+c^{-(a+b)\,.\,\sqrt{-1}}}{2}\right\}$$

$$=\tfrac{1}{2}\,.\cos.(a-b)+\tfrac{1}{2}\,.\cos.(a+b),$$

as appears from formula [12]. This is the same as [17]. Changing b into $\frac{1}{2}\pi+b$, we get [18, 19]. If we change a into $\frac{1}{2}\pi+a$, in [18], we shall get [20].

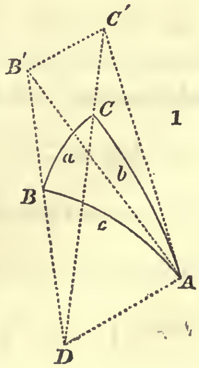
The sum of [17, 20] is [24], and if we change the sign of b, it becomes as in [23]. Putting $a-\frac{1}{2}\pi$ for a in [23, 24], we shall get [21, 22]. Changing a into $\frac{1}{2}\,.\,(a+b)$, b into $\frac{1}{2}\,.\,(a-b)$, in [18], and multiplying the result by 2, we get [25]. Putting in this $-b$ for b, it becomes as in [26]. Increasing the angles a, b, [25, 26], by a right angle and observing that $\sin.\frac{1}{2}\,.\,(a-b)=-\sin.\frac{1}{2}\,.\,(b-a)$, we obtain [27, 28].

Dividing [21] by [23], we get $\tan g.(a+b)=\dfrac{\sin.a\,.\cos.b+\cos.a\,.\sin.b}{\cos.a\,.\cos.b-\sin.a\,.\sin.b}$, dividing also the numerator and denominator by $\cos.a\,.\,\cos.b$, and putting $\dfrac{\sin.a}{\cos.a}=\text{tang.}\,a$, $\dfrac{\sin.b}{\cos.b}=\text{tang.}\,b$, we obtain [29]. Changing the sign of b it becomes as in [30]. Putting $b=a$, in [29], we get [30′], &c. Making $b=a$, in [21, 23] we get [31, 32]. Substituting, in [32], $\cos.^2 a=1-\sin.^2 a$, $\sin.^2 a=1-\cos.^2 a$, we obtain [33, 34].

Dividing [26] by [25], [26] by [27], [25] by [27], [25] by [28], [28] by [27], we shall get the formulas [35—39] respectively. Putting $a=0$, in [39] we get [40], also $b=0$, in [36] produces [41, 41′], and changes [38] into [42, 42′]. Formula [60] is derived from [678a]; changing in this z into $\frac{1}{2}\pi+z$, we get [61].

SPHERICAL TRIGONOMETRY.

All the formulas of spherical trigonometry, used in this work, may be derived from the theorem [172i], and it has been thought expedient to investigate, in this appendix, some of the most important of these formulas, and to give the following additional demonstration of this theorem.



Let ABC be a spherical triangle, described on the surface of a sphere, whose centre is D, and radius $DA=1$, Draw the lines DBB', DCC', to meet the lines AB', AC', which are drawn through the point A, tangents to the arcs AB, AC, respectively, and forming the plane triangles $DB'C'$, $AB'C'$. Then the angles of the spherical triangle, being denoted by A, B, C, and their opposite

sides by a, b, c, respectively, we shall have $DA = 1$, $DC' = \text{sec.}\, b$, $AC' = \text{tang.}\, b$;
[1345^{7}] $DB' = \text{sec.}\, c$, $AB' = \text{tang.}\, c$; angle $B'AC' =$ spherical angle $BAC = A$, angle $B'DC' = a$; and from [471] or [62] Int. we shall have, in the plane triangle $B'AC'$,

$$B'C'^2 = AB'^2 - 2AB'.AC'.\cos. B'AC' + AC'^2 = \text{tang.}^2 c - 2\,\text{tang.}\, c.\text{tang.}\, b.\cos. A + \text{tang.}^2 b,$$

and in the plane triangle $B'DC'$, we shall have, in like manner,

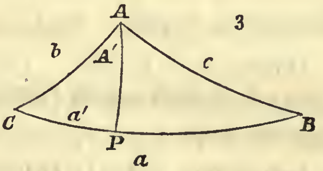
$$B'C'^2 = DB'^2 - 2DB'.DC'.\cos. B'DC' + DC'^2$$
$$= \text{sec.}^2 c - 2\,\text{sec.}\, c.\text{sec.}\, b.\cos. a + \text{sec.}^2 b,$$

subtracting the first expression of $B'C'^2$ from the second, and reducing, by putting $\text{sec.}^2 c - \text{tang.}^2 c = 1$, $\text{sec.}^2 b - \text{tang.}^2 b = 1$, we shall get

$$0 = 2 - 2\,\text{sec.}\, c.\text{sec.}\, b.\cos. a + 2\,\text{tang.}\, c.\text{tang.}\, b.\cos. A.$$

Multiplying this by $\frac{1}{2}\cos. c.\cos. b$, and putting $\cos. c.\text{sec.}\, c = 1$, $\cos. c.\text{tang.}\, c = \sin. c$, &c., we shall obtain the formula [172*i*], $0 = \cos. c.\cos. b - \cos. a + \sin. c.\sin. b.\cos. A$, or $\cos. a = \cos. A.\sin. b.\sin. c + .\cos. b \cos. c$. If in this we change each letter of the triplets, a, b, c; A, B, C; into the next in succession, recommencing the series, when the letter c or C is to be changed, we shall get the following system of equations,

[1345^{8}] $\cos. a = \cos. A.\sin. b.\sin. c + \cos. b.\cos. c,$
[1345^{9}] $\cos. b = \cos. B.\sin. c.\sin. a + \cos. c.\cos. a,$
[1345^{10}] $\cos. c = \cos. C.\sin. a.\sin. b + \cos. a.\cos. b.$



The first of these equations gives $\cos. A = \frac{\cos. a - \cos. b.\cos. c}{\sin. b.\sin. c}$, hence

$$1 - \cos. A = 2\sin.^2 \tfrac{1}{2} A = \frac{(\cos. b.\cos. c + \sin. b.\sin. c) - \cos. a}{\sin. b.\sin. c} = \frac{\cos.(b-c) - \cos. a}{\sin. b.\sin. c}$$
$$= \frac{2\sin. \frac{1}{2}(a-b+c).\sin. \frac{1}{2}(a+b-c)}{\sin. b.\sin. c},$$

and if we put $s = \frac{1}{2}(a+b+c)$, we shall get,

[1345^{11}] $$\sin.^2 \tfrac{1}{2} A = \frac{\sin. \frac{1}{2}(a-b+c).\sin. \frac{1}{2}(a+b-c)}{\sin. b.\sin. c} = \frac{\sin.(s-b).\sin.(s-c)}{\sin. b.\sin. c}.$$

The same value of $\cos. A$ gives

$$1 + \cos. A = 2\cos.^2 \tfrac{1}{2} A = \frac{\cos. a - (\cos. b.\cos. c - \sin. b.\sin. c)}{\sin. b.\sin. c} = \frac{\cos. a - \cos.(b+c)}{\sin. b.\sin. c}$$
$$= \frac{2\sin. \frac{1}{2}(b+c+a).\sin. \frac{1}{2}(b+c-a)}{\sin. b.\sin. c},$$

hence

$$\cos.^2 \tfrac{1}{2} A = \frac{\sin. \tfrac{1}{2} . (b + c + a) . \sin. \tfrac{1}{2} . (b + c - a)}{\sin. b . \sin. c} = \frac{\sin. s . \sin. (s - a)}{\sin. b . \sin. c} . \qquad [1345^{12}]$$

Dividing the preceding expression of $\sin.^2 \tfrac{1}{2} A$ by that of $\cos.^2 \tfrac{1}{2} A$, we get

$$\text{tang.}^2 \tfrac{1}{2} A = \frac{\sin. \tfrac{1}{2} . (a - b + c) . \sin. \tfrac{1}{2} . (a + b - c)}{\sin. \tfrac{1}{2} . (b + c + a) . \sin. \tfrac{1}{2} . (b + c - a)} = \frac{\sin. (s - b) . \sin. (s - c)}{\sin. s . \sin. (s - a)} . \qquad [1345^{13}]$$

These three formulas to find $\tfrac{1}{2} A$, are very convenient for the use of logarithms.

If we multiply together the formulas [1345$^{11, 12}$], putting

$$\sin.^2 \tfrac{1}{2} A . \cos.^2 \tfrac{1}{2} A = (\sin. \tfrac{1}{2} A . \cos. \tfrac{1}{2} A)^2 = (\tfrac{1}{2} \sin. A)^2,$$

we shall get, $\tfrac{1}{4} \sin.^2 A = \dfrac{\sin. s . \sin. (s - a) . \sin. (s - b) . \sin. (s - c)}{\sin.^2 b . \sin.^2 c}$. Dividing this by $\tfrac{1}{4} \sin.^2 a$, and taking the square root, we shall find,

$$\frac{\sin. A}{\sin. a} = \frac{2 . \sqrt{\sin. s . \sin. (s - a) . \sin. (s - b) . \sin. (s - c)}}{\sin. a . \sin. b . \sin. c} . \qquad [1345^{14}]$$

The second member of this expression is symmetrical in a, b, c, and without altering its value, we may change A, a, into B, b, or C, c, and the contrary; by which means we shall obtain,

$$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c} , \qquad [1345^{15}]$$

which is the well known formula, that the sine of the side is proportional to the sine of the opposite angle.

Substituting $\cos. c$, [1345^{10}] in [1345^{8}], and then putting $\cos.^2 b = 1 - \sin.^2 b$, we shall get,

$$\begin{aligned} \cos. a &= \cos. A . \sin. b . \sin. c + \cos. b . \{\cos. C . \sin. a . \sin. b + \cos. a . \cos. b\} \\ &= \cos. A . \sin. b . \sin. c + \sin. a . \sin. b . \cos. b . \cos. C + \cos. a . (1 - \sin.^2 b). \end{aligned}$$

Rejecting $\cos. a$, which occurs in both members, with the same sign, and dividing by $\sin. b$, we shall obtain the first of the two following equations; the two others being deduced from this, by changing successively, each letter of the triplets a, b, c, A, B, C, into the next in succession, as before.

$$\cos. a . \sin. b = \cos. A . \sin. c + \sin. a . \cos. b . \cos. C, \qquad [1345^{16}]$$

$$\cos. b . \sin. c = \cos. B . \sin. a + \sin. b . \cos. c . \cos. A, \qquad [1345^{17}]$$

$$\cos. c . \sin. a = \cos. C . \sin. b + \sin. c . \cos. a . \cos. B. \qquad [1345^{18}]$$

Substituting in the first of these equations, the value $\sin. c = \dfrac{\sin. a . \sin. C}{\sin. A}$, [1345^{15}];

dividing by $\sin. a$, putting $\frac{\cos. a}{\sin. a} = \cot. a$, and $\frac{\cos. A}{\sin. A} = \cot. A$, we get the first of the following equations, from which the other two may be derived, by the change of letters, as above.

[1345^{19}] $$\cot. a \,.\, \sin. b = \cot. A \,.\, \sin. C + \cos. b \,.\, \cos. C,$$

[1345^{20}] $$\cot. b \,.\, \sin. c = \cot. B \,.\, \sin. A + \cos. c \,.\, \cos. A,$$

[1345^{21}] $$\cot. c \,.\, \sin. a = \cot. C \,.\, \sin. B + \cos. a \,.\, \cos. B.$$

If we change $A, a,$ into $C, c,$ and the contrary, in [1345^{17}], we shall get

$$\cos. b \,.\, \sin. a = \cos. B \,.\, \sin. c + \sin. b \,.\, \cos. a \,.\, \cos. C.$$

Substituting this value of $\sin. a \,.\, \cos. b$, in [1345^{16}], it will become

$$\cos. a \,.\, \sin. b = \cos. A \,.\, \sin. c + \cos. C \,.\, (\cos. B \,.\, \sin. c + \sin. b \,.\, \cos. a \,.\, \cos. C), \quad \text{or}$$

$$\cos. a \,.\, \sin. b \,.\, (1 - \cos.^2 C) = \cos. A \,.\, \sin. c + \cos. B \,.\, \cos. C \,.\, \sin. c.$$

Substituting in the first member $1 - \cos.^2 C = \sin.^2 C$, and using

$$\sin. b \,.\, \sin. C = \sin. c \,.\, \sin. B, \quad [1345^{15}],$$

the whole will become divisible, by $\sin. c$, and we shall get

$$\cos. a \,.\, \sin. C \,.\, \sin. B = \cos. A + \cos. B \,.\, \cos. C,$$

which is the same as the first of the three following equations, the second and third being derived from it, by changing the order of the letters as above,

[1345^{22}] $$\cos. A = \cos. a \,.\, \sin. B \,.\, \sin. C - \cos. B \,.\, \cos. C,$$

[1345^{23}] $$\cos. B = \cos. b \,.\, \sin. C \,.\, \sin. A - \cos. C \,.\, \cos. A,$$

[1345^{24}] $$\cos. C = \cos. c \,.\, \sin. A \,.\, \sin. B - \cos. A \,.\, \cos. B.$$

The whole of spherical trigonometry is comprised in the formulas [1345$^{8-24}$], but in some cases it will be convenient to use an auxiliary angle, in the manner hereafter to be explained.

If we compare the two formulas [1345^{8,22}], we shall find that they become identical, by [1345^{25}] changing $a, b, c, A, B, C,$ into $\pi - A$, $\pi - B$, $\pi - C$, $\pi - a$, $\pi - b$, $\pi - c$, respectively, π being two right angles. We may therefore substitute the one of these triangles for the other. This second triangle is called the *supplemental triangle*. The sides and angles of the first being changed respectively into the supplements of the angles, and the supplements of the sides of the second. This is a well known property of spherical triangles.

Supplemental triangle.

If B be a right angle, the equations $[1345^{9,15,17,23,21,22}]$, will give the six following equations, comprising the whole of rectangular spherical trigonometry.

Rectangular spherical trigonometry.

$$\cos. b = \cos. a \,.\, \cos. c, \qquad [1345^{27}]$$

$$\sin. b = \frac{\sin. a}{\sin. A} = \frac{\sin. c}{\sin. C}, \qquad [1345^{28}]$$

$$\left.\begin{aligned} \text{tang.}\, c &= \text{tang.}\, b \,.\, \cos. A, \\ \text{tang.}\, a &= \text{tang.}\, b \,.\, \cos. C, \end{aligned}\right\} \qquad [1345^{29}]$$

$$\cos. b = \cot. A \,.\, \cot. C, \qquad [1345^{30}]$$

$$\left.\begin{aligned} \cot. C &= \cot. c \,.\, \sin. a, \\ \cot. A &= \cot. a \,.\, \sin. c, \end{aligned}\right\} \qquad [1345^{31}]$$

$$\left.\begin{aligned} \cos. A &= \sin. C \,.\, \cos. a, \\ \cos. C &= \sin. A \,.\, \cos. c. \end{aligned}\right\} \qquad [1345^{32}]$$

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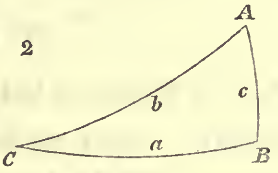

In several cases of oblique trigonometry, it will be necessary to introduce a subsidiary angle to facilitate the computation by logarithms.

1. If b, c, C, be given to find a, we may assume the auxiliary angle a', so that

$$\text{tang.}\, a' = \cos. C \,.\, \text{tang.}\, b\,; \qquad [1345^{33}]$$

which by putting $\text{tang.}\, b = \frac{\sin. b}{\cos. b}$, will give $\cos. b \,.\, \text{tang.}\, a' = \cos. C \,.\, \sin. b$. Substituting this in $[1345^{10}]$, we get

$$\cos. c = \cos. b \,.\, (\text{tang.}\, a' \,.\, \sin. a + \cos. a) = \frac{\cos. b}{\cos. a'} \,.\, (\sin. a' \,.\, \sin. a + \cos. a' \,.\, \cos. a) = \frac{\cos. b \,.\, \cos. (a - a')}{\cos. a'};$$

hence,

$$\cos. (a - a') = \frac{\cos. c \,.\, \cos. a'}{\cos. b}; \qquad [1345^{34}]$$

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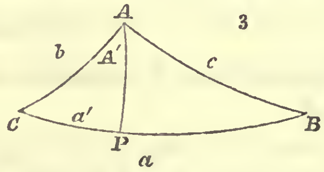


from which $a - a'$, and then a may be determined.

The same process answers for the case where b, a, C, are given to find c. For having computed a', we get $a - a'$, and then $\cos. c = \frac{\cos. b \,.\, \cos. (a - a')}{\cos. a'}$. $[1345^{35}]$

It is evident from $[1345^{29,33}]$ that this subsidiary angle a' is equal to the segment CP of the base, formed by letting fall the perpendicular AP upon the base BC. $[1345^{36}]$

If we change A, a, into C, c, and the contrary, in $[1345^{20}]$, we shall get the following formula, to determine B by a, b, C, $\cot. b \,.\, \sin. a = \cot. B \,.\, \sin. C + \cos. a \,.\, \cos. C$, hence $\cot. B = \frac{\cot. b \,.\, \sin. a}{\sin. C} - \cos. a \,.\, \cot. C$. Substituting $\cot. b = \cos. C \,.\, \cot. a'$,

$[1345^{33}]$, it becomes

$$\cot. B = \cot. C . (\cot. a' . \sin. a - \cos. a) = \frac{\cot. C}{\sin. a'} . (\cos. a' . \sin. a - \sin. a' . \cos. a)$$

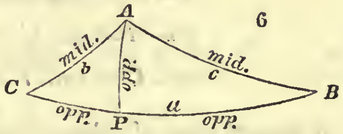
$$= \frac{\cot. C}{\sin. a'} . \sin. (a - a'),$$

therefore,

$[1345^{37}]$ $$\cot. B = \frac{\cot. C . \sin. (a - a')}{\sin. a'} .$$

2. If $b, A, C,$ are given to find c, we shall get, by changing, in $[1345^{20}]$, $c, C,$ into $b, B,$ and the contrary, $\cot. c . \sin. b = \cot. C . \sin. A + \cos. b . \cos. A$; or

$$\cot. c = \frac{\cot. C . \sin. A}{\sin. b} + \cot. b . \cos. A.$$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Now assuming the subsidiary angle A', such that

$[1345^{38}]$ $$\cot. A' = \cos. b . \tan g. C ;$$

we shall get $\cot. C = \cos. b . \text{tang}. A'$, hence,

$$\cot. c = \frac{\cos. b . \text{tang}. A' . \sin. A}{\sin. b} + \cot. b . \cos. A = \cot. b . (\text{tang}. A' . \sin. A + \cos. A)$$

$$= \frac{\cot. b}{\cos. A'} . (\sin. A' . \sin. A + \cos. A . \cos. A') = \frac{\cot. b . \cos. (A - A')}{\cos. A'},$$

that is,

$[1345^{39}]$ $$\cot. c = \frac{\cot. b . \cos. (A - A')}{\cos. A'} .$$

If $b, c, C,$ are given to find A, we may use the same subsidiary angle A', and then the preceding formula will give,

$[1345^{40}]$ $$\cos. (A - A') = \cot. c . \text{tang}. b . \cos. A' ;$$

from which we may compute $A - A'$; with which, and A', we shall obtain the value of A.

It may be observed, that the subsidiary angle A' is the same as the angle CAP, formed by the side CA, and the perpendicular AP, let fall upon BC.

If $c, A, B,$ are given to find C, we may take an auxiliary angle, such that

$$\cot. A'' = \cos. c . \text{tang}. B, \qquad \text{whence} \qquad \cos. c . \sin. B = \cos. B . \cot. A'',$$

substituting this in $[1345^{24}]$, we get

$$\cos. C = \cos. B . (\sin. A . \cot. A'' - \cos. A)$$

$[1345^{41}]$ $$= \frac{\cos. B}{\sin. A''} . (\sin. A . \cos. A'' - \cos. A . \sin. A'') = \frac{\cos. B . \sin. (A - A'')}{\sin. A''},$$

and it is evident that the subsidiary angle A'', is the same as the angle BAP.

If it be required to find B from b, A, C, we may use the subsidiary angle A', [1345^{38}], from which we shall get also $A-A'$. Now if we substitute, in [1345^{23}], the expression [1345^{38}], $\cos. b=\cot. A'.\cot. C$, we shall get

$$\cos. B=\cot. A'.\cos. C.\sin. A-\cos. C.\cos. A=\cos. C.(\cot. A'.\sin. A-\cos. A)$$

$$=\frac{\cos. C}{\sin. A'}.(\cos. A'.\sin. A-\sin. A'.\cos. A)=\frac{\cos. C.\sin.(A-A')}{\sin. A'},$$

hence,

$$\cos. B=\frac{\cos. C.\sin.(A-A')}{\sin. A'}. \qquad [1345^{42}]$$

If b, B, C, were given to find A, we must find A', as before, and then

$$\sin.(A-A')=\frac{\cos. B.\sin. A'}{\cos. C}, \qquad [1345^{43}]$$

from which we may compute $A-A'$, and thence A.

From [1345^{13}] we get $\tang. \tfrac{1}{2} A=\left(\frac{\sin.(s-b).\sin.(s-c)}{\sin. s.\sin.(s-a)}\right)^{\frac{1}{2}}$, and by changing A, a, into B, b, and the contrary, which does not affect $s=\tfrac{1}{2}.(a+b+c)$, we find

$$\tang. \tfrac{1}{2} B=\left(\frac{\sin.(s-a).\sin.(s-c)}{\sin. s.\sin.(s-b)}\right)^{\frac{1}{2}}. \qquad [1345^{44}]$$

The product of these two expressions is $\tang. \tfrac{1}{2} A.\tang. \tfrac{1}{2} B=\frac{\sin.(s-c)}{\sin. s}$; and if we change B, b, into C, c, and the contrary, it will become $\tang. \tfrac{1}{2} A.\tang. \tfrac{1}{2} C=\frac{\sin.(s-b)}{\sin. s}$; [1345^{45}]
again changing A, a, into B, b, and the contrary, we shall obtain from this last expression $\tang. \tfrac{1}{2} B.\tang. \tfrac{1}{2} C=\frac{\sin.(s-a)}{\sin. s}$. Taking the sum and difference of these values, we shall get

$$(\tang. \tfrac{1}{2} A+\tang. \tfrac{1}{2} B).\tang. \tfrac{1}{2} C=\frac{\sin.(s-b)+\sin.(s-a)}{\sin. s},$$

$$(\tang. \tfrac{1}{2} A-\tang. \tfrac{1}{2} B).\tang. \tfrac{1}{2} C=\frac{\sin.(s-b)-\sin.(s-a)}{\sin. s}, \qquad [1345^{46}]$$

$$1+\tang. \tfrac{1}{2} A.\tang. \tfrac{1}{2} B=\frac{\sin. s+\sin.(s-c)}{\sin. s},$$

$$1-\tang. \tfrac{1}{2} A.\tang. \tfrac{1}{2} B=\frac{\sin. s-\sin.(s-c)}{\sin. s}.$$

Substituting these values in the expressions $\tang. \tfrac{1}{2}.(A\pm B)=\frac{\tang. \frac{1}{2} A\pm\tang. \frac{1}{2} B}{1\mp\tang. \frac{1}{2} A.\tang. \frac{1}{2} B}$, [29, 30] Int., we shall get

$[1345^{47}]$

$$\mathrm{tang.}\tfrac{1}{2}.(A+B).\mathrm{tang.}\tfrac{1}{2}C=\frac{\sin.(s-b)+\sin.(s-a)}{\sin.s-\sin.(s-c)},$$

$$\mathrm{tang.}\tfrac{1}{2}.(A-B).\mathrm{tang.}\tfrac{1}{2}C=\frac{\sin.(s-b)-\sin.(s-a)}{\sin.s+\sin.(s-c)},$$

but from [25, 26] Int. we have

$$\sin.(s-b)+\sin.(s-a)=2\sin.\tfrac{1}{2}.(2s-a-b).\cos.\tfrac{1}{2}.(a-b)=2\sin.\tfrac{1}{2}c.\cos.\tfrac{1}{2}.(a-b),$$

$$\sin.s-\sin.(s-c)=2\sin.\tfrac{1}{2}c.\cos.\tfrac{1}{2}.(2s-c)=2\sin.\tfrac{1}{2}c.\cos.\tfrac{1}{2}.(a+b),$$

$$\sin.(s-b)-\sin.(s-a)=2\sin.\tfrac{1}{2}.(a-b).\cos.\tfrac{1}{2}.(2s-a-b)=2\sin.\tfrac{1}{2}.(a-b).\cos.\tfrac{1}{2}c,$$

$$\sin.s+\sin.(s-c)=2\sin.\tfrac{1}{2}.(2s-c).\cos.\tfrac{1}{2}c=2\sin.\tfrac{1}{2}.(a+b).\cos.\tfrac{1}{2}c.$$

Substituting these in $[1345^{47}]$, and rejecting the factors common to the numerators and denominators, we shall obtain the following formulas of Napier,

Napier's formulas.

$[1345^{48}]$ $$\mathrm{tang.}\tfrac{1}{2}.(A+B).\mathrm{tang.}\tfrac{1}{2}C=\frac{\cos.\frac{1}{2}.(a-b)}{\cos.\frac{1}{2}.(a+b)},$$

$[1345^{49}]$ $$\mathrm{tang.}\tfrac{1}{2}.(A-B).\mathrm{tang.}\tfrac{1}{2}C=\frac{\sin.\frac{1}{2}.(a-b)}{\sin.\frac{1}{2}.(a+b)}.$$

If in these formulas, and in $[1345^{11-13}]$, we change the values as in $[1345^{25}]$, so as to correspond to the supplemental triangle, we shall easily obtain, by slight reductions, and putting $S=\frac{1}{2}.(A+B+C)$, the following expressions, of which the two first were discovered by Napier,

$[1345^{50}]$ $$\mathrm{tang.}\tfrac{1}{2}(a+b).\cot.\tfrac{1}{2}c=\frac{\cos.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)},$$

Napier's formulas.

$[1345^{51}]$ $$\mathrm{tang.}\tfrac{1}{2}(a-b).\cot.\tfrac{1}{2}c=\frac{\sin.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)},$$

$[1345^{52}]$ $$\cos.^2\tfrac{1}{2}a=\frac{\cos.\frac{1}{2}(A-B+C).\cos.\frac{1}{2}(A+B-C)}{\sin.B.\sin.C}=\frac{\cos.(S-B).\cos.(S-C)}{\sin.B.\sin.C},$$

$[1345^{53}]$ $$\sin.^2\tfrac{1}{2}a=\frac{-\cos.\frac{1}{2}(A+B+C).\cos.\frac{1}{2}(B+C-A)}{\sin.B.\sin.C}=\frac{-\cos.S.\cos.(S-A)}{\sin.B.\sin.C},$$

$[1345^{54}]$ $$\mathrm{tang.}^2\tfrac{1}{2}a=\frac{-\cos.\frac{1}{2}(A+B+C).\cos.\frac{1}{2}(B+C-A)}{\cos.\frac{1}{2}(A-B+C).\cos.\frac{1}{2}(A+B-C)}=\frac{-\cos.S.\cos.(S-A)}{\cos.(S-B).\cos.(S-C)}.$$

If in the preceding formulas we suppose a, b, c, to be infinitely small in comparison with the radius of the sphere, or unity, we may put $\sin.a=a$, $\sin.b=b$, $\sin.c=c$, $\cos.a=1$, $\cos.b=1$, $\mathrm{tang.}a=a$, &c., and we may, by this means, obtain several theorems of plane trigonometry. Thus if $s=\frac{1}{2}(a+b+c)$, we shall get from [1345^{11-15}], the following, **Plane trigonometry.**

$$\sin.^2 \tfrac{1}{2} A=\frac{(s-b).(s-c)}{bc}, \qquad [1345^{55}]$$

$$\cos.^2 \tfrac{1}{2} A=\frac{s.(s-a)}{bc}, \qquad [1345^{56}]$$

$$\mathrm{tang.}^2 \tfrac{1}{2} A=\frac{(s-b).(s-c)}{s.(s-a)}, \qquad [1345^{57}]$$

$$\sin. A=\frac{2.\sqrt{s.(s-a).(s-b).(s-c)}}{bc}, \qquad [1345^{57\prime}]$$

$$\frac{\sin. A}{a}=\frac{\sin. B}{b}=\frac{\sin. C}{c}. \qquad [1345^{58}]$$

If we retain the second powers of a, b, c, we may put, as in [43, 44] Int., $\sin. b=b$, $\sin. c=c$, $\cos. a=1-\frac{1}{2}a^2$, $\cos. b=1-\frac{1}{2}b^2$, $\cos. c=1-\frac{1}{2}c^2$. Substituting these in [1345^8], it will become

$$1-\tfrac{1}{2}a^2=bc.\cos. A+(1-\tfrac{1}{2}b^2).(1-\tfrac{1}{2}c^2), \qquad [1345^{58\prime}]$$

whence by reduction $\cos. A=\frac{b^2+c^2-a^2}{2bc}$, which is the same as [62] Int. In like manner we may obtain other formulas.

Many other combinations of the angles $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, $\frac{1}{2}a$, $\frac{1}{2}b$, $\frac{1}{2}c$, may be found in several works on Trigonometry. Gauss published several formulas of a nature somewhat similar to those in [1345^{48-51}], which he has often used, though the common formulas would answer the same purpose, and sometimes be shorter. Delambre in his Astronomie, Vol. I, p. 164, observes that he had given several of these theorems in the *Connoisance des Tems*, 1808, before the publication of the work of Gauss, and that he had suppressed the demonstrations, supposing the theorems would not be very useful. Considering the remarks of Delambre as essentially correct, and wishing to abridge this part of the work, I have not inserted any of these formulas, which may however be easily demonstrated, if it should be found necessary, by the methods here given.

For the more easy recollection of the formulas of spherical trigonometry, Lord Napier supposed a rectangular spherical triangle to consist of *five circular parts*, namely, the *two sides*, *the complement of the hypotenuse*, and *the complement of the two oblique angles*, which he named *adjacent*, or *opposite*, according to their position with respect to each other. In this method the right angle is not considered as one of the circular parts, neither is it supposed to separate the sides. In all cases two of these parts are given to find the third. If the three parts join, that which is in the middle is called the *middle part*; if they do not all join, two of them must, and that which is separate is called the *middle part*, and the other **Napier's circular parts.** [1345^{59}]

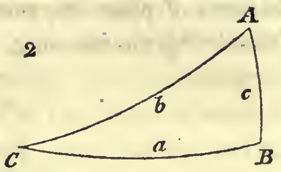
[1345^{60}] two *opposite parts*, as in figures 4, 5. Then putting the radius equal to unity, the equations given by Napier will become

[1345^{61}] *Sine of middle part = Rectangle of the tangents of the adjacent parts*
= Rectangle of the cosines of the opposite parts,

It may be of assistance in remembering these rules, that the first letters of the words adjacent and opposite, are the same as those of the second letters of the words tangents and cosines, with which they are respectively combined. The demonstration of the equations [1345^{61}] may easily be obtained, by applying them to the preceding formulas Thus in [1345^{27}] the complement of the hypotenuse $\frac{1}{2}\pi - b$, is the middle part, and a, c, opposite parts, as in fig. 4; $\frac{1}{2}\pi$ being a right angle. In [1345^{29}], second formula, if we put, as in fig. 5, $\frac{1}{2}\pi - C$, for the middle part, a and $\frac{1}{2}\pi - b$ will be the adjacent parts. In [1345^{30}], fig. 4, $\frac{1}{2}\pi - b$ is the middle part, and $\frac{1}{2}\pi - A$,
[1345^{62}] $\frac{1}{2}\pi - C$ adjacent parts. In [1345^{31}], first formula, a is the middle part and $\frac{1}{2}\pi - C$, c the adjacent parts. Lastly, in [1345^{32}], first formula, $\frac{1}{2}\pi - A$, is the middle part, and $\frac{1}{2}\pi - C$, a, the opposite parts. In this way the expressions [1345^{61}], will be found to include all the preceding formulas in rectangular spherical trigonometry, except [1345^{28}], which depends on the well known formula, that the sine of a side is proportional to the sine of its opposite angle.

Application of Napier's circular parts to oblique trigonometry.

This method may be applied to the solutions of cases in oblique spherical trigonometry, by dividing the triangle ACB, fig. 6—9, into two rectangular triangles, CPA, BPA, by means of a perpendicular AP, let fall from the angular point A, upon the opposite side or base BC; the perpendicular being so chosen as to *make two of the*
[1345^{63}] *given quantities fall in one of the rectangular triangles*, or in other words, *the perpendicular ought to be let fall from the end of a given side as CA, and opposite to a given angle C*; so that all the parts of this triangle are either given or may be computed, by the formulas [1345$^{27-32}$], or the equivalent ones [1345^{61}]. Each of these rectangular triangles CPA, BPA, contains,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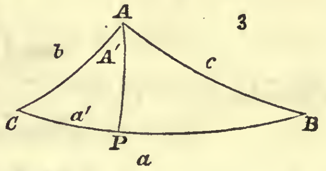
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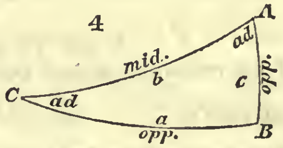
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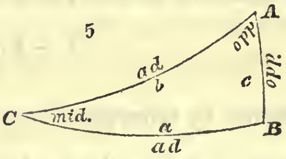
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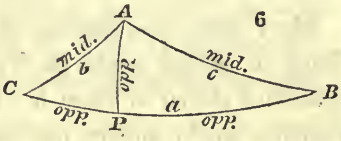
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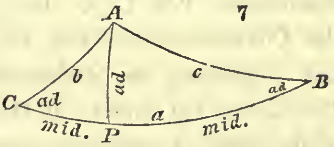
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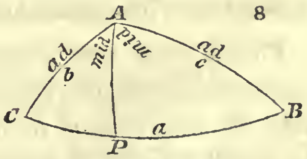
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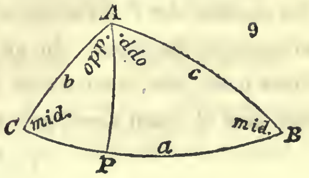
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as above, five circular parts, *the perpendicular AP being counted in each, and bearing in both the same name*; therefore the parts of each triangle, similarly situated with respect to the perpendicular, must have the same names, as is evident from the inspection of the figures. If in the triangle APB, we put M for the middle part, A for the adjacent part, and P for the opposite part; also m, a, p, for the corresponding parts of the triangle CPA; supposing the perpendicular AP to be an adjacent part, the rules of Napier, [1345⁶¹], will give $\text{tang.}\,AP = \frac{\sin. M}{\text{tang.}\,A}$, in the triangle CPA; and $\text{tang.}\,AP = \frac{\sin.\,m}{\text{tang.}\,a}$, in the triangle BPA; hence $\frac{\sin. M}{\text{tang.}\,A} = \frac{\sin.\,m}{\text{tang.}\,a}$, therefore

$$\sin. M : \text{tang.}\,A :: \sin.\,m : \text{tang.}\,a. \qquad [1345^{64}]$$

But if AP be an opposite part, we shall have by [1345⁶¹], $\cos.\,AP = \frac{\sin. M}{\cos. B} = \frac{\sin.\,m}{\cos.\,b}$, hence $\sin. M : \cos. B :: \sin.\,m : \cos.\,b$; and we shall have, for solving these cases of oblique spherical trigonometry, this rule, the sine of the middle parts are proportional to the tangents of the adjacent parts, or to the cosines of the opposite parts. Therefore we shall have, for solving all the cases of rectangular spherical trigonometry, and all except two cases of oblique angled spherical trigonometry, the following formulas,

$$\textit{Sine middle part} \left\{ \begin{matrix} = \\ \propto \end{matrix} \right\} \begin{matrix} \textit{Tangents of the adjacent parts.} \\ \textit{Cosines of the opposite parts.} \end{matrix} \qquad [1345^{65}]$$

These expressions, when applied to rectangular spherical triangles, denote, as above, that the sine of the middle part is *equal* to the product of the tangents of the adjacent parts, or to the product of the cosines of the opposite parts of the same triangle. When applied to an oblique angled spherical triangle, they denote that the sines of the middle part are *proportional* to the tangents of the adjacent parts; or that the sines of the middle parts are proportional to the cosines of the opposite parts of the same triangle; observing that the [1345⁶⁶] perpendicular being common to both triangles APB, APC, and bearing the same name in each of them, *must not be used in these proportions, nor counted as a middle part*; it not being necessary to compute the value of the perpendicular in making these calculations.

The two cases not included in the formulas [1345⁶⁵], are, *First*, where the question is between two sides and the opposite angles, which can be solved by the noted theorem [1345⁶⁷] [1345¹⁵]. *Second*, where three sides are given to find an angle, or three angles to find a side, this last being included in the former by using the supplemental triangle. These calculations may be made by means of the formulas [1345¹¹⁻¹³] or [1345⁵²⁻⁵⁴].

The manner of using Napier's method in rectangular trigonometry is well known. The rules for oblique trigonometry are the same as were given in a paper I published in the third volume of the Memoirs of the American Academy of Arts and Sciences, and may be illustrated by the following examples.

First, Let AB, AC, and C be given, to find BC, and the angles A, B. In the
[1345^{68}] first rectangular triangle CPA, we must compute the segment CP, by means of AC, C, [1345^{29}]. Mark the three parts AC, CP, AP, of the first triangle, as in fig. 6, and the second triangle APB, in a similar manner. Then the rule [1345^{65}],

$$\text{sin. mid.} \propto \text{cos. opp.} \quad \text{gives}$$

$$\text{sin.}\,(\text{co.}\,AC) : \text{cos.}\,CP :: \text{sin.}\,(\text{co.}\,AB) : \text{cos.}\,BP, \qquad \text{or}$$

[1345^{69}]

$$\text{cos.}\,AC : \text{cos.}\,CP :: \text{cos.}\,AB : \text{cos.}\,BP,$$

being the same as [1345^{34}]. Having CP, BP, we get $BC = CP + BP$, noticing the signs, and then the angles A, B, may be found by [1345^{15}].

If we mark the triangle as in fig. 7, and use the rule, $\text{sin. mid.} \propto \text{tang. adj.}$, we shall get

[1345^{70}]

$$\text{sin.}\,CP : \text{tang.}\,(\text{co.}\,C) :: \text{sin.}\,BP : \text{tang.}\,(\text{co.}\,B), \qquad \text{or}$$

$$\text{sin.}\,CP : \text{cot.}\,C :: \text{sin.}\,BP : \text{cot.}\,B,$$

as in [1345^{37}].

If the side BC be not required, but merely the angle A, we may compute the angle CAP, [1345^{30}], instead of the segment CP. Then marking the triangles as in fig. 8, we shall have, from the rule $\text{tang. adj.} \propto \text{sin. mid.}$,

$$\text{tang.}\,(\text{co.}\,AC) : \text{sin.}\,(\text{co.}\,CAP) :: \text{tang.}\,(\text{co.}\,AB) : \text{sin.}\,(\text{co.}\,BAP), \qquad \text{or}$$

[1345^{71}]

$$\text{cot.}\,AC : \text{cos.}\,CAP :: \text{cot.}\,AB : \text{cos.}\,BAP,$$

as in [1345^{39}]. Having the segments CAP, BAP, we easily obtain the angle $CAB = CAP + BAP$, noticing the signs; and we may then mark the triangles as in fig. 9, and the rule $\text{cos. opp.} \propto \text{sin. mid.}$, will give

$$\text{cos.}\,(\text{co.}\,CAP) : \text{sin.}\,(\text{co.}\,ACP) :: \text{cos.}\,(\text{co.}\,BAP) : \text{sin.}\,(\text{co.}\,ABP), \qquad \text{or}$$

[1345^{72}]

$$\text{sin.}\,CAP : \text{cos.}\,ACP :: \text{sin.}\,BAP : \text{cos.}\,ABP,$$

as in [1345^{41}].

Second, Let AC, BC, and the included angle C, be given, to find AB and the angles A, B. Having found the segment CP, as above, [1345^{68}], we shall get $BP = BC - CP$, noticing the signs. Marking the triangles as in fig. 7, and using the rule, $\text{sin. mid.} \propto \text{tang. adj.}$, we shall find, as before, [1345^{70}],

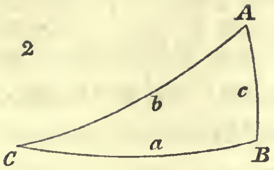
[1345^{73}]

$$\text{sin.}\,CP : \text{tang.}\,(\text{co.}\,C) :: \text{sin.}\,BP : \text{tang.}\,(\text{co.}\,B),$$

as in [1345^{37}], and as in [1345^{69}], we shall get, by using fig. 6,

$$\text{cos.}\,CP : \text{sin.}\,(\text{co.}\,AC) :: \text{cos.}\,BP : \text{sin.}\,(\text{co.}\,AB);$$

then A may be found as in [1345^{15}].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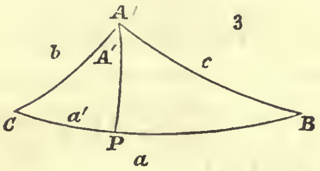
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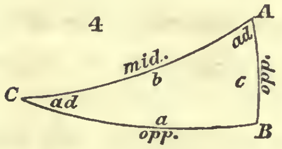
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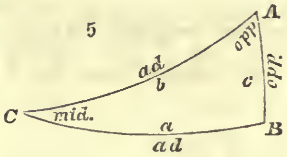
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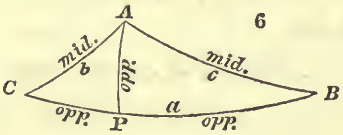
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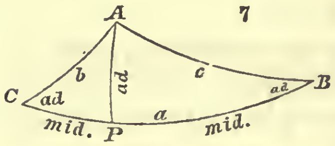
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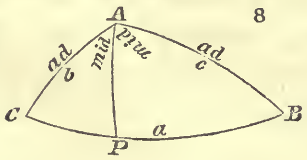
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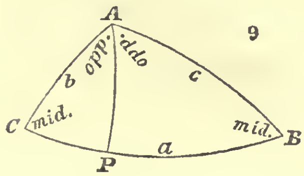
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Third. Given B, C, and the side AC; to find the rest. The segment CP being found as before, we get, in fig. 7, by using the rule tang. adj. $\propto$ sin. mid.,

tang. (co. C) : sin. CP : : tang. (co. B) : sin. BP, $[1345^{74}]$

$[1345^{37}]$. If we mark the triangles as in fig. 6, and use the rule cos. opp. $\propto$ sin. mid., we get

cos. CP : sin. (co. CA) : : cos. BP : sin. (co. AB),

$[1345^{35}]$. Otherwise we may compute as before, $[1345^{30}]$, the angle CAP, and then marking the triangles as in fig. 9, and using sin. mid. $\propto$ cos. opp., we shall get

sin. (co. C) : cos. (co. CAP) : : sin. (co. B) : cos. (co. CAP),

or cos. C : sin. CAP : : cos. B : sin. BAP, $[1345^{75}]$

$[1345^{41}]$, hence we get the angle

$$CAB = CAP + BAP,$$

noticing the signs. If we mark the triangles as in fig. 8, we shall get, by using the rule sin. mid. $\propto$ tang. adj.,

sin. (co. CAP) : tang. (co. AC) : : sin. (co. BAP) : tang. (co. AB)

or cos. CAP : cot. AC : : cos. BAP : cot. AB,

$[1345^{39}]$.

Fourth, Given the angles A, C, and the included side AC, to find the rest. Having computed as before, $[1345^{30}]$, the angle CAP, we shall have also, the other segment $BAP = CAB - CAP$, noticing the signs; then marking the triangles as in fig. 9, and using the rule cos. opp. $\propto$ sin. mid., we get

cos. (co. CAP) : sin. (co. C) : : cos. (co. BAP) : sin. (co. B),

or sin. CAP : cos. C : : sin. BAP : cos. B, $[1345^{41}]$. $[1345^{76}]$

If we mark the triangles as in fig. 8, we shall get, by using the rule sin. mid. $\propto$ tang. adj.,

sin. (co. CAP) : tang. (co. AC) : : sin. (co. BAP) : tang. (co. AB), $[1345^{77}]$

or cos. CAP : cot. AC : : cos. BAP : cot. AB,

$[1345^{39}]$.

The computation of a spherical triangle, in which the sides a, b, c, are very small in comparison with the radius of the sphere, and the angles opposite to those sides are respectively A, B, C, may be reduced to the computation of a plane triangle, having the same sides a, b, c, and the opposite angles A', B', C', respectively. For if we neglect terms of the fifth order in a, b, c, we shall have, by formulas [43, 44] Int.

[1345^{78}]

$$\cos. a = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4, \qquad \cos. b = 1 - \tfrac{1}{2}b^2 + \tfrac{1}{24}b^4,$$

$$\cos. c = 1 - \tfrac{1}{2}c^2 + \tfrac{1}{24}c^4, \qquad \sin. b = b - \tfrac{1}{6}b^3, \qquad \sin. c = c - \tfrac{1}{6}c^3.$$

Substituting these in $\cos. A = \dfrac{\cos. a - \cos. b \,.\, \cos. c}{\sin. b \,.\, \sin. c}$, [1345^{8}], we shall get

$$\cos. A = \frac{\tfrac{1}{2}\,.\,(b^2+c^2-a^2) + \tfrac{1}{24}\,.\,(a^4-b^4-c^4) - \tfrac{1}{4}b^2c^2}{bc\,.\,\{1-\tfrac{1}{6}\,.\,(b^2+c^2)\}},$$

or by reduction

[1345^{79}]

$$\cos. A = \frac{b^2+c^2-a^2}{2bc} - \frac{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}{24bc}.$$

If we suppose the radius of the sphere 1, to be infinitely great in comparison with the sides a, b, c, the terms of the second order will vanish in the preceding expression, and A will become A', corresponding to a plane triangle. The expression of $\cos. A'$ thus obtained will be the same as in [1345$^{58'}$] or [62] Int., namely,

[1345^{80}]

$$\cos. A' = \frac{b^2+c^2-a^2}{2bc},$$

from which we get,

[1345^{81}]

$$\sin.^2 A' = 1 - \cos.^2 A' = \frac{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}{4b^2c^2}.$$

Substituting these in [1345^{79}], we get,

[1345^{82}]

$$\cos. A = \cos. A' - \tfrac{1}{6}bc\,.\,\sin.^2 A'.$$

Now in this plane triangle, the perpendicular let fall from the angular point C', upon the opposite side c, is evidently $= b\,.\,\sin. A'$; multiplying this by half the side c, we shall
[1345^{83}] obtain the area of the triangle $s = \tfrac{1}{2}bc\,.\,\sin. A'$. Substituting this in [1345^{82}], it becomes $\cos. A = \cos. A' - \tfrac{1}{3}s\,.\,\sin. A'$, hence by [61] Int. $\cos. A = \cos.(A' + \tfrac{1}{3}s)$, or $A = A' + \tfrac{1}{3}s$, and as the area s does not change by putting B or C for A, and B', C', for A', respectively, we shall have the following system of equations,

[1345^{84}]

$$A = A' + \tfrac{1}{3}s,$$

$$B = B' + \tfrac{1}{3}s,$$

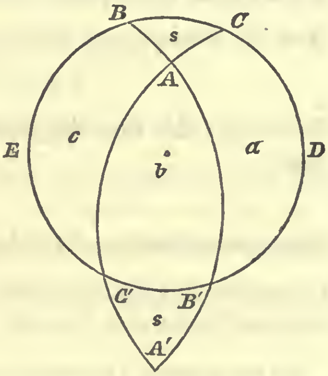
$$C = C' + \tfrac{1}{3}s.$$

The sum of these, putting for $A'+B'+C'$ its value 180^d or π, is $A+B+C=\pi+s$, hence

$$s=A+B+C-\pi.$$ [1345^{85}]

[1345^{86}] *Therefore if we denote by s, the excess of the sum of the three angles of a spherical triangle above two right angles, and subtract one third of this excess from each of the spherical angles, A, B, C, we shall obtain the corresponding angles A′, B′, C′, of a rectilinear plane triangle, the sides of which are equal in length to those of the spherical triangle.* This beautiful theorem, discovered by Le Gendre, is much used in geodetical operations, for reducing the calculations of small spherical triangles to the common operations of plane trigonometry.

Le Gendre's theorem for small spherical triangles.



The area of a spherical triangle ABC, may be found in the following manner. Suppose the radius of the sphere to be 1, corresponding to the circumference of the great circle $2\pi=6{,}2831\ldots$. Then by a well known theorem, the whole spherical surface will be 4π. This also follows from the expression of m, [275*b*], which gives $4\pi dR$, for the mass of the spherical shell of the thickness dR, included between the radii R and $R+dR$, when $R=1$. Now if we suppose any semicircle of this surface, as $BCDB'$ to revolve about its diameter BB' till it make a complete revolution, or 4 right angles $=2\pi$, it will pass over, during this revolution, the whole spherical surface 4π; the ratio of these quantities 2π and 4π being as 1 to 2; and it is evident that the same ratio will obtain, between any other angle as $ABC=B$, and the corresponding space passed over $BDB'AB$, which will therefore be represented by $2B$; so that, in the present notation, *the spherical surface, included between any two semicircles of the sphere, will be represented by twice the angle of inclination of these semicircles.* [1345^{87}]

Continuing the side of the triangle BC so as to complete the great circle $BCDB'C'E$, also the sides BA, CA, till they cross this great circle in B', C', and meet again in the opposite hemisphere at A'; we shall evidently have the arc

$$BAB'=AB'A'=\text{a semicircle},$$ [1345^{88}]

also the arc, $CAC'=AC'A'=$ a semicircle; subtracting from these AB', AC', respectively, we shall get $AB=A'B'$, $AC=A'C'$, and as the angle $BAC=B'A'C'$, we shall have the triangles BAC, $B'A'C'$, equal to each other, therefore $BC=B'C'$, and the surface ABC equal to the surface $A'B'C'$.

Putting $s=$ the surface of the triangle ABC, or $A'B'C'$, $a=$ surface $ACDB'A$, [1345^{89}]

[1345⁹⁰] $b =$ surface $B'AC'$, $c =$ surface $ABEC'$, $A =$ angle BAC, $B =$ angle ABC, $C =$ angle ACB, we shall have, by using the theorem [1345⁸⁷],

[1345⁹¹]
$$\begin{aligned} &\text{surface } BCDB'AB = s + a = 2B, \\ &\text{surface } CBEC'AC = s + c = 2C, \\ &\text{surfaces } BAC + B'AC' = \text{surfaces } B'A'C' + B'AC' \\ &\qquad\qquad = \text{surface } AC'A'B'A = s + b = 2A. \end{aligned}$$

Adding these three quantities together, we shall obtain $3s + a + b + c = 2(A + B + C)$. Now the hemispherical surface 2π, is evidently equal to

$$s + a + b + c, \qquad \text{or} \qquad s + a + b + c = 2\pi.$$

Subtracting this from the preceding equation, we get $2s = 2(A + B + C - \pi)$, and finally,

[1345⁹²]
$$s = A + B + C - \pi.$$

[1345⁹³] *Therefore the surface of a spherical triangle, expressed in squares of the radius, taken as unity, is equal to the spherical excess of the sum of the angles of the triangle above two right angles, expressed in the above notation*, in which 180^d is represented by 3,1415....

[1345⁹⁴] As the quantity s is always positive, we shall have $A + B + C > \pi$. That is, the sum of the three angles of a spherical triangle *exceeds two right angles*; and since each of these angles is less than two right angles, the sum of the three angles must be *less than six right angles.*

If we substitute, in the expression of the area of a plane triangle $\tfrac{1}{2}bc.\sin.A$, [1345⁸³], the expression of $\sin.A$, [1345⁵⁷'] we shall get, for the area of a plane triangle whose sides are a, b, c, the expression

[1345⁹⁵]
$$\text{area} = \sqrt{s.(s-a).(s-b).(s-c)} = \tfrac{1}{4}.\sqrt{(a+b+c).(-a+b+c).(a-b+c).(a+b-c)}.$$

ON THE SYMBOL $\sqrt{-1}$.

The imaginary symbol $\sqrt{-1}$ occurs frequently in this work, particularly in the use of circular arcs, and as the principles, upon which the use of it is founded, are not commonly explained in the elementary works in this country, it may be proper to make a few remarks on the application of it to the calculus of sines and cosines of circular arcs, from which the propriety of employing it will very evidently appear.

If we, for brevity, denote this symbol by e, so that $e = \sqrt{-1}$, we must always put $e^2 = -1$, $e^3 = -\sqrt{-1}$, $e^4 = 1$, $e^5 = \sqrt{-1}$, and generally, $e^{4m} = 1$,

$e^{4m+1} = \sqrt{-1}$, $e^{4m+2} = -1$, $e^{4m+3} = -\sqrt{-1}$, m being any integral number whatever. This is conformable to the usual rules of multiplication in algebra, and must be considered as a *definition* of this symbol, and of the manner of using it, and not as a *demonstration* of its properties. It is also to be understood, *as a part of this definition*, that in all cases the symbol $\sqrt{-1}$, or e, is to be operated upon by addition, subtraction, multiplication, division, &c., according to the usual rules of algebra. Thus the sum of a and $b\sqrt{-1}$, is $a + b\sqrt{-1}$; the product of a by $b\sqrt{-1}$ is $ab\sqrt{-1}$; and the quotient of $b\sqrt{-1}$, divided by a, is $\frac{b\sqrt{-1}}{a}$, or $\frac{b}{a}.\sqrt{-1}$. In like manner the product of the binomials $a + b\sqrt{-1}$ by $c + d\sqrt{-1}$, or $a + be$ by $c + de$, is

$$ac + (ad + bc).e + bde^2 = ac - bd + (ad + bc).\sqrt{-1};$$

these operations being evidently conformable to the principles and definitions here used. Again, from [607c], we have $c^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$, and if $\pm ze$ had been used instead of x, in the development of this exponential, by the common processes of algebra, considering ze as a real quantity, we should have

$$c^{ze} = 1 + ze + \frac{z^2e^2}{1.2} + \frac{z^3e^3}{1.2.3} + \&c., \qquad c^{-ze} = 1 - ze + \frac{z^2e^2}{1.2} - \frac{z^3e^3}{1.2.3} + \&c.,$$

from which we should get

$$\frac{c^{ze} - c^{-ze}}{2e} = z + \frac{z^3e^2}{1.2.3} + \frac{z^5e^4}{1.2.3.4.5} + \&c.,$$

$$\frac{c^{ze} + c^{-ze}}{2} = 1 + \frac{z^2e^2}{1.2} + \frac{z^4e^4}{1.2.3.4} + \&c.,$$

$[1345^{97}]$

and if we now substitute the above values $e^2 = -1$, $e^4 = 1$, &c., in the second members of these expressions, they will represent the values of $\sin. z$, $\cos. z$, [607d, e], which had been found independently of the use of the symbol $\sqrt{-1}$. Therefore we shall have,

$$\sin. z = \frac{c^{z.\sqrt{-1}} - c^{-z.\sqrt{-1}}}{2.\sqrt{-1}}, \qquad \cos. z = \frac{c^{z.\sqrt{-1}} + c^{-z.\sqrt{-1}}}{2},$$

as in [1345^5]; and these must be considered as nothing more than abridged values of $\sin. z$, $\cos. z$, reduced to simple analytical forms, extremely convenient in many trigonometrical calculations. Hence we perceive the real import of these expressions to be nothing more than that if the quantities $c^{z.\sqrt{-1}}$, $c^{-z.\sqrt{-1}}$, or c^{ze}, c^{-ze}, be developed according to the powers of $z\,e$, by the usual rules of development of *real* quantities, the analytical expressions $\frac{c^{ze} - c^{-ze}}{2e}$, $\frac{c^{ze} + c^{-ze}}{2}$, putting $e^2 = -1$, after the development, will accurately represent the values of $\sin. z$, $\cos. z$, respectively, in *real* finite quantities, independent of $\sqrt{-1}$; and there is in fact, no more mystery in the use of this imaginary symbol e, in this manner, and for this purpose, than there is in substituting the abridged expression $(1 - x)^{\frac{1}{2}}$, instead of its equivalent values in an infinite series

$$1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \&c.$$

Having obtained these abridged analytical values of $\sin. z$, $\cos. z$, we shall, by changing z into x, get similar analytical expressions of $\sin. x$, $\cos. x$, and if we wish to obtain the products, or powers, or any functions whatever, of such sines or cosines, we may use these analytical formulas, as has already been done in [1345^6, &c.] What has been said will serve to illustrate briefly the logical principles upon which the use of this symbol is founded, and any one who wishes to pursue the investigation, may consult a valuable paper of Mr. Woodhouse, published in the Philosophical Transactions of London for 1801, in which this subject is fully discussed.

Ira Berry and Lucius A. Thomas, compositors.

Considerable pains have been taken to print this volume as correctly as possible; but several errors of the press have been discovered, by a young friend who has read the work before the publication. Most of these mistakes have been corrected with a pen. The reader is requested to make the following additions and alterations,

Page 43, line 2, *for* by *read* through.

Page 47, line 9, *for* proportional to *read* equal to.

Page 55, [82*a*], in $\Pi . (n, c, \varphi)$, the factor $\Delta . (c, \varphi)$ ought to be in the denominator.

Page 98, [143^{v}], *read*, $d\varphi = \Sigma m . (P . dx + Q . dy + R . dz)$.

Page 133, line 14, *read*, homogeneous cylinder, of an elliptical base.

There may be other mistakes, which have been passed over without notice, as it is extremely difficult to print a work of this kind free from error.

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